Chirplet chains: quasi-physical model for near-optimal detection of GW chirps

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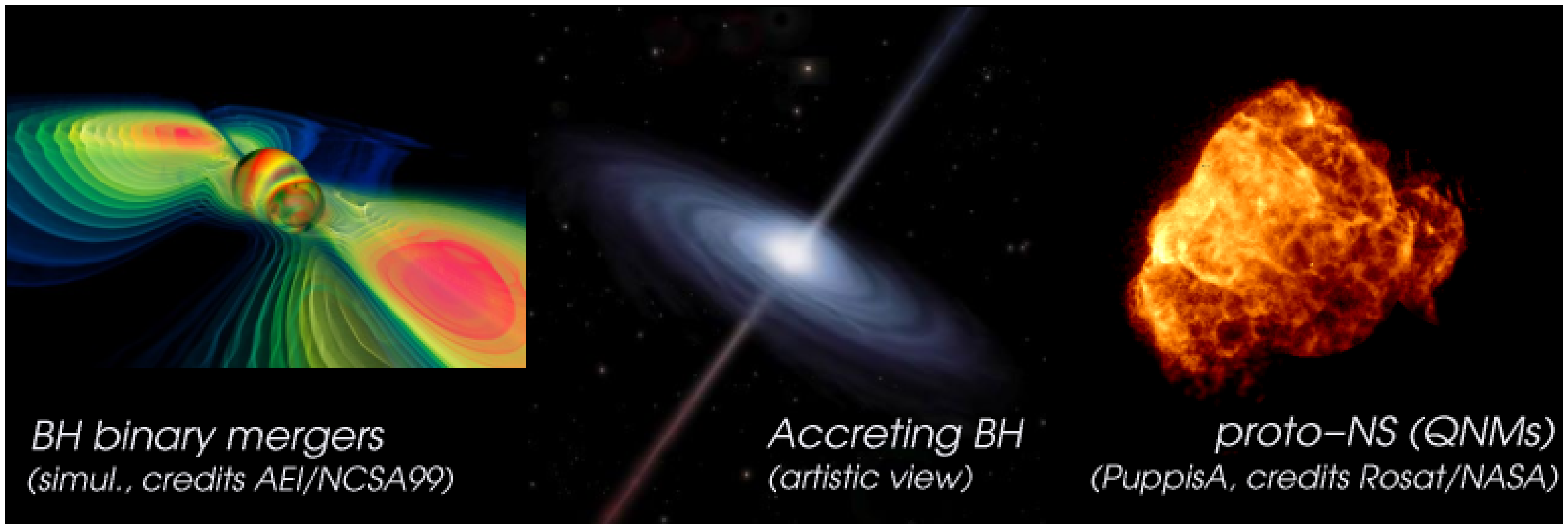


exploratory search vs. targeted search (precise model)

unmodelled GW chirps: $s(t) \equiv A\cos(\phi(t)+\theta)$, $\{A, \theta$ unknowns$\}$
typical duration $T \sim$ few sec in detector bandwidth

$\phi(t)$ is partially/not known

inst. frequency: $f(t) \equiv \dot{\phi}(t)/(2\pi)$
smooth chirps are such that $|\dot{f}(t)| \leq \dot{F}$ and $|\ddot{f}(t)| \leq \ddot{F}$

Chirps and optimal detection

$$\textit{detection problem}: \ (H_0)\ : x_k = n_k \ \textit{vs.}\ (H_1)\ : x_k = s_k + n_k,\ k = 0 \dots N-1$$

chirp $s_k = s(t_s k)$ in n_k white Gaussian noise ($\sigma^2 = 1$)

likelihood ratio: $\lambda = \mathbb{P}(x_k|H_1)/\mathbb{P}(x_k|H_0)$, max. over A and θ, $\phi(t)$ fixed

$$\ell(x,\phi) = \log(\lambda_{\max}) \propto \left(\sum_{k=0}^{N-1} x_k \tilde{c}_k\right)^2 + \left(\sum_{k=0}^{N-1} x_k \tilde{s}_k\right)^2 \lessgtr \eta$$

