RAQIS'07

Recent Advances in Quantum Integrable Systems

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Avant-propos

La conférence *RAQIS'07* (*Recent Advances in Quantum Integrable Systems 2007*) s'est déroulée à Annecy-le-Vieux (France) du 11 au 14 septembre 2007. Environ 60 physiciens venus du monde entier ont participé à ce colloque, qui a abordé les thèmes principaux étudiés en Systèmes Intégrables.

Ce domaine de recherche est très actif et possède une palette amplement variée en ce qui concerne ses applications.

De nombreuses personnes travaillent sur de multiples applications en physique de la matière condensée : de grands progrès ont été faits dans le calcul de quantités physiques, comme par exemple les fonctions de corrélation, à la fois au niveau analytique et au niveau numérique. Dans le même temps, les expérimentateurs de la matière condensée sont devenus capables de réaliser des systèmes quasi-unidimensionnels très proches de modèles de chaînes de spins, ou plus généralement de modèles intégrables. Il est clair que la confrontation des mesures expérimentales et des résultats théoriques est une perspective des plus excitantes et des plus prometteuses pour les années à venir.

Il faut aussi noter l'émergence, ces dernières années, des systèmes intégrables en théorie des cordes et dans les théories de Yang-Mills supersymétriques : beaucoup de travaux ont été dévolus à cette connexion, qui pourrait amener à une compréhension plus approfondie de ces théories.

Les aspects mathématiques des systèmes intégrables ne doivent pas être oubliés. Les structures de Hopf, les twists ou les cocycles recèlent encore de nombreux mystères attendant que leur Livingstone mathématique les dévoile ou leur Stanley physique les retrouve.

La structure algébrique des systèmes intégrables avec bords mérite elle aussi des éclaircissements. Indubitablement, une meilleure compréhension de cette structure facilitera le calcul des quantités physiques liées à ces modèles.

Enfin, les systèmes intégrables avec défauts ont connus ces derniers temps un regain d'intéret, en particulier les théories des champs intégrables sur graphes. Il paraît clair que ce thème de recherche est destiné à se développer, tant au niveau théorique qu'en ce qui concerne les expériences.

Tous ces sujets ont été abordés durant la conférence, et les participants ont apprécié la très haute qualité des séminaires. Ce fut certainement une des raisons principales du succès de la rencontre, et nous voudrions remercier chaleureusement les orateurs pour leurs efforts en ce domaine. L'autre raison essentielle de ce succès réside dans l'atmosphère très détendue et amicale qui a régné durant tout le colloque. Grâce en soit rendue à tous les participants ! Une mention spéciale est à accorder à ces physiciens endurcis qui, malgré des repas riches et des vins des plus fruités, s'en sont allé suivre opiniâtrement les séminaires de l'après-midi.

Une partie du succès est aussi due à la localisation exceptionnelle du LAPTH à Annecy, à la très haute qualité des infrastuctures du LAPP, ainsi qu'à l'efficacité redoutable d'un certain nombre de personnes auxquelles nous voudrions exprimer toute notre reconnaissance.

Un très grand merci à toute l'équipe administrative du LAPTH, Véronique Jonnery, Virginie Malaval et Dominique Turc, qui ont, discrètement et efficacement, aplani toutes les difficultés et résolu tous les problèmes survenus durant la conférence. Il nous faut aussi louer le gigantesque travail accompli par Véronique avant et après la rencontre : elle fut la pierre d'angle sur laquelle se sont appuyés nos rêves quantiques d'une tour Eiffel intégrable.

Les participants ont apprécié le réseau informatique mis à leur disposition. Le crédit en revient à l'équipe informatique du LAPP-LAPTH : Moha Ahbar, Monique Cottin, Alain Derible (qui maintenant profite d'une retraite amplement méritée), Gérard Dromby, Sylvain Garrigues, Frédéric Girault, Muriel Gougerot, Nicole Iribarnes et Patrick Letournel.

Le financement fut quelque peu difficile à obtenir et surtout à maintenir (que Dieu nous préserve de la rémanence du CNRS et des fluides de l'Université !), mais il fut bien sûr essentiel à l'organisation de RAQIS'07. Nous remercions le CNRS, l'Université de Savoie, le Ministère des Affaires Étrangères, le Conseil Général de Haute-Savoie, le Conseil Régional Rhône-Alpes, l'Institut Universitaire de France et le LAPTH pour leur soutien financier.

Enfin, nous voudrions remercier tous les membres du LAPTH pour leur patience et leurs encouragements, sans oublier Patrick Aurenche, directeur du LAPTH, pour son soutien sans faille et Yannis Karyotakis, directeur du LAPP, qui a mis à notre disposition les locaux du LAPP.

Les organisateurs de *RAQIS'07*

Luc Frappat

Eric Ragoucy

Se réjouissant à l'avance de votre prochaine venue...

Forewords

The conference *RAQIS'07* (*Recent Advances in Quantum Integrable Systems 2007*) took place at Annecy-le-Vieux (France) from 11 September to 14 September 2007. About 60 physicists from all over the world attended the meeting, which covered the main topics within the field of Integrable Systems.

The domain is a very active area of research and has a very wide range of applications.

Many people are working on numerous applications in the domain of condensed matter physics: much progress has been done in the calculation of physical quantities, such as correlation functions, at the level of both numerical and analytical calculations. At the same time, experimentalists of condensed matter physics are nowadays able to build effective one-dimensional systems that are very close to spin chain models, or more generally to integrable models. Comparison between theoretical and experimental quantities is certainly one of the most exciting and promising prospect of the next years.

Let us also mention the appearance, these last years, of integrable systems in string and supersymmetric Yang-Mills theories: a lot of work has been done on this connection and it may lead to a deeper understanding of these theories.

The mathematical aspect of integrable systems has not to be put aside. Hopf structures, twists or cocycles keep some mysteries waiting for their mathematical Livingstone and their physical Stanley.

Integrable systems with open boundaries also need some clarifications on the algebraic structure they are based on. Undoubtedly, its understanding will provide new insights on the physical quantities related to these models.

Finally, integrable systems with defects attracted renewed interest these last years, specially integrable field theories on graphs. It seems clear that this part of our field will improve both at theoretical and very experimental levels.

All these topics have been tackled during the conference. Participants have enjoyed the very high quality of the talks on these subjects. It was certainly one of the main reason for the success of the meeting, and we would like to thank the speakers for their efforts. Another main reason undoubtedly lies in the very friendly atmosphere among participants. We would like to express our deep gratitude to speakers and participants for it. A mention is due for these though physicists that, despite the rich lunches and the fruity wines, have been able follow the afternoon talks!

Part of the success was certainly due to the exceptional location of the LAPTH in Annecy, the high quality of the LAPP infrastructure, and the efficiency of many people, to whom we owe our deepest gratefulness.

Many thanks to the administrative staff of LAPTH, Véronique Jonnery, Virginie Malaval and Dominique Turc who discreetly and efficiently smoothed everything during the conference. A special thanks is due to Véronique for the enormous work she did before and after the conference: it was the cornerstone on which we built our Integrable Eiffel tower.

Participants also enjoyed the computer and network falicities during the conference. Credits for them have to be granted to the members of the computing center of LAPP-LAPTH: Moha Ahbar, Monique Cottin, Alain Derible (who now hopefully enjoys a deserved retirement), Gérard Dromby, Sylvain Garrigues, Frédéric Girault, Muriel Gougerot, Nicole Iribarnes and Patrick Letournel.

Financial support was a hard time to get and to keep (Heaven preserve us from the *CNRS's remanence* and the *UdS's 12.5%*!), but was essential in the organisation of RAQIS'07. We would like to thank CNRS, the University of Savoie, the Ministry of Foreign Affairs, the Council of Haute-Savoie, the Council of Rhône-Alpes region, the Institut Universitaire de France and LAPTH for their financial support.

Finally, we would like to thanks LAPTH's members for their encouragements and their patience during the conference. We do not forget Patrick Aurenche, head of LAPTH, for his strong support and advice, and Yannis Karyotakis, head of LAPP, who left the LAPP's premises to our disposal.

RAQIS'07 organizers

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Enjoying in advance your next coming...

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Contents

New results in the XXZ open spin chain

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Abstract

In this review, I describe a recent approach based on the representation theory of the $q-$Onsager algebra which is used to derive exact results for the XXZ open spin chain. The complete spectrum and eigenstates are obtained as rational functions of a single variable which discrete values correspond to the roots of a certain characteristic polynomial. Comments and open problems are also presented.

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Keywords: $q-$Onsager algebra; Tridiagonal algebra; XXZ open spin chain; Boundary integrable models

1 Introduction

Since Sklyanin's seminal work [1] on spin chains with integrable boundary conditions, finding exact results such as the energy spectrum of a model, corresponding eigenstates and correlation functions for elementary excitations has remained an interesting problem in connection with condensed matter or high-energy physics. Among the known integrable open spin chains that have been considered in details, one finds the XXZ open spin chain. For generic boundary conditions, its Hamiltonian reads

$$\begin{aligned} H_{XXZ} &= \sum_{k=1}^{N-1}\Big(\sigma_1^k\sigma_1^{k+1}+\sigma_2^k\sigma_2^{k+1}+\Delta\sigma_3^k\sigma_3^{k+1}\Big) \\ &+ \frac{(q^{1/2}-q^{-1/2})}{(\epsilon_+^{(0)}+\epsilon_-^{(0)})}\Big(\frac{(\epsilon_+^{(0)}-\epsilon_-^{(0)})}{2}\sigma_3^1+\frac{2}{(q^{1/2}-q^{-1/2})}\big(k_+\sigma_+^1+k_-\sigma_-^1\big)\Big) \\ &+ \frac{(q^{1/2}-q^{-1/2})}{(\kappa+\kappa^*)}\Big(\frac{(\kappa-\kappa^*)}{2}\sigma_3^N+2(q^{1/2}+q^{-1/2})\big(\kappa_+\sigma_+^N+\kappa_-\sigma_-^N\big)\Big) \quad (1.1) \end{aligned}$$

where $\sigma_{1,2,3}$ and $\sigma_\pm=(\sigma_1\pm i\sigma_2)/2$ are usual Pauli matrices and $\Delta=(q^{1/2}+q^{-1/2})/2$ denotes the anisotropy parameter. Here, we restrict our attention to the (massless) regime $-1\leq\Delta\leq 1$ i.e. $q=\exp(\phi)$ with ϕ purely imaginary. To exhibit the *six* independent boundary parameters, below we will sometimes use the following parametrization with $\theta,\tilde\theta\in\mathbb{R}$ and $\alpha,\tilde\alpha\in\mathbb{C}$:

$$\begin{aligned} &\epsilon_+^{(0)}=(\epsilon_-^{(0)})^\dagger=\cosh\alpha\ , \qquad k_+=(k_-)^\dagger=-(q^{1/2}-q^{-1/2})e^{i\theta}/2 \qquad \text{(left)}\ , \\ &\kappa^*=(\kappa)^\dagger=-\cosh\tilde\alpha\ , \qquad \kappa_+=-(\kappa_-)^\dagger=-e^{i\tilde\theta}/(2(q^{1/2}+q^{-1/2})) \qquad \text{(right)}\ . \quad (1.2) \end{aligned}$$

Known to be integrable since 1988, for the simplest *diagonal* boundary conditions i.e. $k_\pm=\kappa_\pm\equiv 0$ the algebraic Bethe ansatz approach has been succesfully applied and used to derive the spectrum, eigenstates [1] and more recently correlations functions [2] of the model. The derivation of these results essentially relies on the fact that the pseudo-vacuum (or reference) state of this particular case of open spin chain actually coincides with the one associated with the XXZ spin chain with *periodic* boundary conditions.
For non-diagonal boundary conditions, there is no simple or obvious pseudo-vacuum state. From 2001 to 2007, several attempts [3, 4, 5, 6] were considered in order to circumvent this difficulty: either based on the functional Bethe ansatz approach, or based on the algebraic Bethe ansatz approach using suitable gauge transformations of the double-row monodromy matrix provided the boundary parameters satisfy certain linear relations or q is a root of unity. In any case, either the spectrum was obtained but not the eigenstates or the spectrum and eigenstates were obtained but only for certain sets of boundary conditions. The main advantage of a Bethe-type solution being its efficiency in studying the thermodynamic limit of the spin chain through the link between Bethe equations and non-linear integral equations, it may explain why Bethe ansatz approaches (functional or algebraic) have deserved such attention in all attempts at solving the XXZ open spin chain

(1.1). However, for generic boundary parameters and q, these standard approaches have failed. Note however a rather simple functional approach recently proposed for generic parameters, where the spectrum is written in terms of solutions of a system of *highly transcendental equations* which are not Bethe equations [7].
Motivated by the challenging problem of solving the XXZ open spin chain (1.1) in the most general regime of integrable boundary conditions, in 2005 we decided to explore a *different* and *non-standard* (compared to Bethe ansatz) path which takes its roots in Onsager's original article on the solution of the planar Ising model [8]. A novel approach followed, essentially based on the analysis of the representation theory of the $q-$Onsager algebra. After some efforts [9, 10, 11], we used this approach to derive the spectrum and the eigenstates of the XXZ open spin chain (1.1) for generic boundary conditions and q, as well as we recovered the known results for special relations between left and right boundary parameters [12].
In this review, our main objective will be to present the novel approach and methodology we have developped in the last two years and describe the main results rather than the technical aspects or details which can be found in our papers [9, 10, 11, 12] written in collaboration with K. Koizumi. In addition, comments and further directions are briefly presented.

2 The strategy

The starting point of the analysis was the discovery that the transfer matrix $t_{XXZ}(u)$ generating (1.1) can be written in terms of a finite set of mutually commuting quantities $\mathcal{I}_{2l+1}$ which form a $q-$deformation of the Onsager's or Dolan-Grady hierarchy [13, 14]. Namely, one has:

$$t_{XXZ}(u) = \sum_{l=0}^{N-1} \mathcal{F}_{2l+1}(u)\ \mathcal{I}_{2l+1} + \mathcal{F}_0(u)\ \mathbb{I} \tag{2.1}$$

with

$$\mathcal{I}_{2l+1} = \kappa \mathcal{W}_{-l}^{(N)} + \kappa^* \mathcal{W}_{l+1}^{(N)} + \frac{\kappa_+}{k_+}\mathcal{G}_{l+1}^{(N)} + \frac{\kappa_-}{k_-}\tilde{\mathcal{G}}_{l+1}^{(N)} \tag{2.2}$$

where $\mathcal{F}_{2l+1}(u), \mathcal{F}_0(u)$ are rational functions that have been explicitely calculated in [10]. In this description, note that the remaining boundary parameters in (1.2) enter in the explicit realizations of the $4N$ fundamental operators [10]. Such remarkable structure comes from the fact that the elements of the double-row monodromy matrix (also called Sklyanin's operator) $\mathcal{T}(u)$ - u being the spectral parameter - associated with the XXZ open spin chain (1.1) can be decomposed on a basis of $4N$ operators $\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)}$, $k = 0, 1, ..., N-1$, formally as (details can be found in [10]):

$$\mathcal{T}(u) = \begin{pmatrix} \mathcal{A}(u; \{\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}\}) & \mathcal{B}(u; \{\mathcal{G}_{k+1}^{(N)}\}) \\ \mathcal{C}(u; \{\tilde{\mathcal{G}}_{k+1}^{(N)}\}) & \mathcal{D}(u; \{\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}\}) \end{pmatrix} . \tag{2.3}$$

In addition, using the fact that the elements of the double-row monodromy matrix satisfy the reflection algebra it is possible to show that these (spectral parameter independent) operators - which act on N-tensor product representations of $U_{q^{1/2}}(sl_2)$ - induce a finite dimensional representation of the $q-$deformed analogue of the Onsager infinite dimensional algebra [9] with defining relations [1] for $k,l\in\mathbb{N}$

$$\begin{aligned}
&\big[\mathsf{W}_0,\mathsf{W}_{k+1}\big]=\big[\mathsf{W}_{-k},\mathsf{W}_1\big]=\frac{1}{(q^{1/2}+q^{-1/2})}\big(\tilde{\mathsf{G}}_{k+1}-\mathsf{G}_{k+1}\big)\ ,\\
&\big[\mathsf{W}_0,\mathsf{G}_{k+1}\big]_q=\big[\tilde{\mathsf{G}}_{k+1},\mathsf{W}_0\big]_q=\rho\mathsf{W}_{-k-1}-\rho\mathsf{W}_{k+1}\ ,\\
&\big[\mathsf{G}_{k+1},\mathsf{W}_1\big]_q=\big[\mathsf{W}_1,\tilde{\mathsf{G}}_{k+1}\big]_q=\rho\mathsf{W}_{k+2}-\rho\mathsf{W}_{-k}\ ,\\
&\big[\mathsf{W}_0,\mathsf{W}_{-k}\big]=0\ ,\quad \big[\mathsf{W}_1,\mathsf{W}_{k+1}\big]=0\ ,\\
&\big[\mathsf{G}_{k+1},\mathsf{G}_{l+1}\big]=0\ ,\quad \big[\tilde{\mathsf{G}}_{k+1},\tilde{\mathsf{G}}_{l+1}\big]=0\ ,\quad \big[\tilde{\mathsf{G}}_{k+1},\mathsf{G}_{l+1}\big]+\big[\mathsf{G}_{k+1},\tilde{\mathsf{G}}_{l+1}\big]=0\ .
\end{aligned}\tag{2.4}$$

In light of the above formulation, it was tempting to consider the XXZ open spin chain (1.1) from the point of view of the representation theory of the algebra (2.4) in view of the success of the Onsager's algebra and related representation theory in solving the Ising model [8], generalizations [14] and superintegrable chiral Potts model [15, 16]. Having this in mind, we implemented the following procedure in order to solve the spectral problem for (1.1) or, more generally, the transfer matrix:

• Use the relation between the algebra (2.4) and Terwilliger's tridiagonal algebra [17]. See [9];

• Show that $\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}$ form a tridiagonal pair (see below). See [18, 9, 11];

• Construct the states $\{\psi_{n[j]}^{(N)}\}$ and $\{\varphi_{s[k]}^{(N)}\}$ on which $\mathcal{W}_0^{(N)},\mathcal{W}_1^{(N)}$ act as block tridiagonal matrices. See [11];

• Show that $\mathcal{W}_{-k}^{(N)},\mathcal{W}_{k+1}^{(N)},\mathcal{G}_{k+1}^{(N)},\tilde{\mathcal{G}}_{k+1}^{(N)}$, $k=1,...,N-1$ also act as block tridiagonal matrices; See [12].

• Write the spectral problem for all quantities $\mathcal{I}_{2l+1}$ as:

→ a *coupled system three-term recursion relations*;

→ a *system of second-order $q-$difference equations*;

and derive the spectrum and eigenstates. See [12].

It is important to stress that the above procedure introduced to derive all eigenvalues and eigenstates of (2.1) solely uses the *block tridiagonal* structure of the commuting operators $\mathcal{I}_{2l+1}$ in two suitable basis. However, similarly to the Ising or superintegrable model [16], it is possible to show that the operators $\mathcal{W}_{-k}^{(N)},\mathcal{W}_{k+1}^{(N)},\mathcal{G}_{k+1}^{(N)},\tilde{\mathcal{G}}_{k+1}^{(N)}$, $k=0,...,N$ satisfy a set of linear relations. Exploiting this property should lead to *another* solution - may be simpler - to the spectral problem for the transfer matrix of the XXZ open spin chain (1.1). We will explain this point in the last Section.

[1] The $q-$commutator $[X,Y]_q=q^{1/2}XY-q^{-1/2}YX$ is introduced and the scalar $\rho=(q^{1/2}+q^{-1/2})^2k_+k_-$ for the XXZ open spin chain.

3 Solving the spectral problem

3.1 Mathematical background

Almost nothing being known about the representation theory of the algebra (2.4), we used its connection with Terwilliger's tridiagonal algebra [17] (sometimes called the $q-$Onsager algebra). Indeed, we had previously noticed [10] (see also [18]) that $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ given by

$$\begin{aligned} \mathcal{W}_0^{(N)} &= (k_+\sigma_+ + k_-\sigma_-) \otimes I\!I^{(N-1)} + q^{\sigma_3/2} \otimes \mathcal{W}_0^{(N-1)}, \\ \mathcal{W}_1^{(N)} &= (k_+\sigma_+ + k_-\sigma_-) \otimes I\!I^{(N-1)} + q^{-\sigma_3/2} \otimes \mathcal{W}_1^{(N-1)} \end{aligned} \tag{3.1}$$

with $\mathcal{W}_0^{(0)} \equiv \epsilon_+^{(0)}$, $\mathcal{W}_1^{(0)} \equiv \epsilon_-^{(0)}$ satisfy the so-called $q-$Dolan-Grady relations

$$\begin{aligned} \big[\mathcal{W}_0^{(N)}, \big[\mathcal{W}_0^{(N)}, \big[\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}\big]_q\big]_{q^{-1}}\big] &= \rho\big[\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}\big] \ , \\ \big[\mathcal{W}_1^{(N)}, \big[\mathcal{W}_1^{(N)}, \big[\mathcal{W}_1^{(N)}, \mathcal{W}_0^{(N)}\big]_q\big]_{q^{-1}}\big] &= \rho\big[\mathcal{W}_1^{(N)}, \mathcal{W}_0^{(N)}\big] \end{aligned} \tag{3.2}$$

and induce a representation of the tridiagonal algebra. According to Terwilliger's work [17], it follows that $\mathcal{W}_0^{(N)}, \mathcal{W}_1^{(N)}$ provide an explicit example of *tridiagonal pair*. Roughly speaking, for generic boundary parameters (1.2) and generic values of q it means that there exists a complete basis (resp. 'dual' basis) of $\mathcal{V}$ in which $\mathcal{W}_0^{(N)}$ (resp. $\mathcal{W}_1^{(N)}$) is represented by a diagonal matrix with degeneracies and $\mathcal{W}_1^{(N)}$ (resp. $\mathcal{W}_0^{(N)}$) is represented by a block tridiagonal matrix. Such basis and dual one have been constructed in [11], and are denoted below $\{\psi_{n[j]}^{(N)}\}$ and $\{\varphi_{s[k]}^{(N)}\}$.

Let us recall the main results of [11]. Using the canonical basis $\bigotimes_{j=1}^N |\pm\rangle_j$ on which the nonlocal operators (3.1) act, let us introduce the 2^N ordered states

$$\begin{aligned} \psi_{n[j]}^{(N)} &= \Big(e^{\alpha+(N-1-2n)\phi/2+i\theta}|+\rangle_N + |-\rangle_N\Big) \otimes \psi_{n[j]}^{(N-1)} \\ &\qquad\qquad \text{for} \quad j \in \{1, \dots, \binom{N-1}{n}\} \ , \\ \psi_{n[j]}^{(N)} &= \Big(e^{-\alpha-(N+1-2n)\phi/2+i\theta}|+\rangle_N + |-\rangle_N\Big) \otimes \psi_{n-1[j-\binom{N-1}{n}]}^{(N-1)} \\ &\qquad\qquad \text{for} \quad j \in \{\binom{N-1}{n}+1, \dots, \binom{N}{n}\} \ . \end{aligned} \tag{3.3}$$

For generic boundary parameters and q, these states form a complete[2] basis of $\mathcal{V}$ on which the tridiagonal pair $\mathcal{W}_0^{(N)}$, $\mathcal{W}_1^{(N)}$ acts [11]. Using the explicit expressions (3.1) it is straightforward to exhibit the diagonal and block tridiagonal structure of the operators $\mathcal{W}_0^{(N)}$, $\mathcal{W}_1^{(N)}$, respectively, in this basis:

$$\begin{aligned} \mathcal{W}_0^{(N)}\psi_{n[j]}^{(N)} &= \lambda_n^{(N)}\psi_{n[j]}^{(N)} \qquad \text{with} \qquad \lambda_n^{(N)} = \cosh(\alpha + (N-2n)\phi/2) \ , \\ \mathcal{W}_1^{(N)}\psi_{n[j]}^{(N)} &= \sum_{i=1}^{\binom{N}{n+1}} b_{n[ij]}^{(N,0)}\psi_{n+1[i]}^{(N)} + \sum_{i=1}^{\binom{N}{n}} a_{n[ij]}^{(N,0)}\psi_{n[i]}^{(N)} + \sum_{i=1}^{\binom{N}{n-1}} c_{n[ij]}^{(N,0)}\psi_{n-1[i]}^{(N)} \end{aligned} \tag{3.4}$$

[2] For special relations among the boundary parameters, the representation may become indecomposable. We do not consider such possibility here.

with $n=0,1,...,N$ and $j\in\{1,...,\binom{N}{n}\}$. Note that the explicit form of the coefficients $a^{(N,0)}_{n[ij]}$, $b^{(N,0)}_{n[ij]}$, $c^{(N,0)}_{n[ij]}$ can be found in [11]. Clearly, a similar analysis can also be performed by considering the eigenbasis of $\mathcal{W}_1^{(N)}$ instead of $\mathcal{W}_0^{(N)}$, leading to analogous results. In this case, the 'dual' states given by $\varphi^{(N)}_{s[k]}=\psi^{(N)}_{s[k]}|_{\alpha\to-\alpha^*,\ \phi\to-\phi,\ \theta\to\theta+i\pi}$ form an eigenbasis of $\mathcal{W}_1^{(N)}$ [11].

All higher operators realizing (2.4) being generated recursively from the lower ones [10], there were good reasons to expect that they would act in a simple manner on the eigenbasis of $\mathcal{W}_0^{(N)}$ or $\mathcal{W}_1^{(N)}$. Indeed, proceeding similarly for higher values of $l=1,2,...,N-1$ it is possible to show that *all* nonlocal operators $\mathcal{W}^{(N)}_{-k},\mathcal{W}^{(N)}_{k+1},\mathcal{G}^{(N)}_{k+1},\tilde{\mathcal{G}}^{(N)}_{k+1}$ *also* enjoy a block tridiagonal structure in the basis (3.3). After some straightforward calculations, one finds:

$$
\begin{aligned}
\mathcal{W}^{(N)}_{-l}\psi^{(N)}_{n[j]} &= \sum_{i=1}^{\binom{N}{n}} d^{(N,l)}_{n[ij]}\psi^{(N)}_{n[i]}\ ,\\
\mathcal{W}^{(N)}_{l+1}\psi^{(N)}_{n[j]} &= \sum_{i=1}^{\binom{N}{n+1}} b^{(N,l)}_{n[ij]}\psi^{(N)}_{n+1[i]}+\sum_{i=1}^{\binom{N}{n}} a^{(N,l)}_{n[ij]}\psi^{(N)}_{n[i]}+\sum_{i=1}^{\binom{N}{n-1}} c^{(N,l)}_{n[ij]}\psi^{(N)}_{n-1[i]}\ ,\\
\mathcal{G}^{(N)}_{l+1}\psi^{(N)}_{n[j]} &= \sum_{i=1}^{\binom{N}{n+1}} (q^{1/2}\lambda^{(N)}_n-q^{-1/2}\lambda^{(N)}_{n+1})\,b^{(N,l)}_{n[ij]}\psi^{(N)}_{n+1[i]}+\sum_{i=1}^{\binom{N}{n}} h^{(N,l)}_{n[ij]}\psi^{(N)}_{n[i]}\\
&\quad+\sum_{i=1}^{\binom{N}{n-1}} (q^{1/2}\lambda^{(N)}_n-q^{-1/2}\lambda^{(N)}_{n-1})\,c^{(N,l)}_{n[ij]}\psi^{(N)}_{n-1[i]}\ ,\\
\tilde{\mathcal{G}}^{(N)}_{l+1}\psi^{(N)}_{n[j]} &= \sum_{i=1}^{\binom{N}{n+1}} (q^{1/2}\lambda^{(N)}_{n+1}-q^{-1/2}\lambda^{(N)}_n)\,b^{(N,l)}_{n[ij]}\psi^{(N)}_{n+1[i]}+\sum_{i=1}^{\binom{N}{n}} h^{(N,l)}_{n[ij]}\psi^{(N)}_{n[i]}\\
&\quad+\sum_{i=1}^{\binom{N}{n-1}} (q^{1/2}\lambda^{(N)}_{n-1}-q^{-1/2}\lambda^{(N)}_n)\,c^{(N,l)}_{n[ij]}\psi^{(N)}_{n-1[i]}
\end{aligned}
\qquad (3.5)
$$

where all coefficients can be found in [12]. Note that they have a rather simple form regarding to the recursion on the index l, a fact which may have been anticipated from the existence of linear relations among the operators. See the comments below. In the dual basis $\{\varphi^{(N)}_{s[k]}\}$, similar results are also obtained.

Having two different basis $\{\psi^{(N)}_{n[j]}\},\{\varphi^{(N)}_{s[k]}\}$ on which all mutually commuting operators generating the $q-$Dolan-Grady hierarchy (2.1) act as block tridiagonal matrices with entries known explicitely, we are now ready to consider the spectral problem for the XXZ open spin chain (1.1) formulated using (2.1).

3.2 Eigenstates and spectrum for generic parameters

For generic values of the boundary parameters and q, there is no nontrivial subspace of $\mathcal{V}$ which is left invariant under the action of the nonlocal commuting operators $\mathcal{I}_{2l+1}$. Considering the most general linear combinations of states $\psi^{(N)}_{n[j]}$ or $\varphi^{(N)}_{s[k]}$, it follows that any eigenstate admits two dual representations with respect to the two dual basis on which the elements of the $q-$Onsager algebra act in a block tridiagonal form. Any eigenstate can be written either

$$\Psi(\Lambda_1)=\sum_{n=0}^{N}\sum_{j=1}^{\binom{N}{n}} f^{(+)}_{n[j]}(\Lambda_1)\psi^{(N)}_{n[j]} \qquad \text{or} \qquad \Psi(\Lambda_1)=\sum_{s=0}^{N}\sum_{k=1}^{\binom{N}{s}} f^{(-)}_{s[k]}(\Lambda_1)\varphi^{(N)}_{s[k]}\ , \tag{3.6}$$

where each of the dual families of weights $\{f^{(+)}_{n[j]}\},\{f^{(-)}_{s[k]}\}$ and the restricted set of 'allowed' eigenvalues $\{\Lambda_{1,1},...,\Lambda_{1,2^N}\}$ have to be determined. For generic boundary parameters, the spectrum of $\mathcal{I}_1$ is not degenerate. As $[\mathcal{I}_{2l+1},\mathcal{I}_{2k+1}]=0$, it follows that $\{f^{(+)}_{n[j]}\},\{f^{(-)}_{s[k]}\}$ can be uniquely derived from the constraint:

$$\mathcal{I}_1|\Psi(\Lambda_1)\rangle=\Lambda_1|\Psi(\Lambda_1)\rangle\ . \tag{3.7}$$

For any representation (3.6) choosen, this equation leads to a system of *coupled three-term recurrence relations* for the weights which, in some sense, generalizes the Askey-Wilson recursion relations [3] for $q-$orthogonal polynomials. For $n,s=0,1,...,N$ one has:

$$\begin{aligned}\sum_{m=1}^{\binom{N}{n-1}}\mathcal{B}^{(N,0)}_{n-1[jm]}f^{(N,+)}_{n-1[m]}(\Lambda_1) \;+\; &\sum_{m=1}^{\binom{N}{n+1}}\mathcal{C}^{(N,0)}_{n+1[jm]}f^{(N,+)}_{n+1[m]}(\Lambda_1)\\ +\;&\sum_{m=1}^{\binom{N}{n}}\left(\mathcal{A}^{(N,0)}_{n[jm]}-\Lambda_1\delta_{jm}\right)f^{(N,+)}_{n[m]}(\Lambda_1)=0\ ,\end{aligned} \tag{3.8}$$

or, alternatively,

$$\begin{aligned}\sum_{m=1}^{\binom{N}{s-1}}\mathcal{B}'^{(N,0)}_{s-1[km]}f^{(N,-)}_{s-1[m]}(\Lambda_1) \;+\; &\sum_{m=1}^{\binom{N}{s+1}}\mathcal{C}'^{(N,0)}_{s+1[km]}f^{(N,-)}_{s+1[m]}(\Lambda_1)\\ +\;&\sum_{m=1}^{\binom{N}{s}}\left(\mathcal{A}'^{(N,0)}_{s[km]}-\Lambda_1\delta_{km}\right)f^{(N,-)}_{s[m]}(\Lambda_1)=0\end{aligned} \tag{3.9}$$

where the coefficients $\mathcal{X}^{(N,0)}_{n[jm]},\mathcal{X}'^{(N,0)}_{s[km]}$ $(\mathcal{X}\in\{\mathcal{A},\mathcal{B},\mathcal{C}\})$ can be easily obtained from (3.5) [12]. Having 2^N equations in total, one finds that these weights are rational functions of

[3] For $N=1$, the coefficients are special cases of the Askey-Wilson ones.

the variable $\Lambda_1 \in \{\Lambda_{1,1}, ..., \Lambda_{1,2^N}\}$ where $\Lambda_{1,r}$ are the roots of the characteristic polynomial of degree $d = 2^N$

$$\mathcal{P}(\Lambda_1) = \det[\mathcal{I}_1 - \Lambda_1 I\!I] \ . \tag{3.10}$$

The complete family of eigenstates (3.6) being identified, the spectrum of the transfer matrix can now be derived. For instance, acting with higher operators $\mathcal{I}_{2l+1}$ of the $q-$Dolan-Grady hierarchy on these eigenstates, and taking the scalar product with

$$\tilde{\varphi}^{(N)}_{s[k]} = \psi^{(N)}_{s[k]}|_{\alpha\to -\alpha} \qquad \text{or} \qquad \tilde{\psi}^{(N)}_{n[j]} = \varphi^{(N)}_{n[j]}|_{\alpha^*\to -\alpha^*} \ ,$$

respectively, according to the representation choosen in (3.6) one ends up with a *second-order q−difference equation* that determines all higher necessary eigenvalues Λ_{2l+1}, $l = 1, ..., N-1$ as rational functions of Λ_1. For any choice of s, k, n, j, one writes either

$$\Lambda_{2l+1} = \sum_{m=1}^{\binom{N}{s+1}} \tilde{\mathcal{B}}^{(N,l)}_{s[mk]} \frac{\Psi^{(N,+)}_{[m]}(s+1)}{\Psi^{(N,+)}_{[k]}(s)} + \sum_{m=1}^{\binom{N}{s}} \tilde{\mathcal{A}}^{(N,l)}_{s[mk]} \frac{\Psi^{(N,+)}_{[m]}(s)}{\Psi^{(N,+)}_{[k]}(s)} + \sum_{m=1}^{\binom{N}{s-1}} \tilde{\mathcal{C}}^{(N,l)}_{s[mk]} \frac{\Psi^{(N,+)}_{[m]}(s-1)}{\Psi^{(N,+)}_{[k]}(s)} \ , \tag{3.11}$$

or

$$\Lambda_{2l+1} = \sum_{m=1}^{\binom{N}{n+1}} \tilde{\mathcal{B}}'^{(N,l)}_{n[mj]} \frac{\Psi^{(N,-)}_{[m]}(n+1)}{\Psi^{(N,-)}_{[j]}(n)} + \sum_{m=1}^{\binom{N}{n}} \tilde{\mathcal{A}}'^{(N,l)}_{n[mj]} \frac{\Psi^{(N,-)}_{[m]}(n)}{\Psi^{(N,-)}_{[j]}(n)} + \sum_{m=1}^{\binom{N}{n-1}} \tilde{\mathcal{C}}'^{(N,l)}_{n[mj]} \frac{\Psi^{(N,-)}_{[m]}(n-1)}{\Psi^{(N,-)}_{[j]}(n)} \ , \tag{3.12}$$

where the coefficients can be found in [12] and the functions

$$\Psi^{(N,+)}_{[k]}(s) = \big(\tilde{\varphi}^{(N)}_{s[k]}, \Psi(\Lambda_1)\big) \qquad \text{and} \qquad \Psi^{(N,-)}_{[j]}(n) = \big(\tilde{\psi}^{(N)}_{n[j]}, \Psi(\Lambda_1)\big)$$

have been introduced. To resume, all eigenvalues Λ_{2l+1}, $l = 1, ..., N-1$ are obtained as rational functions of the single variable Λ_1 restricted on the discrete support defined by $\mathcal{P}(\Lambda_1) = 0$ with (3.10). In terms of these, the spectrum of the Hamiltonian (1.1) reads:

$$\begin{aligned} E = & \frac{(q^{1/2} - q^{-1/2})(q^{1/2} + q^{-1/2})^{-1}}{2(\kappa + \kappa^*)(\epsilon^{(0)}_+ + \epsilon^{(0)}_-)} \left(\sum_{l=0}^{N-1} \frac{d\mathcal{F}_{2l+1}(u)}{du}\Big|_{u=1} \Lambda_{2l+1} + \frac{d\mathcal{F}_0(u)}{du}\Big|_{u=1} \right) \\ & - \left(N\Delta + \frac{(q^{1/2} - q^{-1/2})^2}{2(q^{1/2} + q^{-1/2})} \right) \ . \end{aligned}$$

3.3 Truncation and the Bethe ansatz regime of boundary parameters

For certain special relations between left and right boundary parameters, important simplifications occur. These situations arise when some of the off-diagonal *lower* (resp. *upper*)

blocks of the block tridiagonal matrices representing $\mathcal{I}_{2l+1}$ for any $l = 0, 1, ..., N-1$ identically vanish, leading respectively to two different conditions on the boundary parameters. For all l, i, j, m, k and integer p fixed:

$$\begin{aligned} \mathcal{B}^{(N,l)}_{p[ij]} \equiv 0 \qquad & \text{and} \qquad \mathcal{B}'^{(N,l)}_{N-p-1[mk]} \equiv 0 \\ & \Rightarrow \quad \alpha \pm \tilde{\alpha} = -i(\tilde{\theta} - \theta) - (N - 2p - 1)\phi/2 \pmod{2i\pi} \; ; \qquad (3.13) \\ \mathcal{C}^{(N,l)}_{p+1[ij]} \equiv 0 \qquad & \text{and} \qquad \mathcal{C}'^{(N,l)}_{N-p[mk]} \equiv 0 \\ & \Rightarrow \quad \alpha \pm \tilde{\alpha} = i(\tilde{\theta} - \theta) - (N - 2p - 1)\phi/2 \pmod{2i\pi} \, . \qquad (3.14) \end{aligned}$$

In these two special cases, the coupled three-term recursion relations (3.8) or (3.9) determining the weights in the expansions (3.6) become truncated. This means that any eigenstate can be written as a truncated combination of one of the two representations in (3.6). For the first family of boundary conditions (3.13), the eigenstates read:

$$\Psi^{(+)}(\Lambda_1) = \sum_{n=0}^{p} \sum_{j=1}^{\binom{N}{n}} f^{(N,+)}_{n[j]}(\Lambda_1)\psi^{(N)}_{n[j]} \qquad \text{and} \qquad \Psi^{(-)}(\Lambda_1) = \sum_{s=0}^{N-p-1} \sum_{k=1}^{\binom{N}{s}} f^{(N,-)}_{s[k]}(\Lambda_1)\varphi^{(N)}_{s[k]} \tag{3.15}$$

whereas

$$\Psi^{(+)}(\Lambda_1) = \sum_{n=p+1}^{N} \sum_{j=1}^{\binom{N}{n}} f^{(N,+)}_{n[j]}(\Lambda_1)\psi^{(N)}_{n[j]} \qquad \text{and} \qquad \Psi^{(-)}(\Lambda_1) = \sum_{s=N-p}^{N} \sum_{k=1}^{\binom{N}{s}} f^{(N,-)}_{s[k]}(\Lambda_1)\varphi^{(N)}_{s[k]} \tag{3.16}$$

for the second family of boundary conditions (3.14). In others words, the complete family of eigenvalues and eigenstates splits in two sets, each associated with a characteristic polynomial of degree $d_1 < 2^N$ and $d_2 < 2^N$, respectively, such that $d_1 + d_2 = 2^N$. According to the family of boundary conditions one finds:

$$d_1 = \sum_{n=0}^{p} \binom{N}{n} \quad \text{for} \quad (3.13) \qquad \text{and} \qquad d_1 = \sum_{n=p+1}^{N} \binom{N}{n} \quad \text{for} \quad (3.14) \, .$$

Then, the eigenvalues Λ_{2l+1} for $l = 1, ..., N$ are obtained with the same method as for generic boundary parameters.

It is important to mention that the linear relations (3.13), (3.14) among the boundary parameters coincide exactly with the ones for which a Bethe-type solution exists, although the derivation of these relations is here totally different. Also, the splitting of the eigenvalues and eigenstates in two sectors is also confirmed: in the algebraic Bethe ansatz framework [4, 5] (see also [3]), the two sets of eigenstates and eigenvalues are described by two different sets of Bethe equations. This shows that the approach described here provides an alternative solution compared to the Bethe ansatz one for the model (1.1)

with the family of boundary conditions (3.13), (3.14). For these regimes of boundary parameters, a detailed comparison between both approaches is an interesting problem: it should give explicit relations between the solutions of Bethe equations and the roots of the truncated characteristic polynomials.