Gram-Schmidt orthonormalization: $\{\cos\phi_k; \sin\phi_k\} \to \{\tilde{c}_k; \tilde{s}_k\}$

quadratic match filtering

$\ell \leftrightarrow$ *distance* between data and (chirp) model

find smooth chirp providing best fit: $\ell_{\max} = \max_{\text{smooth chirps}}\{\ell\}$

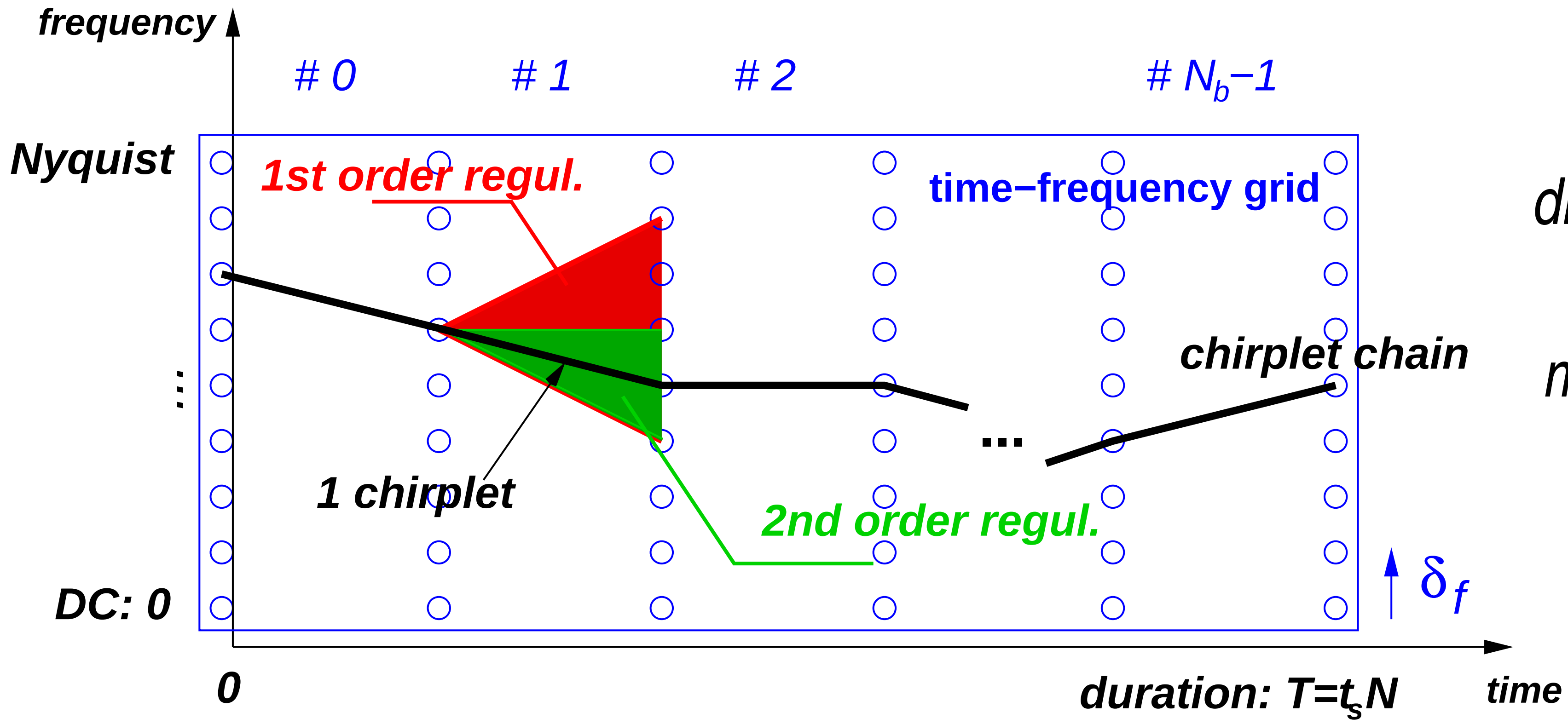
we "receive" $x_k \hat{=} s_k = A\cos(\phi_k + \theta)$ and we "look" with template phase ϕ_k^*