4 Comments and open problems

• **Is the model '*really*' solved?** If one *believes* that solving an integrable model *only* means that one must be able to find its spectrum [4] in terms of solutions of *highly transcendental equations* (Bethe-type equations), then the solution we proposed for the XXZ open spin chain (1.1) [12] is indeed not satisfying at all. And from this point of view, this is not the only integrable model which is out of range[5]! On the other hand, if one thinks that solving an integrable model means that one is able to reduce the complexity of the system using unified mathematical structures (quantum algebra, representation theory) in order to write its spectrum in terms of solutions of simpler equations (for instance the *algebraic* equations with (3.8), (3.9) where $\mathcal{I}_1$ is a *block tridiagonal* matrix) then from this point of view the model (1.1) is solved.

• **Large N and thermodynamic limit?** For generic boundary parameters, a natural question to ask is weither diagonalizing $\mathcal{I}_1$ is simpler and faster than diagonalizing $\mathcal{H}$ and all higher conserved quantities generated from the transfer matrix. Indeed, one has to remember that the eigenstates and eigenvalues of $\mathcal{I}_1$ generate the eigenvalues of all higher quantities $\mathcal{I}_{2l+1}$ for $l = 1, ..., N-1$. Thanks to the block tridiagonal structure of the matrix $\mathcal{I}_1$, finding the spectrum has been reduced to (3.8) or (3.9). Although this simplified formulation helps for numerical calculations (using well-known FORTRAN procedures for block tridiagonal matrices), for $N >> 1$ or in the thermodynamic limit $N \to \infty$ solving *directly* these 2^N coupled three-term recurrence relations is a problem that clearly needs further investigations. Studying infinite dimensional representations of (2.4) might be a possible direction.

• **Another solution using the $q-$Onsager algebra?** The solution here described [12] is essentially based on the block tridiagonal structure of all operators satisfying (2.4) in a certain basis. However, for finite dimensional representations associated with the XXZ

[4] I do not even speak about the eigenstates which can not be derived using the functional Bethe ansatz approach.

[5] Consider for instance the XYZ chain, quantum Toda chain, chiral Potts model,...

open spin chain (1.1) it is also interesting to point out that [6]

$$
\begin{aligned}
&-\frac{(q^{1/2}-q^{-1/2})}{k_+k_-}\omega_0^{(N)}\mathcal{W}_0^{(N)}+\sum_{l=1}^{N}C_{-l+1}^{(N)}\mathcal{W}_{-l}^{(N)}+\epsilon_+^{(N)}I\!I^{(N)}=0\ ,\\
&-\frac{(q^{1/2}-q^{-1/2})}{k_+k_-}\omega_0^{(N)}\mathcal{W}_1^{(N)}+\sum_{l=1}^{N}C_{-l+1}^{(N)}\mathcal{W}_{l+1}^{(N)}+\epsilon_-^{(N)}I\!I^{(N)}=0\ ,\\
&-\frac{(q^{1/2}-q^{-1/2})}{k_+k_-}\omega_0^{(N)}\mathcal{G}_1^{(N)}+\sum_{l=1}^{N}C_{-l+1}^{(N)}\mathcal{G}_{l+1}^{(N)}=0\ ,\\
&-\frac{(q^{1/2}-q^{-1/2})}{k_+k_-}\omega_0^{(N)}\tilde{\mathcal{G}}_1^{(N)}+\sum_{l=1}^{N}C_{-l+1}^{(N)}\tilde{\mathcal{G}}_{l+1}^{(N)}=0\quad .
\end{aligned}
\tag{4.1}
$$

For the reader who is familiar with the Ising and superintegrable chiral Potts models, the spectrum of these models is known to be expressed in terms of the roots of the polynomial (called Baxter's polynomial for the superintegrable Potts model) associated with $\sum_l \alpha_l A_l = 0$ and $\sum_l \alpha_l G_l = 0$ where A_l, G_l are Onsager's algebra generators (see references and details in [16]). By analogy, linear relations of the form (4.1) may be key ingredients in finding another solution – may be simpler than the one we proposed in [12] – to the XXZ open spin chain (1.1).

• **Beyond.** The computation of correlation functions essentially relies on the knowledge of the complete set of eigenstates (in the *massless* regime which is here considered). In view of (3.6), one can easily calculate the action of the elementary local operators on the basis (3.3)

$$
\sigma_a^{(k)}\psi_{n[i]}^{(N)}=\sum_{m=0}^{N}\sum_{j=1}^{\binom{N}{m}}\mathcal{M}_{a\ \ n[i],m[j]}^{(N,k)}\ \psi_{m[j]}^{(N)}
\tag{4.2}
$$

with

$$
\sigma_a^{(k)}=I\!I\otimes\ldots\otimes I\!I\otimes\underbrace{\sigma_a}_{\text{at site } k}\otimes I\!I\otimes\ldots\otimes I\!I\ ,\quad a\in\{1,2,3\},
$$

and the coefficients $\mathcal{M}_{a\ \ n[i],m[j]}^{(N,k)}$ are derived recursively. As an example, it is possible to write one-point functions as rational functions of the variable Λ_1 defined on the discret

[6] One has

$$
\begin{aligned}
C_{-n}^{(N)} &= (-1)^{N-n}(q^{1/2}+q^{-1/2})^{n+1}\Big(\frac{2(q+q^{-1})}{(q^{1/2}+q^{-1/2})}\Big)^{N-n-2}\\
&\times\ \frac{(N-1)!}{(n+1)!(N-n-2)!}\left(\frac{2(q+q^{-1})}{(q^{1/2}+q^{-1/2})}\frac{N}{N-n-1}+\frac{\epsilon_+^{(0)}\epsilon_-^{(0)}(q^{1/2}-q^{-1/2})^2}{k_+k_-(q^{1/2}+q^{-1/2})}\right)
\end{aligned}
$$

.

support $\mathcal{P}(\Lambda_1) = 0$ with (3.10). For small values of $N < 6$ and generic parameters or the (Bethe ansatz) regime of parameters [7] (3.13), (3.14), the results are easily checked numerically. Having a better understanding of solutions to (3.8), (3.9) at large N would then play an essential role for studying correlation functions. In this direction, extending the results of [19] might be helpful.

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Extension of a Borel subalgebra symmetry into the sl_2 loop algebra symmetry for the twisted XXZ spin chain at roots of unity and the Onsager algebra

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Abstract

We discuss a conjecture that the twisted transfer matrix of the six-vertex model at roots of unity with some discrete twist angles should have the sl(2) loop algebra symmetry. As an evidence of this conjecture, we show the following mathematical result on a subalgebra of the sl(2) loop algebra, which we call a Borel subalgebra: any given finite-dimensional highest weight representation of the Borel subalgebra is extended into that of the sl(2) loop algebra, if the parameters associated with it are nonzero. Thus, if operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model at roots of unity generate the Borel subalgebra, then they also generate the sl(2) loop algebra. The result should be useful for studying the connection of the sl(2) loop algebra symmetry to the Onsager algebra symmetry of the superintegrable chiral Potts model.

1 Introduction

Spectral properties of the XXZ spin chain under the twisted boundary conditions have attracted much attention in mathematical physics and condensed matter physics [1, 2, 3, 4, 5]. The XXZ Hamiltonian on a ring of L sites is given by

$$\mathcal{H}_{\rm XXZ} = J \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) . \tag{1.1}$$

where σ_j^α $(\alpha = X, Y, Z)$ are the Pauli matrices defined on the jth site, and they satisfy the twisted boundary conditions:

$$\sigma_{L+1}^{\pm} = \exp(\pm i\phi)\sigma_1^{\pm} , \qquad \sigma_{L+1}^Z = \sigma_1^Z . \tag{1.2}$$

We call the parameter ϕ the twist angle. When $\phi = 0$, conditions (1.2) reduces to the periodic boundary conditions. We define parameter q by $\Delta = (q + q^{-1})/2$. We also introduce twist parameter φ by

$$q^{2\varphi} = \exp(i\phi) . \tag{1.3}$$

It has been shown that when q is a root of unity the XXZ spin chain under the periodic boundary conditions commutes with the sl_2 loop algebra, $U(L(sl_2))$ [6]. (See also, [7, 8, 9, 10, 11].) Through the similar derivation in terms of the Temperley-Lieb algebra as given in [6], it was shown that the twisted XXZ spin chain at roots of unity commutes with the sl_2 loop algebra for $\phi = \pi$, i.e. under the anti-periodic boundary conditions [12]. It was also shown that when q is a root of unity such as $q^{2N} = 1$ and φ is an integer, there exist some operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model [12]. Furthermore, it was pointed out by Korff that in some sectors such operators generate a subalgebra $U(\mathcal{B}_0)$ of the sl_2 loop algebra $U(L(sl_2))$, which we call a Borel subalgebra [13]. Let $x_m^{\pm}$ and h_n for $m, n \in \boldsymbol{Z}$ be the generators of the sl_2 loop algebra $U(L(sl_2))$. Then, the Borel subalgebra $U(\mathcal{B}_0)$ is generated by the following operators: x_k^+, h_k for $k = 0, 1, \ldots$, and x_k^- for $k = 1, 2, \ldots$.

In the paper we show a mathematical result that every highest weight representation of the Borel subalgebra $U(\mathcal{B}_0)$ is extended into that of the sl_2 loop algebra if the parameters associated with the representation are nonzero. It follows from the result that if the twisted transfer matrix has the Borel subalgebra symmetry, then it has also the sl_2 loop algebra symmetry. We thus give a conjecture that the sl_2 loop algebra is generated by the operators constructed in [12] which commute or anti-commute with the twisted transfer matrix of the six-vertex model at roots of unity. Here we note that Benkart and Terwilliger have shown that the action of $U(\mathcal{B}_0)$ on a finite-dimensional irreducible $U(\mathcal{B}_0)$ module extends uniquely to an action of $U_q(L(sl_2))$ on it [14]. The mathematical result in the paper is new for reducible highest weight representations of $U(\mathcal{B}_0)$. We also discuss construction of generators of the Onsager algebra from a highest weight representation of the sl_2 loop algebra. The result should be useful for investigating the connection of the sl_2

loop algebra to the Onsager algebra symmetry of the super-integrable chiral Potts model [15]. Quite recently in an independent research [16], eigenvectors of the superintegrable model associated with the superintegrable chiral Potts model have been studied by making use of the sl_2 loop algebra symmetry of some XXZ spin chain. They should be closely related to Ref. [15], and some results of the present paper should also be relevant.
The content of the paper consists of the following: In section 2, we review the infinite-dimensional symmetries of the twisted transfer matrix of the six-vertex model at roots of unity. In particular, we review operators commuting or anti-commuting with the twisted transfer matrix at roots of unity. In some sectors they generate the Borel subalgebra. In section 3, we show that any given highest weight representation of the Borel subalgebra is extended to that of the sl_2 loop algebra if the associated parameters are nonzero. In section 4, we summarize some results on the infinite-dimensional symmetry of the twisted transfer matrix of the six-vertex model at roots of unity, and then suggest a conjecture that the twisted transfer matrix of the six-vertex model at roots of unity should have the sl_2 loop algebra symmetry. In section 5 , we give a method for constructing a representation of the Onsager algebra from a finite-dimensional highest weight representation of the sl_2 algebra.

2 Infinite dimensional symmetry of the twisted XXZ spin chain

2.1 Definition of the twisted transfer matrix

In order to formulate the twisted transfer matrix of the six-vertex model, we review some formulas of the algebraic Bethe ansatz. The R matrix of the XXZ spin chain is defined by

$$R(z-w) = \begin{pmatrix} f(w-z) & 0 & 0 & 0 \\ 0 & g(w-z) & 1 & 0 \\ 0 & 1 & g(w-z) & 0 \\ 0 & 0 & 0 & f(w-z) \end{pmatrix} \tag{2.1}$$

where $f(z-w)$ and $g(z-w)$ are given by

$$f(z-w) = \frac{\sinh(z-w-2\eta)}{\sinh(z-w)}, \quad g(z-w) = \frac{\sinh(-2\eta)}{\sinh(z-w)}. \tag{2.2}$$

We introduce L operators for the XXZ spin chain

$$L_n(z) = \begin{pmatrix} L_n(z)_1^1 & L_n(z)_2^1 \\ L_n(z)_1^2 & L_n(z)_2^2 \end{pmatrix} = \begin{pmatrix} \sinh\left(z\, I_n + \eta\sigma_n^Z\right) & \sinh 2\eta\, \sigma_n^- \\ \sinh 2\eta\, \sigma_n^+ & \sinh\left(z\, I_n - \eta\sigma_n^Z\right) \end{pmatrix} \tag{2.3}$$

Here I_n and σ_n^a $(n = 1, \ldots, L)$ are acting on the nth vector space V_n. We recall that $\sigma^{\pm}$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^X, \sigma^Y, \sigma^Z$ the Pauli matrices. In terms of the R matrix and L operators, the Yang-Baxter equation is expressed as

$$R(z-w)\,(L_n(z) \otimes L_n(w)) = (L_n(w) \otimes L_n(z))\,R(z-w) \tag{2.4}$$

We define the monodromy matrix $T(z)$ by $T(z) = L_L(z) \cdots L_2(z) L_1(z)$. The monodromy matrix satisfies the Yang-Baxter equations

$$R(z-w)\ (T(z;\{\xi_n\}) \otimes T(w;\{\xi_n\})) = (T(w;\{\xi_n\}) \otimes T(z;\{\xi_n\}))\ R(z-w) \tag{2.5}$$

Let us denote the matrix elements of $T(z)$ as follows:

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{2.6}$$

The twisted transfer matrix $\tau_{6V}(z;\varphi)$ is defined by

$$\tau_{6V}(z;\varphi) = \mathrm{tr}\left(q^{\varphi\sigma_0^z} T(z) \right) = q^{\varphi} A(z) + q^{-\varphi} D(z). \tag{2.7}$$

The twisted Hamiltonian is given by the following logarithmic derivative:

$$\begin{aligned} &\sinh 2\eta \times \frac{d}{dz} \log \tau(z;\varphi)|_{z=\eta} \\ &= \sum_{j=1}^{L-1} \left(2\sigma_j^+ \sigma_{j+1}^- + 2\sigma_j^- \sigma_{j+1}^+ + \cosh 2\eta \sigma_j^Z \sigma_{j+1}^Z \right) \\ &\quad + q^{-2\varphi}\, 2\sigma_L^+ \sigma_1^- + q^{2\varphi}\, 2\sigma_L^- \sigma_1^+ + \cosh 2\eta \sigma_L^Z \sigma_1^Z + L \cosh 2\eta \\ &\quad = \mathcal{H}_{XXZ}(\phi)/J + L\Delta\,. \end{aligned}$$

2.2 Roots of unity conditions

Let us formulate roots of unity conditions explicitly as follows [12, 11].

Definition 2.1 (Roots of unity conditions) *We say that q_0 is a root of unity with $q_0^{2N} = 1$, if one of the three conditions hold: (1) N is odd and q_0 is a primitive Nth root of unity, i.e. $q_0^N = 1$; (2) N is odd and q_0 is a primitive $2N$th root of unity , i.e. $q_0^N = -1$; (3) N is even and q_0 is a primitive $2N$th root of unity , i.e. $q_0^N = -1$.*

Let us denote by $S^Z \pm \varphi$ either $S^Z + \varphi$ or $S^Z - \varphi$. We now consider the condition of $q_0^{2S^Z \pm 2\varphi} = 1$. The values of S^Z and φ are given by integers or half-integers under the twisted boundary conditions.
(1) When N is odd and $q_0^N = 1$, we have $q_0^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod N$ or $S^Z \pm \varphi \equiv N/2 \pmod N$. When $S^Z \pm \varphi \equiv 0 \pmod N$, φ is given by an integer for even L, and a half-integer for odd L. When $S^Z \pm \varphi \equiv N/2 \pmod N$, φ is given by a half-integer for even L, and an integer for odd L.
(2) When N is odd and $q_0^N = -1$, we have $q^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod N$. φ is given by an integer for even L, and a half-integer for odd L.
(3) When N is even and $q_0^N = -1$, we have $q^{2S^Z \pm 2\varphi} = 1$ if and only if $S^Z \pm \varphi \equiv 0 \pmod N$. φ is given by an integer for even L, and a half-integer for odd L.
Here we note that if the number of lattice sites L is given by an even integer, then S^Z takes integral values, while if L is odd, S^Z takes half-integral values.

2.3 Operators commuting with the twisted XXZ Hamiltonian

We now formulate operators commuting or anti-commuting with the twisted transfer matrix of the six-vertex model at roots of unity [12]. We introduce operators $S_j^{\pm}$ and $T_j^{\pm}$ by

$$\begin{aligned} S_j^{\pm} &= q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2}, \\ T_j^{\pm} &= q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \quad (j = 1, 2, \ldots, L). \end{aligned} \tag{2.8}$$

We define $S^{\pm}$ and $T^{\pm}$ by

$$S^{\pm} = \sum_{j=1}^{L} S_j^{\pm}, \quad T^{\pm} = \sum_{j=1}^{L} T_j^{\pm}. \tag{2.9}$$

They are generators of the affine quantum group $U_q(\hat{sl}(2))$.
Let us introduce the q-integer $[n]$ and the q-factorial $[m]!$, respectively, by the following:

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]! = \prod_{k=1}^{m} [k]. \tag{2.10}$$

It is easy to show

$$\begin{aligned} (S^{\pm})^m &= q^{\pm m(m-1)/2}[m]! \sum_{1 \le i_1 < \cdots < i_m \le L} S_{i_1}^{\pm} \cdots S_{i_m}^{\pm}, \\ (T^{\pm})^m &= q^{\mp m(m-1)/2}[m]! \sum_{1 \le i_1 < \cdots < i_m \le L} T_{i_1}^{\pm} \cdots T_{i_m}^{\pm}. \end{aligned} \tag{2.11}$$

The symbols $S^{\pm(N)}$ and $T^{\pm(N)}$ are defined in Ref. [6] by

$$S^{\pm(N)} = \lim_{q \to q_0} \frac{(S^{\pm})^N}{[N]!}, \quad T^{\pm(N)} = \lim_{q \to q_0} \frac{(T^{\pm})^N}{[N]!}. \tag{2.12}$$

Here we define $(S^{\pm})^{(m)}$ and $(T^{\pm})^{(m)}$ for all positive integers m by

$$(S^{\pm})^{(m)} = \lim_{q \to q_0} \frac{(S^{\pm})^m}{[m]!}, \quad (T^{\pm})^{(m)} = \lim_{q \to q_0} \frac{(T^{\pm})^m}{[m]!}. \tag{2.13}$$

Explicitly, we have $(S^{\pm})^{(m)}$ for any positive integer m as follows.

$$\begin{aligned} (S^{\pm})^{(m)} = \sum_{1 \le j_1 < \cdots < j_m \le L} & q_0^{\frac{m}{2}\sigma^Z} \otimes \cdots \otimes q_0^{\frac{m}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q_0^{\frac{(m-2)}{2}\sigma^Z} \otimes \cdots \otimes q_0^{\frac{(m-2)}{2}\sigma^Z} \\ & \otimes \sigma_{j_2}^{\pm} \otimes q_0^{\frac{(m-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_m}^{\pm} \otimes q_0^{-\frac{m}{2}\sigma^Z} \otimes \cdots \otimes q_0^{-\frac{m}{2}\sigma^Z}. \end{aligned} \tag{2.14}$$

Let m and n be integers such that $|m - n| = kN$ for some integer k. When q_0 is a root of unity with $q^{2N} = 1$, we have the following.

(1) In the sectors of $S^Z \equiv -\varphi + n (\mathrm{mod} N)$, we have

$$(S^+)^{(m)}(T^-)^{(n)}\tau(z;\varphi) = q_0^{m-n}\,\tau(z;\varphi)(S^+)^{(m)}(T^-)^{(n)}$$

(2) In the sectors of $S^Z \equiv -\varphi - n (\mathrm{mod} N)$, we have

$$(T^-)^{(m)}(S^+)^{(n)}\tau(z;\varphi) = q_0^{m-n}\,\tau(z;\varphi)(T^-)^{(m)}(S^+)^{(n)}$$

(3) In the sectors of $S^Z \equiv \varphi - n (\mathrm{mod} N)$, we have

$$(S^-)^{(m)}(T^+)^{(n)}\tau(z;\varphi) = q_0^{m-n}\tau(z;\varphi)(S^-)^{(m)}(T^+)^{(n)}$$

(4) In the sectors of $S^Z \equiv \varphi + n (\mathrm{mod} N)$, we have

$$(T^+)^{(m)}(S^-)^{(n)}\tau(z;\varphi) = q_0^{m-n}\tau(z;\varphi)(S^-)^{(m)}(T^+)^{(n)}$$

Here we note that $q_0^N = \pm 1$ when q_0 is a root of unity with $q_0^{2N} = 1$. Thus we have $q_0^{m-n} = (\pm 1)^k$. For simplicity, we have not considered the case when N is odd with $q_0^N = 1$ and $S^Z + \varphi \equiv N/2 \ (\mathrm{mod}\ N)$ or $S^Z - \varphi \equiv N/2 \ (\mathrm{mod}\ N)$.

2.4 Examples

For an illustration, we consider the case of a root of unity where $N = 3$ ($q_0^3 = 1$) and L is even. Some of the operators commuting or anti-commuting with the twisted transfer matrix are given as follows.

(1a) $\varphi = 0$ and $S^Z \equiv 0 \ (\mathrm{mod}\ N)$

$$(S^+)^{(3)}\,, \quad (S^-)^{(3)}\,, \quad (T^+)^{(3)}\,, \quad (T^-)^{(3)}$$

They generate the sl_2 loop algebra [6].

(1b) $\varphi = 0$ and $S^Z \equiv 1 \ (\mathrm{mod}\ N)$:

$$\begin{aligned}&(S^+)^{(4)}(T^-)^{(1)}\,, \quad (T^-)^{(5)}(S^+)^{(2)}\,, \quad (S^-)^{(5)}(T^+)^{(2)}\,, \quad (T^+)^{(4)}(S^-)^{(1)}\,,\\ &(S^+)^{(1)}(T^-)^{(4)}\,, \quad (T^-)^{(2)}(S^+)^{(5)}\,, \quad (S^-)^{(2)}(T^+)^{(5)}\,, \quad (T^+)^{(1)}(S^-)^{(4)}\,, \ldots .\end{aligned}$$

It is conjectured that they generate the sl_2 loop algebra [6].

(1c) $\varphi = 0$ and $S^Z \equiv 2 \ (\mathrm{mod}\ N)$:

$$\begin{aligned}&(S^+)^{(5)}(T^-)^{(2)}\,, \quad (T^-)^{(4)}(S^+)^{(1)}\,, \quad (S^-)^{(4)}(T^+)^{(1)}\,, \quad (T^+)^{(5)}(S^-)^{(2)}\,,\\ &(S^+)^{(2)}(T^-)^{(5)}\,, \quad (T^-)^{(1)}(S^+)^{(4)}\,, \quad (S^-)^{(1)}(T^+)^{(4)}\,, \quad (T^+)^{(2)}(S^-)^{(5)}\,, \ldots .\end{aligned}$$

It is conjectured that they should generate the sl_2 loop algebra [6].

(2a) $\varphi = 1$ and $S^Z \equiv 0 \pmod N$:

$$(S^+)^{(4)}(T^-)^{(1)}, \quad (T^-)^{(5)}(S^+)^{(2)}, \quad (S^-)^{(5)}(T^+)^{(2)}, \quad (T^+)^{(4)}(S^-)^{(1)}, \dots,$$
$$(S^+)^{(1)}(T^-)^{(4)}, \quad (T^-)^{(2)}(S^+)^{(5)}, \quad (S^-)^{(2)}(T^+)^{(5)}, \quad (T^+)^{(1)}(S^-)^{(4)}, \dots.$$

(2b) $\varphi = 1$ and $S^Z \equiv 1 \pmod N$:

$$(S^+)^{(5)}(T^-)^{(2)}, \quad (T^-)^{(4)}(S^+)^{(1)}, \quad (S^-)^{(3)}, \quad (T^+)^{(3)}, \dots,$$
$$(S^+)^{(2)}(T^-)^{(5)}, \quad (T^-)^{(1)}(S^+)^{(4)}, \dots.$$

$(S^-)^{(3)}$ and $(T^+)^{(3)}$ generate a Borel subalgebra [13].

(2c) $\varphi = 1$ and $S^Z \equiv 2 \pmod N$

$$(S^+)^{(3)}, \quad (T^-)^{(3)}, \quad (S^-)^{(5)}(T^+)^{(2)}, \quad (T^+)^{(4)}(S^-)^{(1)}, \dots,$$
$$(S^-)^{(2)}(T^+)^{(5)}, \quad (T^+)^{(1)}(S^-)^{(4)}, \dots. \tag{2.15}$$

$(S^+)^{(3)}$ and $(T^-)^{(3)}$ generate a Borel subalgebra [13].

(3a) $\varphi = 2$ and $S^Z \equiv 0 \pmod N$:

$$(S^+)^{(5)}(T^-)^{(2)}, \quad (T^-)^{(4)}(S^+)^{(1)}, \quad (S^-)^{(5)}(T^+)^{(2)}, \quad (T^+)^{(4)}(S^-)^{(1)}, \dots,,$$
$$(S^+)^{(2)}(T^-)^{(5)}, \quad (T^-)^{(1)}(S^+)^{(4)}, \quad (S^-)^{(2)}(T^+)^{(5)}, \quad (T^+)^{(1)}(S^-)^{(4)}, \dots,.$$

(3b) $\varphi = 1$ and $S^Z \equiv 1 \pmod N$:

$$(S^+)^{(3)}, \quad (T^-)^{(3)}, \quad (S^-)^{(4)}(T^+)^{(1)}, \quad (T^+)^{(5)}(S^-)^{(2)},$$
$$(S^-)^{(1)}(T^+)^{(4)}, \quad (T^+)^{(2)}(S^-)^{(5)}, \dots,.$$

$(S^+)^{(3)}$ and $(T^-)^{(3)}$ generate a Borel subalgebra [13].

(3c) $\varphi = 1$ and $S^Z \equiv 2 \pmod N$:

$$(S^+)^{(4)}(T^-)^{(1)}, \quad (T^-)^{(5)}(S^+)^{(2)}, \quad (S^-)^{(3)}, \quad (T^+)^{(3)}, \dots,,$$
$$(S^+)^{(1)}(T^-)^{(4)}, \quad (T^-)^{(2)}(S^+)^{(5)}, \dots,.$$

$(S^-)^{(3)}$ and $(T^+)^{(3)}$ generate a Borel subalgebra [13].

3 Extension of the Borel subalgebra symmetry

3.1 Definition of the Borel subalgebra of $U(L(sl_2))$

We recall that the Borel subalgebra, $U(\mathcal{B}_0)$, is generated by the following operators:

$$x_k^+, h_k \quad \text{for } k = 0, 1, \dots, \quad \text{and} \quad x_k^- \quad \text{for } k = 1, 2, \dots.$$

They satisfy the defining relations given as follows:

$$\begin{aligned}
[h_j, x_k^+] &= 2x_{j+k}^+ , \quad \text{for} \quad j,k \geq 0, \\
[h_j, x_k^-] &= (-2)x_{j+k}^- , \quad \text{for} \quad j \geq 0 \,\text{and}\, k \geq 1, \\
[x_j^+, x_k^-] &= \delta_{j,k} h_{j+k} , \quad \text{for} \quad j \geq 0 \,\text{and}\, k \geq 1, \\
[h_j, h_k] &= 0 , \quad \text{for} \quad j,k \geq 0, \\
[x_j^+, x_k^+] &= 0 \quad \text{for} \quad j \geq 0 \,\text{and}\, k \geq 0, \\
[x_j^-, x_k^-] &= 0 \quad \text{for} \quad j \geq 1 \,\text{and}\, k \geq 1.
\end{aligned} \tag{3.1}$$

3.2 Highest weight vectors and highest weight parameters

Let us define highest weight vectors of the Borel subalgebra $U(\mathcal{B}_0)$.

Definition 3.1 *In a representation of $U(\mathcal{B}_0)$, we call a vector Ψ a highest weight vector if it is annihilated by all x_k^+'s, i.e. $x_k^+\Psi = 0$ for $k = 0, 1, \ldots$, and is a simultaneous eigenvector of all h_k's, i.e. $h_k\Psi = d_k\Psi$ for $k = 0, 1, \ldots$. We call the set of eigenvalues d_k the highest weight of Ψ. We call the representation generated by a highest weight vector Ψ, the highest weight representation of Ψ. We denote it by $U(\mathcal{B}_0)\Psi$.*

Definition 3.2 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. If $(x_1^-)^{r+1}\Psi = 0$ and $(x_1^-)^r\Psi \neq 0$ for an integer r, we say that x_1^- is nilpotent of degree r in the highest weight representation.*

In a finite-dimensional representation of $U(\mathcal{B}_0)$, x_1^- is nilpotent, i.e. $(x_1^-)^s = 0$ for some integer s. For a highest weight vector in a finite-dimensional representation of $U(\mathcal{B}_0)$, we can define the highest weight polynomial and highest weight parameters a_j similarly as in the case of the sl_2 loop algebra [10, 17, 18].

Definition 3.3 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. By applying the Poincaré-Birkhoff-Witt theorem to $U(\mathcal{B}_0)$, it follows that the highest weight representation $U(\mathcal{B}_0)\Psi$ is decomposed into the direct sum of subspaces with respect to eigenvalues of h_0, and that every vector v in the subspace of weight $d_0 - 2n$ is written as follows:*

$$v = \sum_{1 \leq k_1 \leq \cdots \leq k_n} C_{k_1,\cdots,k_n} x_{k_1}^- \cdots x_{k_n}^- \Psi .$$

We call the subspace of weight $d_0 - 2n$ the sector of degree n.

Proposition 3.4 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. If x_1^- is nilpotent of degree r in the highest weight representation $U(\mathcal{B}_0)\Psi$, then the sector of degree $2r$ in $U(\mathcal{B}_0)\Psi$ is one-dimensional.*

We can show proposition 3.4 through the following lemma [18, 10].

Lemma 3.5 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. We assume that x_1^- is nilpotent of degree r in $U(\mathcal{B}_0)\Psi$. Let us take a non-negative integer n satisfying $n \leq r$. Then, for any set of positive integers, $k_1, \ldots, k_n$, we have*

$$(x_1^-)^{r-n} x_{k_1}^- \cdots x_{k_n}^- \Psi = A_{k_1,\ldots,k_n} (x_1^-)^r \Psi \,. \tag{3.2}$$

Here, $A_{k_1,\ldots,k_n}$ is given by a complex number.

Let us denote by $(X)^{(n)}$ the nth power of operator X divided by the n factorial, i.e. $(X)^{(n)} = X/n!$.

Lemma 3.6 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. If x_1^- is nilpotent of degree r in $U(\mathcal{B}_0)\Psi$, then Ψ is a simultaneous eigenvector of $(x_0^+)^{(n)}(x_1^-)^{(n)}$:*

$$(x_0^+)^{(j)}(x_1^-)^{(j)}\Omega = \lambda_j \Omega \,, \quad \text{for} \quad j = 1, 2, \ldots, r \,. \tag{3.3}$$

Here λ_j are eigenvalues.

Proof. From the Poincaré-Birkhoff-Witt theorem of $U(\mathcal{B}_0)$, it follows that the sector of degree 0 in $U(\mathcal{B}_0)\Psi$ is one-dimensional. Since $(x_0^+)^{(j)}(x_1^-)^{(j)}\Psi$ is in the sector of degree 0 in $U(\mathcal{B}_0)\Psi$, it is proportional to the basis vector Ψ. ∎

Let Ψ be a highest weight vector in a finite-dimensional representation of the Borel subalgebra $U(\mathcal{B}_0)$. We now introduce parameters expressing the highest weight of Ψ. We denote by $\lambda = (\lambda_1, \ldots, \lambda_r)$ the sequence of eigenvalues λ_k which are defined in eq. (3.3). Here we recall that in a finite-dimensional representation, x_1^- is nilpotent of some degree. We define a polynomial $P_\lambda(u)$ by the following relation [10]:

$$P_\lambda(u) = \sum_{k=0}^{r} \lambda_k (-u)^k \,. \tag{3.4}$$

We call it the *highest weight polynomial* of Ψ.
Let us factorize polynomial $P_\lambda(u)$ as follows

$$P_\lambda(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k} \,, \tag{3.5}$$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. We denote by $\boldsymbol{a}$ the sequence of s parameters a_j:

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_s). \tag{3.6}$$

Here we note that r is equal to the sum of multiplicities m_j: $r = m_1 + \cdots + m_s$. We define parameters $\hat{a}_i$ for $i = 1, 2, \ldots, r$, as follows.

$$\hat{a}_i = a_k \quad \text{if } m_1 + m_2 + \cdots + m_{k-1} < i \leq m_1 + \cdots + m_{k-1} + m_k \,. \tag{3.7}$$

Then, the set $\{\hat{a}_j \,|j = 1, 2, \ldots, r\}$ corresponds to the set of parameters a_j with multiplicities m_j for $j = 1, 2, \ldots, s$. We denote by $\hat{\boldsymbol{a}}$ the sequence of r parameters $\hat{a}_i$:

$$\hat{\boldsymbol{a}} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r). \tag{3.8}$$

We call parameters $\hat{a}_i$ the *highest weight parameters* of Ψ. It follows from the definition of highest weight polynomial $\mathcal{P}_\lambda(u)$ given by (3.4) and that of highest weight parameters (3.5) that we have

$$\lambda_n = \sum_{1 \le j_1 < \cdots < j_n \le r} \hat{a}_{j_1} \cdots \hat{a}_{j_n} \,. \tag{3.9}$$

If the highest weight parameters are nonzero, we define $\bar{\lambda}_n$ for $n = 0, 1, \ldots, r$ by

$$\bar{\lambda}_n = \sum_{1 \le j_1 < \cdots < j_n \le r} \hat{a}_{j_1}^{-1} \cdots \hat{a}_{j_n}^{-1} \,. \tag{3.10}$$

We remark that we may call the highest weight polynomial of Ψ and the highest weight parameters of Ψ the *loop-highest weight polynomial* of Ψ and the *loop-highest weight parameters* of Ψ, respectively [19].

3.3 Borel subalgebra generators with parameters

Let $\boldsymbol{\alpha}$ denote a finite sequence of complex parameters such as $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We define generators with n parameters, $x_m^{\pm}(\boldsymbol{\alpha})$ and $h_m(\boldsymbol{\alpha})$, as follows [10, 17, 18]:

$$\begin{aligned} x_m^{\pm}(\boldsymbol{\alpha}) &= \sum_{k=0}^{n} (-1)^k x_{m-k}^{\pm} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \,, \\ h_m(\boldsymbol{\alpha}) &= \sum_{k=0}^{n} (-1)^k h_{m-k} \sum_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \,. \end{aligned} \tag{3.11}$$

Here, in the case of the Borel subalgebra $U(\mathcal{B}_0)$, we define $x_m^{+}(\boldsymbol{\alpha})$ and $h_m(\boldsymbol{\alpha})$ for $m \ge n \ge 0$, and $x_m^{-}(\boldsymbol{\alpha})$ for $m > n \ge 0$.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be arbitrary sequences of n and p parameters, respectively. Here we have $n, p \ge 0$. In terms of generators with parameters we express the defining relations of the Borel subalgebra as follows:

$$[x_\ell^{+}(\boldsymbol{\alpha}), x_m^{-}(\boldsymbol{\beta})] = h_{\ell+m}(\boldsymbol{\alpha}\boldsymbol{\beta}) \,, \quad [h_\ell(\boldsymbol{\alpha}), x_m^{-}(\boldsymbol{\beta})] = -2x_{\ell+m}^{-}(\boldsymbol{\alpha}\boldsymbol{\beta}) \,, \tag{3.12}$$

for $\ell \ge n$ and $m > p$, and

$$[h_\ell(\boldsymbol{\alpha}), x_m^{+}(\boldsymbol{\beta})] = 2x_{\ell+m}^{+}(\boldsymbol{\alpha}\boldsymbol{\beta}) \,. \tag{3.13}$$

for $\ell \ge n$ and $m \ge p$, Here the symbol $\boldsymbol{\alpha}\boldsymbol{\beta}$ denotes the composite sequence of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\boldsymbol{\alpha}\boldsymbol{\beta} = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_p). \tag{3.14}$$

Lemma 3.7 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$. If $x_t^-(\boldsymbol{\alpha})\Psi = 0$ for a positive integer t and a sequence of parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ where $t > n$, we have $h_{t+m}(\boldsymbol{\alpha}) = 0$ and $x_{t+m}^{\pm}(\boldsymbol{\alpha}) = 0$ in the highest weight representation of Ψ for $m \in \mathbf{Z}_{\geq 0}$: for any set of positive integers, $k_1, k_2, \ldots, k_n$, and for $m \in \mathbf{Z}_{\geq 0}$, we have the following:*

$$x_{t+m}^-(\boldsymbol{\alpha})\, x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0\,, \tag{3.15}$$

$$h_{t+m}(\boldsymbol{\alpha})\, x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0\,, \tag{3.16}$$

$$x_{t+m}^+(\boldsymbol{\alpha})\, x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi = 0\,. \tag{3.17}$$

Proof. Following the Poincaré-Birkhoff-Witt theorem, one can show that every vector in the sector of degree n of $U(\mathcal{B}_0)\Psi$ is expressed as a linear combination of monomial vectors $x_{k_1}^- x_{k_2}^- \cdots x_{k_n}^- \Psi$. It is easy to show (3.15). By induction on n, we can show (3.16), and then (3.17). ∎

3.4 Recurrence relations

Let us denote by $\mathcal{B}_0^+$ such a subalgebra of $U(\mathcal{B}_0)$ that is generated by x_k^+ for $k \in \boldsymbol{Z}_{\geq 0}$. Similary as the case of the sl_2 loop algebra [10], we can show the following:

Lemma 3.8 *The following recurrence relations hold for $n \in \boldsymbol{Z}_{\geq 0}$:*

$$(\mathrm{A}_n): \quad (x_0^+)^{(n-1)}(x_1^-)^{(n)} = \sum_{k=1}^{n} (-1)^{k-1} x_k^- (x_0^+)^{(n-k)}(x_1^-)^{(n-k)} \bmod U(\mathcal{B}_0)\mathcal{B}_0^+,$$

$$(\mathrm{B}_n): \quad (x_0^+)^{(n)}(x_1^-)^{(n)} = \frac{1}{n}\sum_{k=1}^{n} (-1)^{k-1} h_k (x_0^+)^{(n-k)}(x_1^-)^{(n-k)} \bmod U(\mathcal{B}_0)\mathcal{B}_0^+,$$

$$(\mathrm{C}_n): \quad [h_j(a), (x_0^+)^{(m)}(x_1^-)^{(m)}] = 0 \bmod U(\mathcal{B}_0)\mathcal{B}_0^+ \quad \text{for } m \leq n \text{ and } j \in \boldsymbol{Z}_{\geq 0}\,.$$

Proposition 3.9 (Reduction relations)

$$x_{r+1+m}^-\,\Psi \;=\; \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k}\, x_{k+m}^-\,\Psi\,, \quad \text{for } m \in \mathbf{Z}_{\geq 0}\,, \tag{3.18}$$

$$d_{r+1+m} \;=\; \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k}\, d_{k+m}\,, \quad \text{for } m \in \mathbf{Z}_{\geq 0}\,. \tag{3.19}$$

Proof. Reduction relation (3.18) for $m = 0$ is derived from (A_{r+1}) of lemma 3.8 and lemma 3.6. Applying h_n for $n \geq 0$ to reduction relation (3.18) for $m = 0$, we have reduction relation (3.18) for $m = n$. Applying x_0^+ to (3.18) from the left, we derive relations (3.19). ∎

Corollary 3.10 *Let Ψ be a highest weight vector of $U(\mathcal{B}_0)$ and $\hat{\boldsymbol{a}} = (\hat{a}_1, \ldots, \hat{a}_r)$ the highest weight parameters. In the highest weight representation of Ψ we have $h_{r+m}(\hat{\boldsymbol{a}}) = 0$, $x^+_{r+m}(\hat{\boldsymbol{a}}) = 0$ and $x^-_{r+1+m}(\hat{\boldsymbol{a}}) = 0$ for $m \in \boldsymbol{Z}_{\geq 0}$.*

Proof. It follows from lemma 3.7 and reduction relations (3.18). ∎

3.5 A theorem on the Borel subalgebra

We first recall a simple fact in linear algebra. Let x_n for $n = 0, 1, \ldots,$ be an infinite sequence of numbers satisfying a linear recurrence relation:

$$x_{n+r} = \sum_{k=1}^{r} \gamma_k x_{n+r-k} \tag{3.20}$$

We denote by $\boldsymbol{x}_n =^t (x_{n+1}, \ldots, x_{n+r})$ Then, for any integer n, there exists a matrix $A^{[n]}$ such that

$$\boldsymbol{x}_n = A^{[n]} \boldsymbol{x}_1. \tag{3.21}$$

Furtheremore, we have for any m the following:

$$\boldsymbol{x}_{n+m} = A^{[n]} \boldsymbol{x}_{m+1}. \tag{3.22}$$

Theorem 3.11 *Let Ψ be a highest weight vector in a finite-dimensional representation of the Borel subalgebra $U(\mathcal{B}_0)$. If all the highest weight parameters of Ψ, i.e. $\hat{a}_j$, are nonzero, then the action of $\mathcal{B}_0$ on Ψ can be extended to that of the sl_2 loop algebra: Suppose that x^-_1 is nilpotent of degree r in the highest weight representation of Ψ. We define $\tilde{h}_0$ by $\tilde{h}_0 = h_0 - d_0 + r$, and x^-_0 by*

$$x^-_0 = \sum_{j=1}^{r} (-1)^{j-1} \bar{\lambda}_j x^-_j \,, \tag{3.23}$$

where $\bar{\lambda}_j$ are given by

$$\bar{\lambda}_j = \sum_{i_1 < \cdots < i_j} \hat{a}^{-1}_{i_1} \cdots \hat{a}^{-1}_{i_j} \,. \tag{3.24}$$

We also define $x^\pm_{-\ell}$ and $h_{-\ell}$ for $\ell \in \mathbf{Z}_{>0}$ by

$$\begin{aligned} x^\pm_{-\ell} &= \sum_{j=1}^{r} (-1)^{j-1} \bar{\lambda}_{j-1} x^\pm_{j-\ell} \,, \\ h_{-\ell} &= \sum_{j=1}^{r} (-1)^{j-1} \bar{\lambda}_{j-1} h_{j-\ell} \,. \end{aligned} \tag{3.25}$$

Then, they satisfy the defining relations of the sl_2 loop algebra in the highest weight representation $U(\mathcal{B}_0)\Psi$.