$$\Delta\ell(\phi,\phi^*) \equiv \frac{\ell(s,\phi) - \ell(s,\phi^*)}{\ell(s,\phi)} \quad \textit{metric} \text{ in chirp sets}$$

at second order in $\Delta\phi_k \equiv \phi_k - \phi_k^*$ (under mild conditions)

$$\Delta\ell(\phi,\phi^*) \approx \frac{1}{N}\sum_{k=0}^{N-1}(\Delta\phi_k - \overline{\Delta\phi})^2 \quad \text{with } \overline{\Delta\phi} = \frac{1}{N}\sum_{k=0}^{N-1}\Delta\phi_k$$

chirplet chains (CC): templates for smooth chirps



discretize the space of smooth chirps ($\dot{F}$, $\ddot{F}$)

make chains of "short" linear chirps (chirplet)

regularity constraints: limit chirping rate

free parameters: N_b, δ_f, N_s', N_s''

if $N_b = \ddot{F}T^{3/2}$ and $N_s' = \dot{F}T/(N_b\delta_f)$,

$$\boxed{\forall \text{ smooth chirps } \phi \; \exists \text{ CC } \phi^* \quad \Delta\ell(\phi,\phi^*) \lesssim \pi^2/48(N_s''\delta_f t_s N)^2}$$

CCs cover set of smooth chirps entirely $\longrightarrow$ the template grid is *tight*!

- feasability/orders of magnitude

toy model, Newtonian inspiral, fix $\Delta\ell \lesssim \pi^2/48 \approx 20\%$ min. match

$$f_s = 2048 \text{ Hz}, \; M \gtrsim 55 M_\odot, \; T \lesssim 2 \text{ s}$$

$$\dot{F} = \dot{f}_{\text{isco}} = 1 \text{ kHz/s}, \; \ddot{F} = \ddot{f}_{\text{isco}} = 32 \text{ kHz/s}^2$$

$\longrightarrow \quad N = 4096$, $N_b = 512$, $\delta_f = f_s/(2N)$, $N_s' = 16$, $N_s'' = 2$

best chirplet chain: exhaustive search intractable

find best match: $\ell_{\max} = \max_{\text{smooth chirps}}\{\ell\} \to \max_{\text{all CCs}}\{\ell\}$

brute force solution: try them all! how many CCs?

$$\#_{cc} \approx 2N'_s N_f (2N''_s + 1)^{N_b - 1} \quad \text{asymptotically } (N \to +\infty)$$

for *toy model*, $\#_{cc} \sim 10^{350}$ — infeasible

exhaustive search needs exponential time!

the game is *not* over: near-optimal search in polynomial time

best chirplet chain: near-optimal search

- *approximation 1*: ϕ_k is CC $\rightarrow \cos\phi_k$ and $\sin\phi_k$ are $\sim \perp$

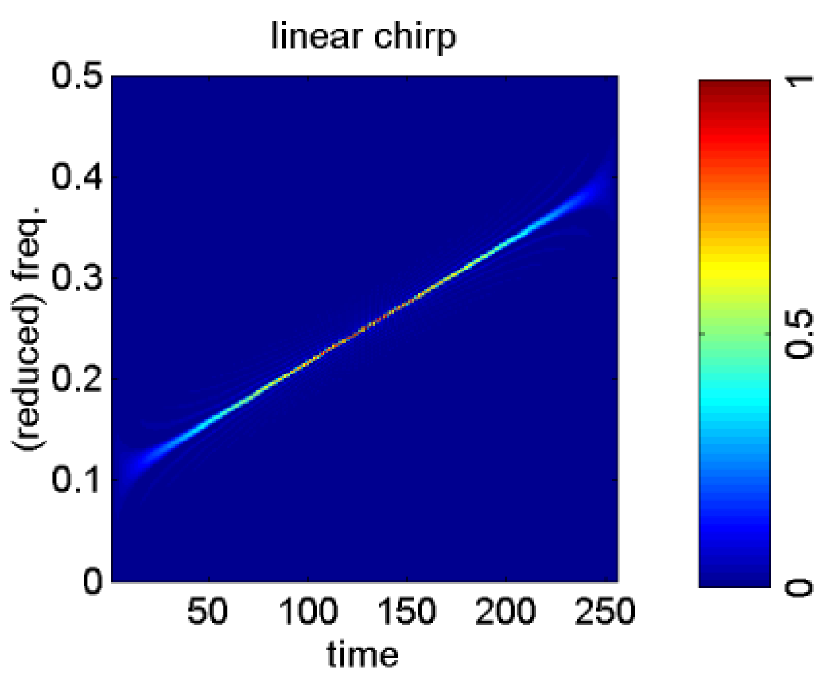
$$\ell \approx \frac{2}{N}\left|\sum_{k=0}^{N-1} x_k e_k\right|^2 \equiv \tilde{\ell} \quad \text{with } e_k = \sqrt{2/N}\exp i\phi_k$$