Proof. If a set of operators x_j^+, h_k for $j = 0, 1, \ldots$, and x_k^- for $k = 1, 2, \ldots$, satisfy the defining relations of the Borel subalgebra, then the set operators with h_0 being replaced by $\tilde{h}_0$ also satisfy the defining relations (3.1). We can also show that x_0^- satisfies the defining relations (3.1). Making use of corollary 3.10, we can show $[x_m^+, x_{-\ell}^-] = h_{m-\ell}$, $[h_m, x_{-\ell}^\pm] = (\pm 2) x_{m-\ell}^\pm$, $[x_{-\ell}^+, x_m^-] = h_{m-\ell}$ and $[h_{-\ell}, x_m^\pm] = (\pm 2) x_{m-\ell}^\pm$ for $m, n \in \mathbf{Z}_{\geq 0}$. Here we express $h_{-\ell}$ and $x_m^\pm$ in terms of linear combinations of h_j and $x_j^\pm$ for $j = 1, 2, \ldots, r$. We calculate commutation relations among h_j and $x_k^\pm$ for $j, k = 1, 2, \ldots, r$, and then show the defining relations through (3.22). For an illustration, we show $[h_{-\ell}, x_m^\pm] = (\pm 2) x_{-\ell+m}^\pm$ as follows.

$$
\begin{aligned}
[h_{-\ell}, x_m^\pm] &= \left[\left(A^{[-\ell-1]} \boldsymbol{h}_0 \right)_1, x_m^\pm \right] \\
&= \sum_{j=1}^{r} A_{1,j}^{[-\ell-1]} [h_j, x_m^\pm] \\
&= (\pm 2) \sum_{j=1}^{r} A_{1,j}^{[-\ell-1]} \, x_{j+m}^\pm \\
&= (\pm 2) \left(A^{[-\ell-1]} \boldsymbol{x}_m^\pm \right)_1 \\
&= (\pm 2) x_{-\ell+m}^\pm \, .
\end{aligned}
$$

Similary, we can show $[x_{-\ell}^+, x_m^-] = h_{-\ell+m}$. Furthermore, we can show $[x_{-m}^+, x_{-\ell}^-] = h_{-m-\ell}$ and $[h_{-m}, x_{-\ell}^\pm] = (\pm 2) x_{-m-\ell}^\pm$ for $m, n \in \mathbf{Z}_{\geq 0}$. ∎

We note that in a finite-dimensional representation of the Borel subalgebra, the highest weight d_0 is not necessarily given by an integer.

We should note that Benkart and Terwilliger (2004) have shown that an irreducible finite-dimensional representation of the Borel subalgebra is extended uniquely to an irreducible representation of the sl_2 loop algebra [14]. Thus, if the highest weight representation is irreducible, then theorem 3.11 should be equivalent to the result [14]. Here we recall that an irreducibility criterion is known for a finite-dimensional highest weight representation of the Borel algebra with nonzero highest weight parameters as follows [18]:

Proposition 3.12 *Let Ψ be a highest weight vector in a finite-dimensional representation of $U(\mathcal{B}_0)$. We denote by $\hat{a}_j$ the highest weight parameters. It generates an irreducible representation if and only we have*

$$
\sum_{j=0}^{s} (-1)^{s-j} \mu_{s-j} x_{j+1}^- \Psi = 0 \, , \tag{3.26}
$$

where μ_k $(k = 1, 2, \ldots, s)$ are given by

$$
\mu_k = \sum_{1 \leq i_1 < \cdots < i_k \leq s} a_{i_1} \cdots a_{i_k} \, .
$$

Thus, in a finite-dimensional highest weight representation of $U(\mathcal{B}_0)$, if the highest weight vector satisfies the condition (3.26), it is irreducible and we can also show by making use of the result of Ref. [14] that it is extended into a finite-dimensional highest weight representation of the sl_2 loop algebra.

4 Application to the twisted XXZ spin chain

4.1 Regular Bethe vectors under the twisted boundary conditions as highest weight of $U(L(sl_2))$

We briefly discuss some important points of the infinite-dimensional symmetry of the twisted XXZ spin chain at roots of unity. Some details will be given elsewhere.
For the periodic XXZ spin chain at a root of unity q_0 with $q_0^{2N} = 1$, Fabricius and McCoy conjectured [9] that every Bethe state should be a highest weight vector of the sl_2 loop algebra. Then, it has been explicitly proved for some sectors of S^Z mod N [11] that every regular Bethe state is a highest weight vector of the sl_2 loop algebra. For the twisted XXZ spin chain, we can also show in some sectors of S^Z mod N that every regular Bethe state is highest weight with respect to the Borel subalgebra $U(\mathcal{B}_0)$.
Let us denote by $t_1, t_2, \ldots, t_R$ solutions of the twisted BA equations

$$\frac{a_{6V}(t_j)}{d_{6V}(t_j)} = q^{-2\varphi} \prod_{k=1, k\neq j}^{R} \frac{f(t_k - t_j)}{f(t_j - t_k)}, \quad \text{for } j = 1, 2, \cdots, R. \tag{4.1}$$

Here $a_{6V}(z)$ and $d_{6V}(z)$ are given by

$$a_{6V}(z) = \sinh^L(z + \eta), \quad d_{6V}(z) = \sinh^L(z - \eta). \tag{4.2}$$

We recall that function $f(z - w)$ is given by

$$f(z - w) = \frac{\sinh(z - w - 2\eta)}{\sinh(z - w)},$$

and $q = \exp(2\eta)$.

Definition 4.1 *Let $t_1, t_2, \ldots, t_R$ satisfy the twisted Bethe ansatz equations (4.1). We call them Bethe roots. We call a set of Bethe roots, $t_1, t_2, \ldots, t_R$, regular, if they are finite and distinct. In terms of a set of regular Bethe roots, $t_1, t_2, \ldots, t_R$, we define the regular Bethe state $|R\rangle$ by*

$$|R\rangle = B(t_1)B(t_2)\cdots B(t_R)\,|0\rangle \tag{4.3}$$

Here $|0\rangle$ denotes the vacuum state in which all spins are up.

We recall the following conjecture [11].

Conjecture 4.2 *For the twisted Bethe ansatz equations (4.1) at a root of unity q_0, every set of regular Bethe roots $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ gives an isolated solution of eqs. (4.1).*

We can show conjecture 4.2 for some particular cases. For instance, it is trivial for such Bethe states with one down-spin.
Assuming conjecture 4.2 we can show the following theorem.

Theorem 4.3 *Let q_0 be a root of unity with $q_0^{2N} = 1$ for some integer N, and φ an integer or a half-integer such that we have $q_0^{2N\varphi} = 1$. Every regular Bethe state $|R\rangle$ gives a highest weight vector of the Borel subalgebra $U(\boldsymbol{B}_0)$ if it is in sector A: $S^Z \equiv +\varphi$ (mod N) or $S^Z \equiv -\varphi$ (mod N) where $q_0^{2N} = 1$, or in sector B: $S^Z \equiv N/2 + \varphi$ (mod N) or $S^Z \equiv N/2 - \varphi$ (mod N) where $q_0^N = 1$ with N odd.*

The proof of theorem 4.3 will be given in a different paper.

4.2 Derivation of the degree of nilpotency for x_1^-

Let us assume that a regular Bethe state $|R\rangle$ of the twisted XXZ spin chain is in such a sector of S^Z where theorem 4.3 holds, i.e. $|R\rangle$ is a highest weight vector of $U(\mathcal{B}_0)$. For a highest weight representation of $U(\mathcal{B}_0)$ generated by $|R\rangle$, it follows from the finite dimensionality that x_1^- is nilpotent.
Let $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ be such a set of regular Bethe roots at a root of unity q_0 with $q_0^{2N} = 1$ that leads to the regular Bethe state $|R\rangle$. Let us define η_0 by $q = \exp 2\eta_0$. We now introduce the following function:

$$Y(v;\varphi) = \sum_{\ell=0}^{N-1} \frac{q_0^{-\varphi(2\ell+1)} \prod_{j=1}^{L} \sinh(v - (2\ell+1)\eta_0)}{\prod_{j=1}^{R} (\sinh(v - \tilde{t}_j - 2\ell\eta_0) \sinh(v - \tilde{t}_j - 2(\ell+1)\eta_0)} \tag{4.4}$$

It follows from the twisted Bethe ansatz equations (4.1) that $Y(v;\varphi)$ is a Laurent polynomial of $\exp 2Nv$. Here we recall that when $\varphi = 0$ it is nothing but the polynomial introduced by Fabricius and McCoy for the XXZ spin chain at roots of unity under the periodic boundary conditions [9, 11].
Making use of the Laurent polynomial $Y(v;\varphi)$, we can show the following.

Proposition 4.4 *Let $|R\rangle$ be a regular Bethe state of the twisted XXZ spin chain at q_0 a root of unity with $q_0^{2N} = 1$ and in a sector of S^Z and with twist parameter φ such that the conditions of theorem 4.3 hold. Then, it is a highest weight vector of $U(\mathcal{B}_0)$, and all the highest weight parameters of $|R\rangle$ are nonzero.*

Let us consider a regular Bethe state $|R\rangle$ in such a sector of S^Z where we have $S^Z \pm \varphi \equiv 0$ (mod N). It follows from theorem 3.11 and proposition 4.4 that the highest weight representation generated by any given regular Bethe state in a sector of $S^Z \pm \varphi \equiv 0$ (mod N) extends to a highest weight representation of the sl_2 loop algebra. Thus, the Borel subalgebra symmetry of the twisted XXZ spin chain is extended to the sl_2 loop algebra symmetry.

4.3 Conjecture of the sl_2 loop algebra symmetry of the twisted transfer matrix

We now present the following conjecture:

Conjecture 4.5 *The twisted XXZ spin chain at roots of unity with twist parameter φ being integers should have the sl_2 loop algebra symmetry in every sector of S^Z mod N.*

5 Representations of the Onsager algebra derived from highest weight representations of $U(L(sl_2))$

The Onsager algebra is generated by operators A_m and G_ℓ ($\ell, m = 0, \pm 1, \pm 2, \ldots$) satisfyng the following defining relations [20, 21, 22, 23, 24, 25]:

$$\begin{aligned} [A_\ell, A_m] &= 4G_{\ell-m}\,, \\ [G_\ell, A_m] &= 2A_{m+\ell} - 2A_{m-\ell}\,, \\ [G_\ell, G_m] &= 0\,. \end{aligned} \tag{5.1}$$

We remark that Davies has shown that if generators A_n satisfy a linear recurrence relation

$$\sum_{k=-n}^{n} \gamma_k A_{k-n} = 0\,, \tag{5.2}$$

then they are expressed in terms of the generators of sl_2 as follows [22]:

$$\begin{aligned} A_m &= 2\sum_{j=1}^{n} \left(e_j^+ \otimes z_j^m + e_j^- \otimes z_j^{-m}\right)\,, \\ G_k &= \sum_{j=1}^{n} \left(h_j \otimes z_j^k - h_j \otimes z_j^{-k}\right) \end{aligned} \tag{5.3}$$

Here $e_j^\pm$ and h_k satisfy

$$[h_j, e_k^\pm] = \pm 2 e_j^\pm\, \delta_{jk}\,, \quad [e_j^+, e_k^-] = h_j\, \delta_{j,k} \tag{5.4}$$

Let $\boldsymbol{\alpha}$ denote a finite sequence of complex parameters such as $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Similarly as (3.11), we define generators with ℓ parameters, $A_m(\boldsymbol{\alpha})$ and $G_m(\boldsymbol{\alpha})$, as follows:

$$\begin{aligned} A_m(\boldsymbol{\alpha}) &= \sum_{k=0}^{\ell} (-1)^k A_{m-k} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,\ell\}} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}\,, \\ G_m(\boldsymbol{\alpha}) &= \sum_{k=0}^{\ell} (-1)^k G_{m-k} \sum_{\{i_1,\ldots,i_k\}\subset\{1,\ldots,\ell\}} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}\,. \end{aligned} \tag{5.5}$$

Let Ω be a highest weight vector in a finite-dimensional representation of the sl_2 loop algebra, $U(L(sl_2))$. We define operators A_m and G_k in terms of generators $x_m^{\pm}$ and h_k of $U(L(sl_2))$ by

$$A_m = x_m^+ + x_m^-, \quad G_k = h_k - h_{-k}. \tag{5.6}$$

Then, operators A_m and G_k satisfy the defining relations of the Onsager algebra. Furthermore, we can show recurrence relations of A_m's.

Proposition 5.1 *Let Ω be a highest weight vector in a finite-dimensional representation of the sl_2 loop algebra, $U(L(sl_2))$. If generators x_m^- of $U(L(sl_2))$ satisfy a recurrence relation, $x_n^-(\boldsymbol{\beta})\Omega = 0$, for a sequence of nonzero parameters $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, then we have*

$$A_n(\boldsymbol{\beta}; \bar{\boldsymbol{\beta}}) = 0. \tag{5.7}$$

Here $\bar{\boldsymbol{\beta}}$ denotes $\bar{\boldsymbol{\beta}} = (\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_n^{-1})$.

In Ref. [15], the sl_2 loop algebra symmetry is derived for a spin-$N/2$ fusion model of the six-vertex model at q_0 being an Nth root of unity, which is associated with the superintegrable chiral Potts model. >From the representations of the sl_2 loop algebra derived from the fusion model, we can thus construct representations of the Onsager algebra. Then, through proposition 5.1, we derive recurrence relations for generators of the Onsager algebra. It should thus be an interesting problem to discuss connections to the Onsager algebra symmetry of the Z_N symmetric Hamiltonian given by von Gehlen and Rittenberg. We shall discuss them elsewhere.

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Parabolic Subalgebras and Invariant Differential Operators

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Abstract

We present an update of an explicit canonical construction of invariant differential operators. We discuss in more detail one of the main ingredients - the parabolic subalgebras. We pay special attention to the peculiarities of the construction when complex simple Lie algebras are considered as real Lie algebras.

1 Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory, (for a recent review, cf. e.g., [2]). Thus, it is important for the applications in physics to study systematically such operators.

In the present paper we start with the classical situation, with the representation theory of semisimple Lie groups, where there are lots of results by both mathematicians and physicists. We shall follow a procedure in representation theory in which such operators appear canonically [3] and which has been generalized to the supersymmetry setting [4] and to quantum groups [5]. We should also mention that this setting is most appropriate for the classification of unitary representations of superconformal symmetry in various dimensions, for generalization to the infinite-dimensional setting, and is also an ingredient in the AdS/CFT correspondence.

Although the scheme was developed some time ago there is still missing explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the representations are induced.

Just in passing, we shall mention that parabolic subalgebras found applications in quantum groups, (in particular, for the quantum deformations of noncompact Lie algebras), and in integrable systems.

2 Canonical construction

Let G be a noncompact semisimple Lie group. Let K denote a maximal compact subgroup of G. Then we have an Iwasawa decomposition $G = KAN$, where A is abelian simply connected, a vector subgroup of G, N is a nilpotent simply connected subgroup of G preserved by the action of A. Further, let M be the centralizer of A in K. Then the subgroup $P_0 = MAN$ is a minimal parabolic subgroup of G. A parabolic subgroup $P = M'A'N'$ is any subgroup of G (including G itself) which contains a minimal parabolic subgroup. The number of non-conjugate parabolic subgroups is 2^r, where $r = \text{rank}\, A$, cf., e.g., [6]. Note that in general M' is a reductive Lie group with structure: $M' = M_d M_s M_a$, where M_d is a finite group, M_s is a semisimple Lie group, M_a is an abelian Lie group central in M'.

The importance of the parabolic subgroups stems from the fact that the representations induced from them generate all (admissible) irreducible representations of G [7]. In fact, it is enough to use only the so-called *cuspidal* parabolic subgroups $P = M'A'N'$, singled out by the condition that $\text{rank}\, M' = \text{rank}\, M' \cap K$ [8, 9], so that M' has discrete series representations [10].[1] However, often induction from a non-cuspidal parabolic is also

[1] The simplest example of cuspidal parabolic subgroup is P_0 when $M' = M$ is compact. In all other

convenient.

Let $P = M'A'N'$ be a parabolic subgroup. Let ν be a (non-unitary) character of A', $\nu \in \mathcal{A}'^*$. If P is cuspidal, let μ fix a discrete series representation D^μ of M' on the Hilbert space V_μ, or the so-called limit of a discrete series representation (cf. [11]).[2]

Although not strictly necessary, sometimes it is convenient to induce from non-cuspidal P (especially if P is a maximal parabolic). In that case, we use any non-unitary finite-dimensional irreducible representation D^μ of M' on the linear space V_μ.

More than this, except in the case of induction from limits of discrete series, we can always work with finite-dimensional representations V_μ by the so-called translation. Namely, when P is non-minimal and cuspidal, then instead of the inducing discrete series representation of M' we can consider the finite-dimensional irrep of M' which lies on the same orbit of the Weyl group (in other words, has the same Casimirs).

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G. (These are called *generalized principal series representations* (or limits thereof) in [11].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \;\mid\; \mathcal{F}(gman) = e^{\lambda(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \tag{2.1}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The special property of the functions of $\mathcal{C}_\chi$ is called *right covariance* [3] (or *equivariance*). Because of this covariance the functions $\mathcal{F}$ actually do not depend on the elements of the parabolic subgroup $P = M'A'N'$.

The elementary representation (ER) $\mathcal{T}^\chi$ acts in $\mathcal{C}_\chi$ as the left regular representation (LRR) by:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G\ . \tag{2.2}$$

One can introduce in $\mathcal{C}_\chi$ a Fréchet space topology or complete it to a Hilbert space (cf. [6]). We shall need also the infinitesimal version of LRR:

$$(X_L\mathcal{F})(g) \;\doteq\; \frac{d}{dt}\mathcal{F}(\exp(-tX)g)\Big|_{t=0} \tag{2.3}$$

where, $\mathcal{F} \in \mathcal{C}_\chi$, $g \in G$, $X \in \mathcal{G}$; then we use complex linear extension to extend the definition to a representation of $\mathcal{G}^{\mathbb{C}}$.

The ERs differ from the LRR (which is highly reducible) by the specific representation spaces $\mathcal{C}_\chi$. In contrast, the ERs are generically irreducible. The reducible ERs form a measure zero set in the space of the representation parameters μ, ν. (Reducibility here is topological in the sense that there exist nontrivial (closed) invariant subspace.) The irreducible components of the ERs (including the irreducible ERs) are called *subrepresentations*.

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cases M' is non-compact.

[2] In general, μ is a actually a triple $(\epsilon, \sigma, \delta)$, where ϵ is the signature of the character of M_d, σ gives the unitary character of M_a, δ fixes a discrete or finite-dimensional irrep of M_s on V_μ (the latter depends only on δ).

The other feature of the ERs which makes them important for our considerations is a highest (or lowest) weight module structure associated with them [3]. For this we shall use the right action of $\mathcal{G}^{\mathbb{C}}$ (the complexification of $\mathcal{G}$) by the standard formula:

$$(\hat{X}\mathcal{F})(g) \doteq \frac{d}{dt}\mathcal{F}(g\exp(tX))\Big|_{t=0} \tag{2.4}$$

where, $X \in \mathcal{G}^{\mathbb{C}}$, $\mathcal{F} \in \mathcal{C}_\chi$, $g \in G$, which is defined first for $X \in \mathcal{G}$ and then is extended to $\mathcal{G}^{\mathbb{C}}$ by linearity. Note that this action takes $\mathcal{F}$ out of $\mathcal{C}_\chi$ for some X but that is exactly why it is used for the construction of the intertwining differential operators.
We can show this property in all cases when V_μ is a highest weight module, e.g., the case of the minimal parabolic subalgebra and when $(M', M' \cap K)$ is a Hermitian symmetric pair. In fact, we agreed that, except when inducing from limits of discrete series, the space V_μ will be finite-dimensional.
Then V_μ has a highest weight vector v_0. Using this we introduce $\mathbb{C}$-valued realization $\tilde{\mathcal{T}}^\chi$ of the space $\mathcal{C}_\chi$ by the formula:

$$\varphi(g) \equiv \langle v_0, \mathcal{F}(g)\rangle \tag{2.5}$$

where $\langle , \rangle$ is the M-invariant scalar product in V_μ. [If $M' = M_0$ is abelian or discrete then V_μ is one–dimensional and $\tilde{\mathcal{C}}_\chi$ coincides with $\mathcal{C}_\chi$; so we set $\varphi = \mathcal{F}$.] On these functions the right action of $\mathcal{G}^{\mathbb{C}}$ is defined by:

$$(\hat{X}\varphi)(g) \equiv \langle v_0, (\hat{X}\mathcal{F})(g)\rangle \tag{2.6}$$

Part of the main result of our paper [3] is:
Proposition: The functions of the $\mathbb{C}$-valued realization $\tilde{\mathcal{T}}^\chi$ of the ER $\mathcal{C}_\chi$ satisfy:

$$\begin{aligned} &\hat{X}\varphi = \Lambda(X)\cdot\varphi, \quad X \in \mathcal{H}^{\mathbb{C}}, \Lambda \in (\mathcal{H}^{\mathbb{C}})^* \\ &\hat{X}\varphi = 0, \quad X \in \mathcal{G}^{\mathbb{C}}_+, \end{aligned} \tag{2.7}$$

where $\Lambda = \Lambda(\chi)$ is built canonically from χ,[3] $\mathcal{G}^{\mathbb{C}}_\pm$ are from the standard triangular decomposition $\mathcal{G}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}}_+ \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}^{\mathbb{C}}_-$.[4]
Note that conditions (2.7) are the defining conditions for the highest weight vector of a highest weight module (HWM) over $\mathcal{G}^{\mathbb{C}}$ with highest weight Λ. Of course, it is enough to impose (2.7) for the *simple* root vectors X^+_j.
Furthermore, special properties of a class of highest weight modules, namely, Verma modules, are immediately related with the construction of invariant differential operators. To be more specific let us recall that a Verma module is a highest weight module V^Λ with highest weight Λ, induced from one-dimensional representations of the Borel subalgebra

[3]It contains all the information from χ, except about the character ϵ of the finite group M_d. In the case of G being a complex Lie group we need two weights to encode χ, cf. Section 3.

[4]Note that we are working here with highest weight modules instead of the lowest weight modules used in [3]; also the weights are shifted by ρ with respect to the notation of [3].

$\mathcal{B} = \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}_+^{\mathbb{C}}$. Thus, $V^\Lambda \cong U(\mathcal{G}_-^{\mathbb{C}})v_0$, where v_0 is the highest weight vector, $U(\mathcal{G}_-^{\mathbb{C}})$ is the universal enveloping algebra of $\mathcal{G}_-^{\mathbb{C}}$.[5] Verma modules are universal in the following sense: every irreducible HWM is isomorphic to a factor-module of the Verma module with the same highest weight.

Generically, Verma modules are irreducible, however, we shall be mostly interested in the reducible ones since these are relevant for the construction of differential equations. We recall the Bernstein-Gel'fand-Gel'fand [13] criterion (for semisimple Lie algebras) according to which the Verma module V^Λ is reducible iff

$$2\langle \Lambda + \rho, \beta \rangle = m \langle \beta, \beta \rangle \tag{2.8}$$

holds for some $\beta \in \Delta^+$, $m \in \mathbb{N}$, where Δ^+ denotes the positive roots of the root system $(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$, ρ is half the sum of the positive roots Δ^+.

Whenever (2.8) is fulfilled there exists [12] in V^Λ a unique vector v_s, called *singular vector*, which has the properties (2.7) of a highest weight vector with shifted weight $\Lambda - m\beta$:

$$Xv_s = (\Lambda - m\beta)(X) \cdot v_s, \quad X \in \mathcal{H}^{\mathbb{C}} \tag{2.9}$$

$$Xv_s = 0, \quad X \in \mathcal{G}_+^{\mathbb{C}} \tag{2.10}$$

The general structure of a singular vector is [3]:

$$v_s = P_{m\beta}(X_1^-, \ldots, X_\ell^-)\, v_0 \tag{2.11}$$

where $P_{m\beta}$ is a homogeneous polynomial in its variables of degrees mk_i, where $k_i \in \mathbb{Z}_+$ come from the decomposition of β into simple roots: $\beta = \sum k_i \alpha_i$, $\alpha_i \in \Delta_S$, the system of simple roots, X_j^- are the root vectors corresponding to the negative roots $(-\alpha_j)$, α_j being the simple roots, $\ell = \mathrm{rank}_{\mathbb{C}} \mathcal{G}^{\mathbb{C}} = \dim_{\mathbb{C}} \mathcal{H}^{\mathbb{C}}$ is the (complex) rank of $\mathcal{G}^{\mathbb{C}}$.[6]

It is obvious that (2.11) satisfies (2.9a), while conditions (2.9b) fix the coefficients of $P_{m\beta}$ up to an overall multiplicative nonzero constant.

Now we are in a position to define the differential intertwining operators for semisimple Lie groups, corresponding to the singular vectors.

Let the signature χ of an ER be such that the corresponding $\Lambda = \Lambda(\chi)$ satisfies (2.8) for some $\beta \in \Delta^+$ and some $m \in \mathbb{N}$.[7] Then there exists an intertwining differential operator [3]:

$$\mathcal{D}_{m\beta} : \tilde{\mathcal{T}}^\chi \longrightarrow \tilde{\mathcal{T}}^{\chi'} \tag{2.12}$$

where χ' is such that $\Lambda' = \Lambda'(\chi') = \Lambda - m\beta$.

The most important fact is that (2.12) is explicitly given by [3]:

$$\mathcal{D}_{m\beta}\, \varphi(g) = P_{m\beta}(\hat{X}_1^-, \ldots, \hat{X}_\ell^-)\, \varphi(g) \tag{2.13}$$

[5] For more mathematically precise definition, cf. [12].

[6] A singular vector may also be written in terms of the full Cartan-Weyl basis of $\mathcal{G}_-^{\mathbb{C}}$.

[7] If β is a real root, (i.e., $\beta|_{\mathcal{H}_m^{\mathbb{C}}} = 0$, where $\mathcal{H}_m$ is the Cartan subalgebra of $\mathcal{M}$), then some conditions are imposed on the character ϵ representing the finite group M_d [14].

where $P_{m\beta}$ is the same polynomial as in (2.11) and $\hat{X}_j^-$ denotes the action (2.4).
One important simplification is that in order to check the intertwining properties of the operator in (2.13) it is enough to work with the infinitesimal versions of (2.1) and (2.2), i.e., work with representations of the Lie algebra. This is important for using the same approach to superalgebras and quantum groups, and to any other (infinite-dimensional) (super-)algebra with triangular decomposition.

3 More on parabolic subalgebras

Let $\mathcal{G}$ be the Lie algebra of G, θ be the Cartan involution in $\mathcal{G}$, and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the Cartan decomposition of $\mathcal{G}$, so that $\theta X = X, X \in \mathcal{K}, \theta X = -X, X \in \mathcal{P}$; $\mathcal{K}$ is the maximal compact subalgebra of $\mathcal{G}$.
Let $\mathcal{K}, \mathcal{A}, \mathcal{N}^{\pm}, \mathcal{M}, \mathcal{P}_0, \mathcal{P}'$ be the Lie algebras of the G-subgroups introduced in the previous subsection $K, A, N^{\pm}, M, P_0, P'$, resp.
With this data we can introduce the Iwasawa decomposition of $\mathcal{G}$:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N} \ , \quad \mathcal{N} = \mathcal{N}^{\pm} \tag{3.1}$$

Next we recall the Bruhat decomposition [15]:

$$\mathcal{G} = \mathcal{N}^+ \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^- \tag{3.2}$$

with minimal parabolic subalgebra: $\mathcal{P}_0 = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^-$. (Note that we may take equivalently $\mathcal{N}^+$ instead of $\mathcal{N}^-$.)
We mention an important class of real Lie algebras, the *split real forms*. For these we can use the same basis as for the corresponding complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$, but over $\mathbb{R}$. Restricting $\mathbb{C} \longrightarrow \mathbb{R}$ one obtains the Bruhat decomposition of $\mathcal{G}$ (with $\mathcal{M} = 0$) from the triangular decomposition of $\mathcal{G}^{\mathbb{C}} = \mathcal{G}^+ \oplus \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}^-$, and obtains the minimal parabolic subalgebras $\mathcal{P}_0$ from the Borel subalgebra $\mathcal{B} = \mathcal{H}^{\mathbb{C}} \oplus \mathcal{G}^+$, (or $\mathcal{G}^-$ instead of $\mathcal{G}^+$). Furthermore, in this case $\dim_{\mathbb{R}} \mathcal{K} = \dim_{\mathbb{R}} \mathcal{N}^{\pm}$.
A *standard parabolic subalgebra* is any subalgebra $\mathcal{P}'$ of $\mathcal{G}$ containing $\mathcal{P}_0$. The number of standard parabolic subalgebras, including $\mathcal{P}_0$ and $\mathcal{G}$, is 2^r.
Thus, if $r = 1$ the only nontrivial parabolic subalgebra is $\mathcal{P}_0$.
Thus, further in this section $r > 1$.
Any standard parabolic subalgebra is of the form:

$$\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'^- \ , \tag{3.3}$$

so that $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{N}'^- \subseteq \mathcal{N}^-$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}$ (mod$\mathcal{A}'$); $\mathcal{N}'^-$ is comprised from the negative root spaces of the restricted root system $\Delta_{\mathcal{A}'}$ of $(\mathcal{G}, \mathcal{A}')$. The decomposition (3.3) is called the Langlands decomposition of $\mathcal{P}'$. One also has the analogue of the Bruhat decomposition (3.2):

$$\mathcal{G} = \mathcal{N}'^+ \oplus \mathcal{A}' \oplus \mathcal{M}' \oplus \mathcal{N}'^- \tag{3.4}$$

where $\mathcal{M}' \supseteq \mathcal{M}$, $\mathcal{A}' \subseteq \mathcal{A}$, $\mathcal{N}'^{\pm} \subseteq \mathcal{N}^{\pm}$; $\mathcal{M}'$ is the centralizer of $\mathcal{A}'$ in $\mathcal{G}$ (mod$\mathcal{A}'$); $[\mathcal{A}', \mathcal{N}'^{\pm}] = \mathcal{N}'^{\pm}$, $\mathcal{N}'^{+} = \theta\mathcal{N}'^{-}$.
We mention also Cartan subalgebras $\mathcal{H}$ of $\mathcal{G}$ related to Bruhat decompositions: $\mathcal{H}_0 = \mathcal{H}_m \oplus \mathcal{A}$, where $\mathcal{H}_m$ is a Cartan subalgebra of $\mathcal{M}$, or more generally: $\mathcal{H}' = \mathcal{H}'_m \oplus \mathcal{A}'$, where $\mathcal{H}'_m$ is a Cartan subalgebra $\mathcal{M}'$.
Finally, we mention Cartan subalgebras $\mathcal{H}$ of $\mathcal{G}$ related to the Iwasawa decomposition: $\mathcal{H}_i = \mathcal{H}_k \oplus \mathcal{A}''$, where $\mathcal{H}_k$ is a Cartan subalgebra of $\mathcal{K}$, dim $\mathcal{A}'' = \text{rank}\,\mathcal{G} - \text{rank}\,\mathcal{K}$.

Reminder 1: [Harish-Chandra] A noncompact semisimple Lie algebra $\mathcal{G}$ has discrete series representations iff $\text{rank}\,\mathcal{G} = \text{rank}\,\mathcal{K}$.$\diamondsuit$

Reminder 2: A noncompact semisimple Lie algebra $\mathcal{G}$ has highest and lowest weight representations iff $(\mathcal{G}, \mathcal{K})$ is a Hermitian symmetric pair, i.e., iff the maximal compact subalgebra $\mathcal{K}$ is reductive and contains a $u(1)$ factor.
All these algebras have discrete series representations.$\diamondsuit$

4 The complex simple Lie algebras considered as real Lie algebras

Let $\mathcal{G}_c$ be a complex simple Lie algebra of dimension d and (complex) rank ℓ and triangular decomposition over $\mathbb{C}$:

$$\mathcal{G}_c = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{4.1}$$

We have $\dim_{\mathbb{C}} \mathcal{G}_c = d$, $\text{rank}_{\mathbb{C}}\,\mathcal{G}_c = \dim_{\mathbb{C}} \mathcal{H} = \ell$, $\dim_{\mathbb{C}} \mathcal{N}^{\pm} = (d-\ell)/2$. Considered as real Lie algebras we have: $\dim_{\mathbb{R}} \mathcal{G}_c = 2d$, $\text{rank}_{\mathbb{R}}\,\mathcal{G}_c = \dim_{\mathbb{R}} \mathcal{H} = 2\ell$, $\dim_{\mathbb{R}} \mathcal{K} = d$, $\text{rank}_{\mathbb{R}}\,\mathcal{K} = \ell$, $\dim_{\mathbb{R}} \mathcal{N}^{\pm} = d - \ell$. Note that the maximal compact subalgebra $\mathcal{K}$ of $\mathcal{G}_c$ is isomorphic to the compact real form $\mathcal{G}_k$ of $\mathcal{G}_c$.
• Thus, the complex simple Lie algebras do not have discrete series representations (since $\text{rank}_{\mathbb{R}}\,\mathcal{G}_c \neq \text{rank}_{\mathbb{R}}\,\mathcal{K}$).
• Complex simple Lie algebras considered over $\mathbb{R}$ do not have highest (or lowest) weight representations (since $\mathcal{K}$ is simple).
Let H_j, $j = 1, \ldots, \ell$, be a basis of $\mathcal{H}$, i.e., $\mathcal{H}$ =c.l.s.$\{H_j, j = 1, \ldots, \ell\}$, (where c.l.s. stands for complex linear span), such that each ad(H_j) has only real eigenvalues. Let $\mathcal{A} \doteq \mathcal{H}_{\mathbb{R}}$ =r.l.s.$\{H_j, j = 1, \ldots, \ell\}$, where r.l.s. stands for real linear span. Then the Iwasawa decomposition of $\mathcal{G}_c$ is:

$$\mathcal{G}_c = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}\ , \quad \mathcal{N} = \mathcal{G}^{\pm} \tag{4.2}$$

The commutant $\mathcal{M}$ of $\mathcal{A}$ in $\mathcal{K}$ is:

$$\mathcal{M} = u(1) \oplus \cdots \oplus u(1)\ , \qquad \ell \ \text{ factors} \tag{4.3}$$

The basis of $\mathcal{M}$ consists of the vectors $\{i\,H_j, j = 1, \ldots, \ell\}$.

The Bruhat decomposition of $\mathcal{G}_c$ is:

$$\mathcal{G}_c = \mathcal{G}^+ \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{G}^- \tag{4.4}$$

Comparing (4.1) and (4.4) we see that

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{A} \tag{4.5}$$

• The complex simple Lie algebras, considered as real noncompact Lie algebras, do not have non-minimal cuspidal parabolic subalgebras.
Thus, it is enough to consider elementary representations induced from the minimal parabolic subgroup $P_0 = MAN$, where $M \cong U(1) \times \ldots \times U(1)$, ($\ell$ factors), $A \cong SO(1,1) \times \ldots \times SO(1,1)$, ($\ell$ factors), $N \cong \exp \mathcal{G}^{\pm}$.
Thus, the signature $\chi = [\mu, \nu]$, consists of ℓ integer numbers $\mu_i \in \mathbb{Z}$ giving the unitary character $\mu = (\mu_1, \ldots, \mu_\ell)$ of M, and of ℓ complex numbers $\nu_i \in \mathbb{C}$ giving the character $\nu = (\nu_1, \ldots, \nu_\ell)$ of A, $\nu_j = \nu(H_j)$.
Thus, if $H = \sum_j \sigma_j H_j$, $\sigma_j \in \mathbb{R}$, is a generic element of $\mathcal{A}$, then for the corresponding factor in (2.1) we have $e^{\nu(H)} = \exp \sum_j \sigma_j \nu_j$.
Analogously, if $m = \exp i \sum_j \phi_j H_j \in M$, $\phi_j \in \mathbb{R}$, then we have $D^\mu(m^{-1}) = \exp i \sum_j \phi_j \mu_j$.
Thus, the right covariance property (2.1) becomes:

$$\mathcal{F}(gman) = \exp \sum_j (\sigma_j \nu_j + i \phi_j \mu_j) \cdot \mathcal{F}(g) \tag{4.6}$$

To relate with the general setting of the previous subsection we must introduce two weight functionals: $\Lambda, \tilde{\Lambda}$, such that $\Lambda(H_j) = \lambda_j$, $\tilde{\Lambda}(H_j) = \tilde{\lambda}_j$.
Let us use (4.5) and $H = \sum_j (\sigma_j + i\phi_j) H_j \in \mathcal{H}$.
Thus the elementary representations (in particular, the right covariance conditions) for a complex semisimple Lie group G_c are given by:

$$\begin{aligned} \mathcal{C}_{\Lambda,\tilde{\Lambda}} &= \{ \mathcal{F} \in C^\infty(G_c) \mid \mathcal{F}(gman) = \exp\Big(\Lambda(H) + \tilde{\Lambda}(\bar{H})\Big) \cdot \mathcal{F}(g) = \\ &= \exp \sum_j \Big((\sigma_j + i\phi_j)\lambda_j + (\sigma_j - i\phi_j)\tilde{\lambda}_j \Big) \cdot \mathcal{F}(g) \ , \\ &\qquad \nu_j = \lambda_j + \tilde{\lambda}_j \ , \quad \mu_j = \lambda_j - \tilde{\lambda}_j \in \mathbb{Z} \ \} \end{aligned} \tag{4.7}$$

and the last condition in (4.7) stresses that we have uniqueness on the compact subgroup M of the Cartan subgroup $H_c = MA$ of G_c.
The ERs for which $\tilde{\Lambda} = 0$ are called *holomorphic*, and those for which $\Lambda = 0$ are called *antiholomorphic*.
Thus, we see that the complex case is richer than the real one. Indeed, there are *two* Verma modules associated with an ER, one 'holomorphic' V^Λ and one 'antiholomorphic' $V^{\tilde{\Lambda}}$.

The ER is reducible when either V^{Λ} or $V^{\tilde{\Lambda}}$ are reducible, i.e., when (2.8) holds for either Λ or $\tilde{\Lambda}$.
More information can be found in [8] from where we mention some important statements:
• All irreducible representations of a complex semisimple Lie group are obtained as subrepresentations of the elementary representations induced from the minimal parabolic subgroup.
• All finite-dimensional irreps are obtained as subrepresentations when all $\lambda_j, \tilde{\lambda}_j \in \mathbb{Z}_+$.

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Systematic derivation of boundary Lax pairs

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Abstract

We systematically derive the Lax pair formulation for both discrete and continuum integrable classical theories with consistent boundary conditions.