good approx if CC doesn't come "too" close from DC nor Nyquist

- *go to time-frequency*: discrete TF Wigner-Ville (cf. our poster)

$$\text{Moyal:} \quad \tilde{\ell} = \frac{1}{2N}\sum_{n=0}^{N-1}\sum_{m=0}^{2N-1} W_x(n,m)W_e(n,m)$$

- *approximation 2*: W_e is almost Dirac $\approx \delta(m - m_n^{(cc)})$



path integral: $\tilde{\tilde{\ell}} = \sum_{n=0}^{N-1} W_x(n, m_n^{(cc)})$

$$\max\{\tilde{\tilde{\ell}}\} = \text{longest path problem}$$

dynamic programming solves it in polynomial time

best chirplet chain: near-optimal search

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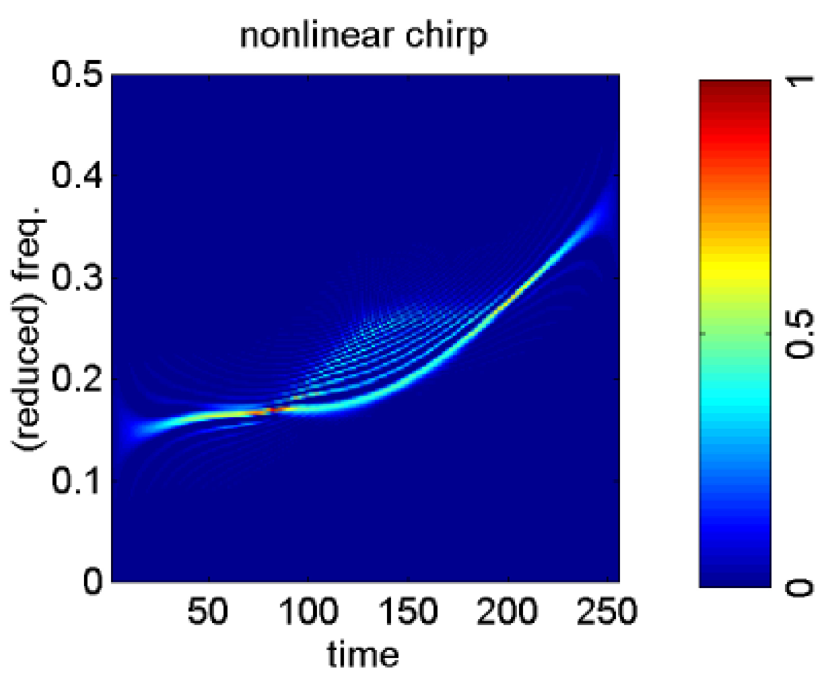