1 The discrete case

Quadratic Poisson structures first appeared ain the well-known Sklyanin bracket [1]. A more general form, characterized by a pair of respectively skew symmetric and symmetric matrices $(r,\ s)$ appeared in [2] in the formulation of consistent Poisson structures for non-ultralocal classical integrable field theories. Finally it was shown [3] that this was the natural quadratic form a la Sklyanin for a non-skew-symmetric r-matrix, reading:

$$\Big\{L_1,\ L_2\Big\} = \Big[r - r^{\pi},\ L_1 L_2\Big] + L_1(r + r^{\pi})L_2 - L_2(r + r^{\pi})L_1. \tag{1.1}$$

A typical situation when one considers naturally a quadratic Poisson structure for the Lax matrix occurs when considering discrete or continuous integrable systems where the Lax matrix depends on either a discrete or a continuous variable; the Lax pair is thus associated to a point on the space-like lattice or continuous line [4, 5]. Let us first examine the discrete case where one considers a finite set of Lax matrices L_n labelled by $n \in \mathbb{N}$. Lax representation of classical dynamical evolution equations [6] is one key ingredient in the modern theory of classical integrable systems [7]–[11] together with the associated notion of classical r-matrix [12, 13]. Introduce the Lax pair $(L,\ A)$ for discrete integrable models [9] (see also [14] for statistical systems), and the associated auxiliary problem (see e.g. [4])

$$\begin{aligned} \psi_{n+1} &= L_n\ \psi_n \\ \dot{\psi}_n &= A_n\ \psi_n. \end{aligned} \tag{1.2}$$

From the latter equations one may immediately obtain the discrete zero curvature condition:

$$\dot{L}_n = A_{n+1}\ L_n - L_n\ A_n. \tag{1.3}$$

The monodromy matrix arises from the first equation (1.2) (see e.g. [4])

$$T_a(\lambda) = L_{aN}(\lambda) \dots L_{a1}(\lambda) \tag{1.4}$$

where index a denotes the auxiliary space, and the indices $1, \dots, N$ denote the sites of the one dimensional classical discrete model.
Consider now a skew symmetric classical r-matrix which is a solution of the classical Yang-Baxter equation [12, 13]

$$\Big[r_{12}(\lambda_1 - \lambda_2),\ r_{13}(\lambda_1) + r_{23}(\lambda_2)\Big] + \Big[r_{13}(\lambda_1),\ r_{23}(\lambda_2)\Big] = 0, \tag{1.5}$$

and let L satisfy the associated Sklyanin bracket

$$\Big\{L_a(\lambda),\ L_b(\mu)\Big\} = \Big[r_{ab}(\lambda - \mu),\ L_a(\lambda) L_b(\mu)\Big]. \tag{1.6}$$

It is then immediate that that (1.4) also satisfies (1.6). Use of the latter equation shows that the quantities $trT(\lambda)^n$ provide charges in involution, that is

$$\Big\{tr\ T^n(\lambda),\ tr\ T^m(\mu)\Big\} = 0 \tag{1.7}$$

which again is trivial by virtue of (1.6). In the simple sl_2 case the only non trivial quantity is $trT(\lambda) = t(\lambda)$, that is the usual "bulk" transfer matrix. In the bulk case in particular the zero curvature condition (1.3) is realized by L_n and [4]:

$$A_n(\lambda,\mu) = t^{-1}(\lambda)\ tr_a\{T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\} \tag{1.8}$$

where we define

$$T_a(n,m;\lambda) = L_{an}(\lambda)L_{an-1}(\lambda)\dots L_{am}(\lambda), \qquad n > m. \tag{1.9}$$

We now generalize the procedure described in [4] for periodic boundary conditions to the case of generic integrable "boundary conditions". We propose a construction of two types of monodromy and transfer matrices, and associated Lax-type evolution equations, albeit incorporating a supplementary set of non-dynamical parameters encapsulated into a "reflection" matrix $K(\lambda)$. Any physical interpretation of the K-matrix as a description of the "boundary properties" may not be appropriate in all cases. We should stress that this is the first time to our knowledge (see also [15]) that such an investigation is systematically undertaken. There are several related studies regarding particular examples of open spin chains [16, 17], however the derivation of the corresponding Lax pair is restricted to the Hamiltonian only and not to all associated integrals of motion. In this study we present a generic description independent of the choice of model, and we derive the Lax pair for each one of the entailed boundary integrals of motion.

Subsequently we shall deal with two types of classical algebras, which are derived from two known types of consistent quantum boundary conditions. These boundary conditions are known as soliton preserving (SP), (see e.g. [18]–[22]), and soliton non-preserving (SNP) [23, 24, 25]. SNP boundary conditions have been also introduced and studied for integrable quantum lattice systems [26]–[30]. From the algebraic perspective the two types of boundary conditions are associated with two distinct algebras, i.e. the reflection algebra [18] and the twisted Yangian respectively [31, 32]. It will be convenient for our purposes here to introduce some useful notation:

$$\begin{aligned}
&\hat{r}_{ab}(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \qquad \hat{r}_{ab}(\lambda) = r_{ba}^{t_a t_b}(\lambda) \quad \text{for SNP} \\
&r^*_{ab}(\lambda) = r_{ab}(\lambda) \quad \text{for SP}, \qquad r^*_{ab}(\lambda) = r_{ba}^{t_b}(-\lambda) \quad \text{for SNP} \\
&\hat{r}^*_{ab}(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \qquad \hat{r}^*_{ab}(\lambda) = r_{ab}^{t_a}(-\lambda) \quad \text{for SNP}
\end{aligned} \tag{1.10}$$

The two types of monodromy matrices will respectively represent the classical version of the reflection algebra $\mathbb{R}$, and the twisted Yangian $\mathbb{T}$ written in the compact form: (see

e.g. [18, 2]):

$$\Big\{\mathcal{T}_1(\lambda_1),\ \mathcal{T}_2(\lambda_2)\Big\} = r_{12}(\lambda_1-\lambda_2)\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2) - \mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)\hat{r}_{12}(\lambda_1-\lambda_2) + \mathcal{T}_1(\lambda_1)\hat{r}^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_2(\lambda_2) - \mathcal{T}_2(\lambda_2)r^*_{12}(\lambda_1+\lambda_2)\mathcal{T}_1(\lambda_1) \quad (1.11)$$

where $\hat{r}$, r^*, $\hat{r}^*$ are defined in (1.10). In most cases, such as the $A^{(1)}_{\mathcal{N}-1}$ r-matrices $r_{12}^{t_1 t_2} = r_{21}$ implying that in the SNP case $r^*_{ab} = \hat{r}^*_{ab}$. In the case of the Yangian r-matrix $r_{12} = r_{21}$, hence all the expressions above may be written in a more symmetric form.
In order to construct representations of (1.11) yielding a generating function of integrals of motion one now introduces c-number (non-dynamical) representations satisfying the purely algebraic condition (1.11) since:

$$\Big\{K_1^{\pm}(\lambda_1),\ K_2^{\pm}(\lambda_2)\Big\} = 0. \quad (1.12)$$

Taking now as $T(\lambda)$ any bulk monodromy matrix (1.4) built from local L matrices obeying (1.6) and defining in addition

$$\hat{T}(\lambda) = T^{-1}(-\lambda) \quad \text{for SP}, \qquad \hat{T}(\lambda) = T^t(-\lambda) \quad \text{for SNP}. \quad (1.13)$$

one shows that representations of the corresponding algebras $\mathbb{R}$, $\mathbb{T}$, are given by the following expression see e.g. [18, 33]:

$$\mathcal{T}(\lambda) = T(\lambda)\ K^-(\lambda)\ \hat{T}(\lambda). \quad (1.14)$$

For a detailed proof see e.g. [33].
Define now as generating function of the involutive quantities

$$t(\lambda) = tr\{K^+(\lambda)\ \mathcal{T}(\lambda)\}. \quad (1.15)$$

Due to (1.11) it is shown that [18, 33]

$$\Big\{t(\lambda_1),\ t(\lambda_2)\Big\} = 0, \quad \lambda_1,\ \lambda_2 \in \mathbb{C}. \quad (1.16)$$

Usually one considers the quantity ln $t(\lambda)$ to get *local* integrals of motion, however for the examples we are going to examine here the expansion of $t(\lambda)$ is enough to provide the associated local quantities as will be transparent in the subsequent section. Finally one shows that time evolution of the local Lax matrix L_n under generating Hamiltonian action of $t(\lambda)$ is given by:

$$\dot{L}_n(\mu) = \mathbb{A}_{n+1}(\lambda,\mu)\ L_n(\mu) - \mathbb{A}_n(\lambda,\mu)\ L_n(\mu), \quad (1.17)$$

where $\mathbb{A}_n$ is the modified (boundary) quantity,

$$\mathbb{A}_n(\lambda,\mu) = tr_a\Big(K_a^+(\lambda)\ T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\ K_a^-(\lambda)\ \hat{T}_a(\lambda) + K_a^+(\lambda)\ T_a(\lambda)\ K_a^-(\lambda)\ \hat{T}_a(1,n-1;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n,N;\lambda)\Big) \quad (1.18)$$

where $T(n,m;\lambda)$ is defined in (1.9) and

$$\hat{T}(m,n;\lambda) = \hat{L}_{am}(\lambda)\dots\hat{L}_{an}(\lambda) \qquad n > m. \tag{1.19}$$

To prove (1.18) we need in addition to (1.6) one more fundamental relation i.e.

$$\Big\{\hat{L}_a(\lambda), L_b(\mu)\Big\} = \hat{L}_a(\lambda)\hat{r}^*_{ab}(\lambda)L_b(\mu) - L_b(\mu)\hat{r}^*_{ab}(\lambda+\mu)\hat{L}_a(\lambda). \tag{1.20}$$

Taking into account (1.6) and the latter expressions we derive:

$$\begin{aligned}\Big\{t(\lambda),\ L_{bn}(\mu)\Big\} &= tr_a\Big(K^+_a(\lambda)\ T_a(N,n+1;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n,1;\lambda)\ K^-_a(\lambda)\ \hat{T}(\lambda)\\
&+ K^+_a(\lambda)\ T_a(\lambda)\ K^-_a(\lambda)\hat{T}_a(1,n;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n+1,N;\lambda)\Big)L_{bn}(\mu)\\
&- L_{bn}(\mu)\ tr_a\Big(K^+_a(\lambda)\ T_a(N,n;\lambda)\ r_{ab}(\lambda-\mu)\ T_a(n-1,1;\lambda)\ K^-_a(\lambda)\hat{T}(\lambda)\\
&+ K^+_a(\lambda)\ T_a(\lambda)\ K^-_a(\lambda)\ \hat{T}_a(1,n-1;\lambda)\ \hat{r}^*_{ab}(\lambda+\mu)\ \hat{T}_a(n,N;\lambda)\Big). \end{aligned}\tag{1.21}$$

Expression (1.18) is readily extracted from (1.21).
Special care should be taken at the boundary points $n = 1$ and $n = N+1$. Indeed we set: $T(N,N+1,\lambda) = T(0,1,\lambda) = \hat{T}(1,0,\lambda) = \hat{T}(N+1,N,\lambda) = \mathbb{I}$. We should stress that the derivation of the boundary Lax pair is universal, namely the expressions (1.18) are generic and independent of the choice of L, r. Note that a different construction of representations of (1.11) was already given in a very general setting in [34]. It is related to the formulation of non-ultralocal integrable field theories on a lattice and extends the analysis of [2].

1.1 Example

We shall now examine a simple example, i.e. the open generalized DST model, which may be seen as a lattice version of the generalized (vector) NLS model, (see also [35, 36, 33]) for further details). The open Toda chain will also be discussed as a limit of the DST model. We shall explicitly evaluate the "boundary" Lax pairs for the first integrals of motion. We focus here on the special case of the simplest rational non-dynamical r-matrices [37]

$$r(\lambda) = \frac{\mathbb{P}}{\lambda} \quad \text{where} \quad \mathbb{P} = \sum_{i,j=1}^{N} E_{ij}\otimes E_{ji} \tag{1.22}$$

$\mathbb{P}$ is the permutation operator, and $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.
The Lax operator of the $gl(\mathcal{N})$ DST model has the following form:

$$L(\lambda) = (\lambda - \sum_{j=1}^{\mathcal{N}-1} x^{(j)}X^{(j)})E_{11} + b\sum_{j=2}^{\mathcal{N}} E_{jj} + b\sum_{j=2}^{\mathcal{N}} x^{(j-1)}E_{1j} - \sum_{j=2}^{\mathcal{N}} X^{(j-1)}E_{j1} \tag{1.23}$$

with $x_n^{(j)}$, $X_n^{(j)}$ being canonical variables. In [33] the first non-trivial integral of motion for the SNP case, choosing the simplest consistent value $K^{\pm} = \mathbb{I}$ was explicitly computed:

$$\mathcal{H} = -\frac{1}{2}\sum_{n=1}^{N}\mathbb{N}_n^2 - b\sum_{n=1}^{N}\sum_{j=1}^{\mathcal{N}-1} X_n^{(j)}x_{n+1}^{(j)} - \frac{1}{2}\sum_{j=1}^{\mathcal{N}-1}(X_N^{(j)}X_N^{(j)} + b^2 x_1^{(j)})$$
$$\text{where} \quad \mathbb{N}_n = \sum_{j=1}^{\mathcal{N}-1} x_n^{(j)}X_n^{(j)}. \tag{1.24}$$

Our aim is now to determine the modified Lax pair induced by the non-trivial integrable boundary conditions. We shall focus here on the case of SNP boundary conditions, basically because in the particular example we consider here such boundary conditions are technically easier to study. Taking into account (1.18) we explicitly derive the modified Lax pair for the generalized DST model with SNP boundary conditions. Indeed, after expanding (1.18) in powers of λ^{-1}, and recalling (1.22) we obtain the quantity associated to the Hamiltonian (1.24)

$$\mathbb{A}_n^{(2)} = \lambda E_{11} - \sum_{j\neq 1} X_{n-1}^{(j-1)}E_{j1} + b\sum_{j\neq 1} x_n^{(j-1)}E_{1j}, \quad n \in \{2,\dots N\}$$
$$\mathbb{A}_1^{(2)} = \lambda E_{11} - b\sum_{j\neq 1} x_1^{(j-1)}E_{j1} + b\sum_{j\neq 1} x_1^{(j-1)}E_{1j},$$
$$\mathbb{A}_{N+1}^{(2)} = \lambda E_{11} - \sum_{j\neq 1} X_N^{(j-1)}E_{j1} + \sum_{j\neq 1} X_N^{(j-1)}E_{1j}. \tag{1.25}$$

It is worth stressing that in the sl_2 case the SP and SNP boundary conditions coincide given that

$$L^{-1}(-\lambda) = V\ L^t(-\lambda)\ V, \qquad V = \text{antid}(1,\dots,1). \tag{1.26}$$

The equations of motion associated to the Hamiltonian (1.24) may be readily extracted by virtue of

$$\dot{L} = \Big\{\mathcal{H}^{(2)},\ L\Big\}. \tag{1.27}$$

Alternatively the equations of motion may be derived from the zero curvature condition, which the modified Lax pair satisfies. It is clear that to each one of the higher local charges a different quantify $\mathbb{A}_n^{(i)}$ is associated. Both equations (1.27), (1.3) lead naturally to the same equations of motion, which for example in the sl_2 case read as:

$$\dot{x}_n = x_n^2X_n + bx_{n+1}, \qquad \dot{X}_n = -x_nX_n^2 - bX_{n-1}, \quad n \in \{2,\dots N-1\}$$
$$\dot{x}_1 = x_1^2X_1 + bx_2, \qquad \dot{X}_1 = -x_1X_1^2 - bx_1$$
$$\dot{x}_N = x_N^2X_N + X_N, \qquad \dot{X}_N = -x_NX_N^2 - bX_{N-1}. \tag{1.28}$$

Note that the Toda model [38] may be seen as an appropriate limit of the sl_2 DST model (see also [39]), and the corresponding boundary Hamiltonian, Lax pair and equations of motion are easily obtained from the expressions above (for more details see [33]).

2 The continuous case

Let us now recall the basic notions regarding the Lax pair and the zero curvature condition for a continuous integrable model following essentially [4]. Define the transition matrix T as being a solution of the following set of equations (see e.g. [4])

$$\frac{\partial T}{\partial x} = \mathbb{U}(x,t,\lambda)T \tag{2.1}$$

$$\frac{\partial T}{\partial t} = \mathbb{V}(x,t,\lambda)T \tag{2.2}$$

$$T(x,x,t,\lambda) = \mathbb{I} \tag{2.3}$$

$\mathbb{U}$, $\mathbb{V}$ being in general $n \times n$ matrices with entries defined as functions of complex valued dynamical fields, their derivatives, and the spectral parameter λ. The solution of (2.1) for $x > y$ may be written as:

$$T(x,y,\lambda) = \mathcal{P}exp\Big\{ \int_y^x \mathbb{U}(x',t,\lambda)dx' \Big\}. \tag{2.4}$$

The fact that T is a solution of equation (2.1) will be extensively used to get the relevant integrals of motion. Compatibility conditions of the two differential equations (2.1), (2.2) lead to the zero curvature condition [8]–[10]

$$\dot{\mathbb{U}} - \mathbb{V}' + \Big[\mathbb{U},\ \mathbb{V}\Big] = 0, \tag{2.5}$$

giving rise to the corresponding classical equations of motion of the system under consideration.

Hamiltonian formulation of the equations of motion is available again under the r-matrix approach. In this picture the underlying classical algebra is manifestly analogous to the quantum case. The existence of the Poisson structure for $\mathbb{U}$ realized by the classical r-matrix, satisfying the classical Yang-Baxter equation (1.5), guarantees the integrability of the classical system. Indeed assuming that the operator $\mathbb{U}$ satisfies the following ultralocal form of Poisson brackets

$$\Big\{\mathbb{U}_a(x,\lambda),\ \mathbb{U}_b(y,\mu)\Big\} = \Big[r_{ab}(\lambda-\mu),\ \mathbb{U}_a(x,\lambda) + \mathbb{U}_b(y,\mu)\Big]\ \delta(x-y), \tag{2.6}$$

then $T(x,y,\lambda)$ satisfies (1.6), and consequently one may readily show for a system on the full line:

$$\Big\{ \ln tr\{T(x,y,\lambda_1)\},\ \ln tr\{T(x,y,\lambda_2)\}\Big\} = 0 \tag{2.7}$$

i.e. the system is integrable, and the charges in involution –local integrals of motion– are obtained by expansion of the generating function $\ln tr\{T(x,y,\lambda)\}$, based essentially on the fact that T satisfies (2.1).

Our aim here is to consider integrable models on the interval with consistent "boundary conditions", and derive rigorously the Lax pairs associated to the entailed boundary local integrals of motion as a continuous extension of the discrete case described previously. We briefly describe this process below for any classical integrable system on the interval. In this case one constructs a modified transition matrix $\mathcal{T}$, based on Sklyanin's formulation and satisfying again the Poisson bracket algebras $\mathbb{R}$ or $\mathbb{T}$. To construct the generating function of the integrals of motion one also needs c-number representations of the algebra $\mathbb{R}$ or $\mathbb{T}$ satisfying (1.11), (1.12). The modified transition matrices, realizing the corresponding algebras $\mathbb{R}$, $\mathbb{T}$ are given by (1.14), where now T defined in (2.4) and $\hat{T}$ in (1.13). The generating function of the involutive quantities is defined in (1.15) and one one shows in this case as well:

$$\Big\{ t(x,y,t,\lambda_1),\ t(x,y,t,\lambda_2) \Big\} = 0, \quad \lambda_1,\ \lambda_2 \in \mathbb{C}. \tag{2.8}$$

In the case of open boundary conditions, exactly as in the discrete integrable models, we prove (for more details on the proof see [15])

$$\Big\{ \mathcal{T}_a(0,-L,\lambda),\ \mathbb{U}_b(x,\mu) \Big\} = \mathbb{M}'_a(x,\lambda,\mu) + \Big[\mathbb{M}_a(x,\lambda,\mu),\ \mathbb{U}_b(x,\ \mu) \Big] \tag{2.9}$$

where we define

$$\begin{aligned} \mathbb{M}(x,\lambda,\mu) &= T(0,x,\lambda) r_{ab}(\lambda-\mu) T(x,-L,\lambda) K^-(\lambda) \hat{T}(0,-L,\lambda) \\ &+ T(0,-L,\lambda) K^-(\lambda) \hat{T}(x,-L,\lambda) \hat{r}^*_{ab}(\lambda+\mu) \hat{T}(0,x,\lambda). \end{aligned} \tag{2.10}$$

Finally bearing in mind the definition of $t(\lambda)$ and (2.9) we conclude with:

$$\Big\{ \ln\ t(\lambda),\ \mathbb{U}(x,\mu) \Big\} = \frac{\partial \mathbb{V}(x,\lambda,\mu)}{\partial x} + \Big[\mathbb{V}(x,\lambda,\mu),\ \mathbb{U}(x,\mu) \Big] \tag{2.11}$$

where

$$\mathbb{V}(x,\lambda,\mu) = t^{-1}(\lambda)\ tr_a \Big(K^+(\lambda)\ \mathbb{M}_a(x,\lambda,\mu) \Big). \tag{2.12}$$

As in the discrete case particular attention should be paid to the boundary points $x = 0,\ -L$. Indeed, for these two points one has to simply take into account that $T(x,x,\lambda) = \hat{T}(x,x,\lambda) = \mathbb{I}$. Moreover, the expressions derived in (2.10), (2.12) are universal, that is independent of the choice of model.