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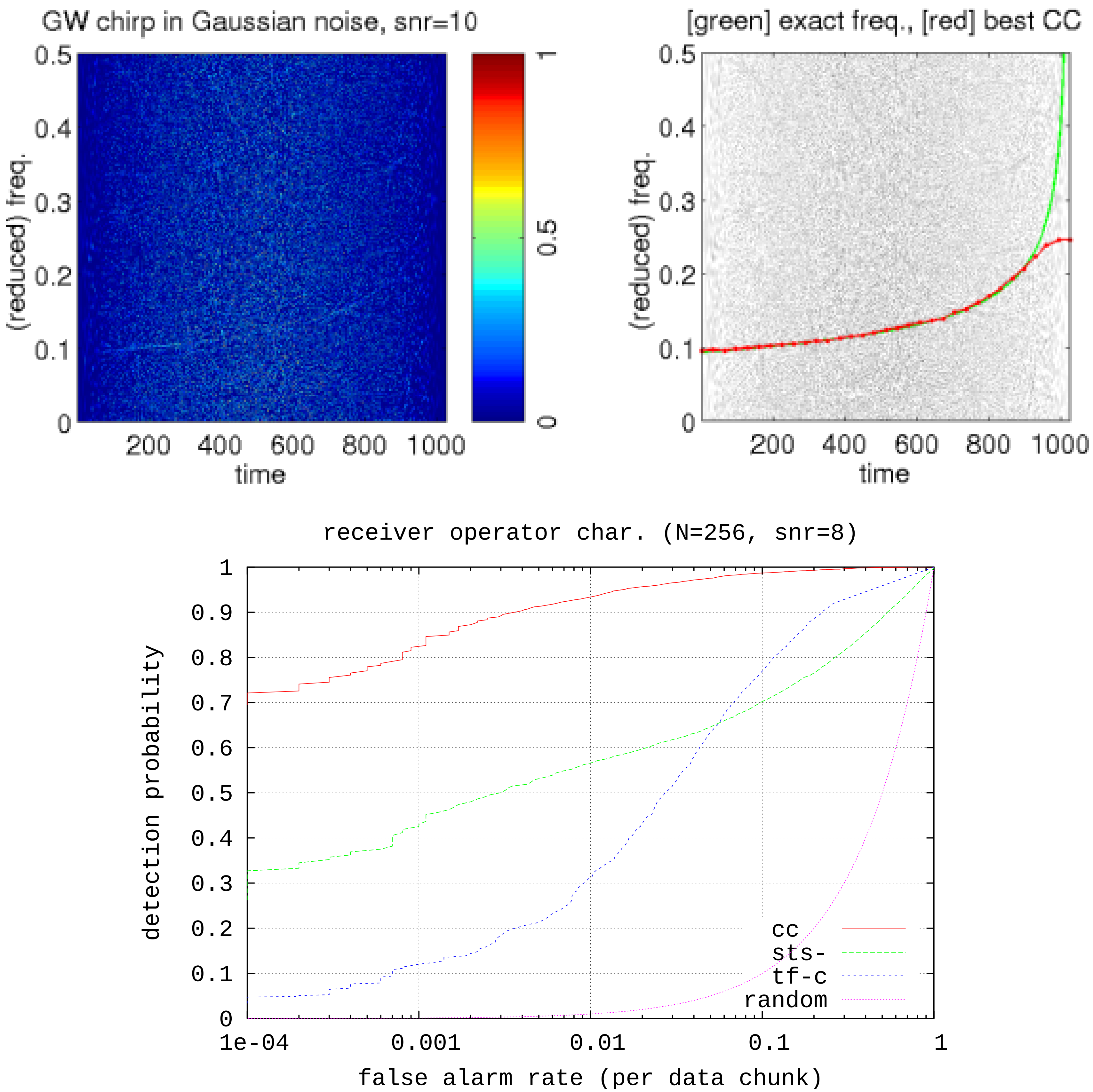
best chirplet chain: check and comparison



`sts`= *Signal Track Search* (Balasubramanian/Anderson: PRD102001, 2001)

`tfc`= *TFCluster* (J. Sylvestre: PRD102004, 2002)

concluding remarks

- *smooth chirps* = general model of "near physical" GW chirps
- *chirplet chains* = *tight* template grid for smooth chirps

helps to understand the geometry and dimension of chirp sets

- original time-frequency search method:

$$\text{a bench of} \sim N_f N_s''^{N_b} \text{ matched filters (CCs=templates)}$$
$$\text{estimate the max in } O(N_s' N^2) \text{ fl.p. operations}$$

- tractable in real time with moderate comput. resources (few tenth of GFlops for interesting cases)