2.1 Example

We shall now examine a particular example associated to the rational r-matrix (1.22), that is the $gl_{\mathcal{N}}$ NLS model. Although in [33] an extensive analysis for both types of boundary conditions is presented, here we shall focus on the simplest diagonal ($K^{\pm} = \mathbb{I}$) boundary conditions. The Lax pair is given by the following expressions [4, 40]:

$$\mathbb{U} = \mathbb{U}_0 + \lambda \mathbb{U}_1, \quad \mathbb{V} = \mathbb{V}_0 + \lambda \mathbb{V}_1 + \lambda^2 \mathbb{V}_2 \tag{2.13}$$

where

$$\mathbb{U}_1 = \frac{1}{2i}(\sum_{i=1}^{\mathcal{N}-1} E_{ii} - E_{\mathcal{N}\mathcal{N}}), \qquad \mathbb{U}_0 = \sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i E_{i\mathcal{N}} + \psi_i E_{\mathcal{N}i})$$
$$\mathbb{V}_0 = i\sum_{i,\ j=1}^{\mathcal{N}-1}(\bar{\psi}_i\psi_j E_{ij} - |\psi_i|^2 E_{\mathcal{N}\mathcal{N}}) - i\sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i' E_{i\mathcal{N}} - \psi_i' E_{\mathcal{N}i}),$$
$$\mathbb{V}_1 = -\mathbb{U}_0, \quad \mathbb{V}_2 = -\mathbb{U}_1 \tag{2.14}$$

and $\psi_i,\ \bar{\psi}_j$ satisfy[1]:

$$\Big\{\psi_i(x),\ \psi_j(y)\Big\} = \Big\{\bar{\psi}_i(x),\ \bar{\psi}_j(y)\Big\} = 0, \qquad \Big\{\psi_i(x),\ \bar{\psi}_j(y)\Big\} = \delta_{ij}\ \delta(x-y). \tag{2.16}$$

The boundary Hamiltonian for the generalized NLS model may be expressed as (see [33])

$$\begin{aligned}\mathcal{H} &= \int_{-L}^{0} dx \sum_{i=1}^{\mathcal{N}-1}\Big(\kappa|\psi_i(x)|^2\sum_{j=1}^{\mathcal{N}-1}|\psi_j(x)|^2 + \psi_i'(x)\bar{\psi}_i'(x)\Big)\\ &- \sum_{i=1}^{\mathcal{N}-1}\Big(\psi_i'(0)\bar{\psi}_i(0) + \psi_i(0)\bar{\psi}_i'(0)\Big) + \sum_{i=1}^{\mathcal{N}-1}\Big(\psi_i'(-L)\bar{\psi}_i(-L) + \psi_i(-L)\bar{\psi}_i'(-L)\Big)\end{aligned} \tag{2.17}$$

One sees here that the K-matrix indeed contributes as a genuine boundary effect. The Hamiltonian, obtained as one of the charges in involution (see e.g. [33] for further details) provides the classical equations of motion by virtue of:

$$\frac{\partial\psi_i(x,t)}{\partial t} = \Big\{\mathcal{H}(0,-L),\ \psi_i(x,t)\Big\}, \quad \frac{\partial\bar{\psi}_i(x,t)}{\partial t} = \Big\{\mathcal{H}(0,-L),\ \bar{\psi}_i(x,t)\Big\},$$
$$-L \le x \le 0. \tag{2.18}$$

Indeed considering the Hamiltonian $\mathcal{H}$, we end up with the following set of equations with Dirichlet type boundary conditions

$$i\frac{\partial\psi_i(x,t)}{\partial t} = -\frac{\partial^2\psi_i(x,t)}{\partial^2 x} + 2\kappa\sum_{j=1}^{\mathcal{N}-1}|\psi_j(x,t)|^2\psi_i(x,t)$$
$$\psi_i(0) = \psi_i(-L) = 0 \qquad i \in \{1,\ldots,\mathcal{N}-1\}. \tag{2.19}$$

For a detailed and quite exhaustive analysis of the various integrable boundary conditions of the NLS model see [33].

As mentioned our ultimate goal here is to derive the boundary Lax pair, in particular the $\mathbb{V}$ operator. Hereafter we shall focus on the SP case with the simplest boundary conditions

[1] The Poisson structure for the generalized NLS model is defined as:

$$\Big\{A,\ B\Big\} = i\sum_i\int_{-L}^{L} dx\Big(\frac{\delta A}{\delta\psi_i(x)}\ \frac{\delta B}{\delta\bar{\psi}_i(x)} - \frac{\delta A}{\delta\bar{\psi}_i(x)}\ \frac{\delta B}{\delta\psi_i(x)}\Big) \tag{2.15}$$

i.e. $K^{\pm} = \mathbb{I}$. For any $gl_{\mathcal{N}}$ r-matrix we may expand (2.12), taking also into account (1.22), in powers of λ^{-1} (we refer the interested reader to [33, 15] for technical details) and we obtain $\mathbb{V}^{(3)}(x,\lambda)$ –the bulk part– coincides with $\mathbb{V}$ defined in (2.13), (2.14), and for the boundary points $x_b \in \{0,\ -L\}$ in particular:

$$\begin{aligned}\mathbb{V}^{(3)}(x_b,\lambda) &= -\frac{\lambda^2}{2i}\Big(\sum_{i=1}^{\mathcal{N}-1} E_{ii} - E_{\mathcal{N}\mathcal{N}}\Big) + i\sum_{i,j=1}^{\mathcal{N}-1}\bar{\psi}_i(x_b)\psi_j(x_b)E_{ij} \\ &\quad -i\sum_{i,j=1}^{\mathcal{N}-1}\Big(\bar{\psi}_i'(x_b)E_{i\mathcal{N}} - \psi_i'(x_b)E_{\mathcal{N}i}\Big). \qquad (2.20)\end{aligned}$$

We may alternatively rewrite the latter formula as:

$$\mathbb{V}^{(3)}(x_b,\lambda) = \mathbb{V}(x_b,\lambda) + i\sum_{i=1}^{\mathcal{N}-1}|\psi_i(x_b)|^2 E_{\mathcal{N}\mathcal{N}} + \lambda\sum_{i=1}^{\mathcal{N}-1}(\bar{\psi}_i(x_b)E_{i\mathcal{N}} + \psi_i(x_b)E_{\mathcal{N}i}). \qquad (2.21)$$

The last two terms additional to $\mathbb{V}$ (2.13), (2.14) are due to the non-trivial boundary conditions; of course more complicated boundary conditions would lead to more intricate modifications of the Lax operator $\mathbb{V}$.

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Generalised integrable Hubbard models

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Abstract

We construct the XX and Hubbard-like models based on unitary superalgebras $gl(N|M)$ generalizing Shastry's and Maassarani's approach.
We introduce the R-matrix of the $gl(N|M)$ XX-type model; the one of the Hubbard-like model is defined by "coupling" two independent XX models. In both cases, we show that the R-matrices satisfy the Yang-Baxter equation. We derive the corresponding local Hamiltonian in the transfer matrix formalism and we determine its symmetries.
A perturbative calculation "à la Klein and Seitz" is performed. Some explicit examples are worked out. We give a description of the two-particle scattering.

1 Introduction

The Hubbard model was introduced in order to study strongly correlated electrons [1, 2] and, since then, it has been widely studied, essentially due to its connection with condensed matter physics. It has been used to describe the Mott metal-insulator transition [3, 4], high T_c superconductivity [5, 6], band magnetism [7] and chemical properties of aromatic molecules [8]. The literature on the Hubbard model being rather large, we do not aim at being exhaustive and rather refer to the books [9, 10] and references therein. Exact results have been mostly obtained in the case of the one-dimensional model, which enters the framework of our study. In particular, the 1D model eigenvalues have been obtained by means of the coordinate Bethe Ansatz in the celebrated paper by Lieb and Wu [11].

One of the main motivations for the present study of the Hubbard model and its generalisations is the fact that it has recently appeared in the context of $N = 4$ super Yang-Mills theory. Indeed, it was noticed in [12] that the Hubbard model at half-filling, when treated perturbatively in the coupling, reproduces the long-ranged integrable spin chain of [13] as an effective theory. It thus provides a localisation of the long-ranged spin chain model and gives a potential solution to the problem of describing interactions which are longer than the length of the spin chain. The Hamiltonian of this chain was conjectured in [13] to be an all-order description of the dilatation operator of $N = 4$ super Yang-Mills in the $su(2)$ subsector. That is, the energies of the spin chain are conjectured to be the anomalous dimensions of the gauge theory operators in this subsector. In relation to this, an interesting approach to the Hubbard model is given in [14] that leads to the evaluation of energies for the antiferromagnetic state and allows one to control the order of the limits of large coupling and large length of the operators/large angular momentum.

There may be the possibility that some integrable extension of the Hubbard model could be put in relation to other subsectors of the $N = 4$ super Yang-Mills theory. Here we will discuss a general approach to constructing a number of supersymmetric Hubbard models. Each of these models can be treated perturbatively and thus gives rise to an integrable long-ranged spin chain as an effective theory.

Other supersymmetric generalisations of the Hubbard model have been constructed, see e.g. [15, 16]. These approaches mainly concern high T_c superconductivity models and their relation with the $t - J$ model. They essentially use the $gl(1|2)$ or $gl(2|2)$ superalgebras, which appear as the symmetry algebras of the Hamiltonian of the model. Our approach however is different and is based on the QISM framework. It ensures the integrability of the model and allows one to obtain local Hubbard-like Hamiltonians for general $gl(N|M)$ superalgebras. They can be interpreted in terms of 'electrons' after a Jordan–Wigner transformation.

In this review paper we revisit and slightly extend the results of [17], our goal here being not to reproduce the calculations but to focus on the main ideas of our approach.

The plan of the paper is as follows. Section 2 is devoted to sketch a number of facts for

the ordinary Hubbard model. In section 3, we define universal XX models. We introduce the corresponding Hamiltonians and determine the symmetries of the model. In section 4, we summarise the construction of the associated Hubbard-like model, in the Shastry and Maassarani approach. From the transfer matrix we obtain the Hamiltonian; we also discuss the symmetries. In section 5 we perform a second order perturbative computation *à la* Klein and Seitz [18]. Then, we define the Jordan-Wigner transformation, section 6, used in section 7 to give some examples where we write explicitly the Hamiltonians in the $gl(2|2)$, $gl(4)$ and $gl(4|4)$ cases. We finish in section 8 with a study of two-particle interactions.

2 Hubbard model

The one-dimensional Hubbard model introduced by [1, 2] describes hopping electrons on a lattice, with an ultralocal repulsive potential that implements a screened Coulomb repulsion, with $U > 0$. The Hamiltonian is given by

$$H = -t \sum_{i=1}^{L} \sum_{\rho=\uparrow,\downarrow} \left(e^{i\phi} c^{\dagger}_{\rho,i} c_{\rho,i+1} + e^{-i\phi} c^{\dagger}_{\rho,i+1} c_{\rho,i} \right) + U \sum_{i=1}^{L} \left(1 - 2n_{\uparrow,i}\right)\left(1 - 2n_{\downarrow,i}\right) \tag{2.1}$$

We will always use periodic boundary conditions.
In $\mathcal{N} = 4$-SYM theory this model was first observed in [12], where a magnetic flux ϕ of Aharonov-Bohm type was included. In that paper, the potential term was written in a slightly different but equivalent form. The relation between couplings was identified and the system was taken at half-filling; for our needs we just observe that the ratio t/U corresponds to the coupling g

$$\frac{t}{U} = \frac{g}{\sqrt{2}}\,. \tag{2.2}$$

We observe that the Hamiltonian is Hermitian if $\phi \in \mathbb{R}$. In the following we will work with this flux equal to zero.
The underling algebraic structure leads us to superalgebras: on each site i the fermionic structure

$$\{c_{\rho,i}, c^{\dagger}_{\rho',j}\} = \delta_{\rho,\rho'}\delta_{i,j} \qquad \{c_{\rho,i}, c_{\rho',j}\} = \{c^{\dagger}_{\rho,i}, c^{\dagger}_{\rho',j}\} = 0 \tag{2.3}$$

is a realisation of the super-Lie algebra $gl(1|1) \oplus gl(1|1)$. The full model algebra is obtained by L-times the tensor product of the one site structure. We can easily represent the fermionic structure by a graded tensor product of Pauli matrices, written here with the standard notation for basis matrices $E_{\alpha\beta}$ to emphasise the grading:

$$E_{12;\rho,i} = c_{\rho,i}\,, \quad E_{21;\rho,i} = c^{\dagger}_{\rho,i}\,, \quad E_{22;\rho,i} = n_{\rho,i} = c^{\dagger}_{\rho,i} c_{\rho,i}\,, \quad E_{11;\rho,i} = 1 - n_{\rho,i} = c_{\rho,i} c^{\dagger}_{\rho,i} \tag{2.4}$$

$$E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

When it occurs, the second pair of labels ρ, i indicates the spin polarisation ρ and the site i. The matrices $E_{12}\,, E_{21}$ are taken of fermionic character (they satisfy anticommutation relations whatever their spin and space labels are) and $E_{11}\,, E_{22}$ are taken of bosonic character (they always enter commutation relations whatever their spin and space labels are). The relation (2.4) is a graded Jordan-Wigner transformation[1] and respects periodic boundary conditions[2]. We now rewrite the Hamiltonian in the spin chain language

$$\begin{aligned} H &= -t \sum_{i=1}^{L} \sum_{\rho=\uparrow,\downarrow} \left(E_{21;\rho,i}\ E_{12;\rho,i+1} + E_{21;\rho,i+1}\ E_{12;\rho,i} \right) \\ &+ U \sum_{i=1}^{L} \left(E_{11;\uparrow,i} - E_{22;\uparrow,i} \right) \left(E_{11;\downarrow,i} - E_{22;\downarrow,i} \right) \end{aligned} \tag{2.5}$$

and we split it into the sum of the two polarisations

$$H = H_{\mathrm{XX}}^{\uparrow} + H_{\mathrm{XX}}^{\downarrow} + U \sum_{i=1}^{L} \left(E_{11;\uparrow,i} - E_{22;\uparrow,i} \right) \left(E_{11;\downarrow,i} - E_{22;\downarrow,i} \right) \ ; \tag{2.6}$$

$$H_{\mathrm{XX}}^{\rho} = -t \sum_{i=1}^{L} \left(E_{21;\rho,i}\ E_{12;\rho,i+1} + E_{21;\rho,i+1}\ E_{12;\rho,i} \right) \ .$$

Taking one polarisation of the kinetic term we easily see that

$$E_{21;\rho,i}\ E_{12;\rho,i+1} + E_{21;\rho,i+1}\ E_{12;\rho,i} = \frac{1}{2} \Big[E_{x;\rho,i}\ E_{x;\rho,i+1} + E_{y;\rho,i}\ E_{y;\rho,i+1} \Big] \tag{2.7}$$

$$E_{x;\rho,i} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\rho,i} \ , \quad E_{y;\rho,i} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{\rho,i}$$

namely we see the appearance of a (graded) XX spin chain Hamiltonian[3] (or better two XX spin chains, one for each polarisation) within the Hubbard model.

It turns out that the breaking of (2.6) into the Hamiltonian of two XX models plus a potential will allow us to generalise this model to higher algebraic structures by maintaining its main property: integrability[4].

A first hint of integrability of the Hubbard model came from the coordinate Bethe Ansatz solution obtained by Lieb and Wu [11] but a full understanding of it by the existence of an infinite set of commuting charges came much later. A complete set of eigenstates was constructed in [19] using the $SO(4)$ symmetry of the 1D Hubbard Hamiltonian.

[1] The ordinary Jordan-Wigner transformation is $c^{\dagger}_{\uparrow,i} = \sigma^{-}_{\uparrow,i} \prod_{k>i} \sigma^{z}_{\uparrow,i}$ for the up polarisation; an additional term occurs for the down polarisation.

[2] The standard one violates periodicity.

[3] At this point it should be clear that the difference between graded and non graded cases appears when boundary effects are observed; the thermodynamic limit usually ignores such terms, being sensitive to bulk contributions only.

[4] The flux ϕ does not affect integrability properties.

Within the framework of the quantum inverse scattering method, an R-matrix was first constructed by Shastry [20, 21] and Olmedilla et al. [22], by coupling (decorated) R-matrices of two independent XX models, through a term depending on the coupling constant U of the Hubbard potential. The proof of the Yang–Baxter relation for the R-matrix was given by Shiroishi and Wadati [23]. With a standard construction, a transfer matrix can be constructed by taking the trace of a tensor product of R-matrices. The Yang-Baxter equation guarantees that the transfer matrix is the generating functional of an infinite set of commuting charges. One of these charges is the Hamiltonian (2.6) itself. The construction of the R-matrix was then generalised in the $gl(N)$ case by Maassarani et al., first for the XX model [24] and then for the $gl(N)$ Hubbard model [25, 26]. Within the QISM framework, the eigenvalues of the transfer matrix of the Hubbard model were found using the algebraic Bethe Ansatz together with certain analytic properties in [27, 28, 29].

3 Universal XX models

We generalize the construction given in [24, 29, 17] to the case of an arbitrary representation space $\mathcal{V}$, possibly infinite dimensional. We will use the standard auxiliary space notation, i.e. to any operator $A \in \mathrm{End}(\mathcal{V})$, we associate the operator $A_1 = A \otimes \mathbb{I}$ and $A_2 = \mathbb{I} \otimes A$ in $\mathrm{End}(\mathcal{V}) \otimes \mathrm{End}(\mathcal{V})$. More generally, when considering expressions in $\mathrm{End}(\mathcal{V})^{\otimes k}$, A_j, $j = 1, \ldots, k$ will act trivially in all spaces $\mathrm{End}(\mathcal{V})$, but the j^{th} one.
To deal with superalgebras, we will also need a $\mathbb{Z}_2$ grading $[.]$ on $\mathcal{V}$, such that $[v] = 0$ will be associated to bosonic states $v \in \mathcal{V}$ and $[v] = 1$ to fermionic ones.
We will also assume the existence of a (super-)trace operator, defined on a subset of $\mathrm{End}(\mathcal{V})$ and obeying cyclicity. When $\mathcal{V}$ is finite dimensional, $\dim(\mathcal{V}) = K$, $\mathrm{End}(\mathcal{V})$ is a matrix algebra or super-algebra so that the trace operator is the usual trace or supertrace of $K \times K$ matrices. When $\mathcal{V}$ is infinite dimensional, the definition of a trace operator is more delicate and we will just assume that it exists and is cyclic, for the operators we use.
The construction of a universal XX model is mainly based on general properties of a given projector and a permutation. Our main projectors are chosen in $\mathrm{End}(\mathcal{V})$ as being

$$\pi : \mathcal{V} \to \mathcal{W} \quad , \quad \widetilde{\pi} = \mathbb{I} - \pi : \mathcal{V} \to \widetilde{\mathcal{W}} \quad \text{with} \quad \mathcal{V} = \mathcal{W} \oplus \widetilde{\mathcal{W}} \tag{3.1}$$

In the tensor product of two vector spaces we take the (possibly graded) permutation

$$P_{12} : \begin{cases} \mathcal{V} \otimes \mathcal{V} & \to \ \mathcal{V} \otimes \mathcal{V} \\ v_1 \otimes v_2 & \to \ (-1)^{[v_1][v_2]}\, v_2 \otimes v_1 \end{cases} \tag{3.2}$$

For example, in the superalgebra $gl(N|M)$ a possible choice is

$$\pi = \sum_{j \neq N, N+M} E_{jj} \quad , \quad \widetilde{\pi} = \mathbb{I} - \pi = E_{NN} + E_{N+M,N+M} \tag{3.3}$$

3.1 R-matrix

From the previous operators, one can construct an R-matrix acting on $\mathcal{V} \otimes \mathcal{V}$

$$R_{12}(\lambda) = \Sigma_{12}\, P_{12} + \Sigma_{12}\, \sin\lambda + (\mathbb{I}\otimes\mathbb{I} - \Sigma_{12})\, P_{12}\, \cos\lambda \tag{3.4}$$

where Σ_{12} is built on the projection operators:

$$\Sigma_{12} \quad = \quad \pi_1\, \widetilde{\pi}_2 + \widetilde{\pi}_1\, \pi_2 \tag{3.5}$$

It is easy to show that Σ_{12} is also a projector in $\mathcal{V} \otimes \mathcal{V}$: $(\Sigma_{12})^2 = \Sigma_{12}$.
Let us introduce the operator C:

$$C = \pi - \widetilde{\pi}\,. \tag{3.6}$$

It obeys $C^2 = \mathbb{I}$ and is related to the R-matrix through the equalities

$$\Sigma_{12} = \frac{1}{2}(1 - C_1 C_2) \quad \text{and} \quad \mathbb{I}\otimes\mathbb{I} - \Sigma_{12} = \frac{1}{2}(1 + C_1 C_2) \tag{3.7}$$

In [17] we gave proof of a number of useful properties of the R-matrix. Essentially the same proofs work also for the slightly more general formulation given here. The main properties are unitarity, regularity, and Yang–Baxter equation (YBE), that guarantees us that we have an integrable model

$$R_{12}(\lambda_{12})\, R_{13}(\lambda_{13})\, R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})\, R_{13}(\lambda_{13})\, R_{12}(\lambda_{12})$$
$$\text{where} \quad \lambda_{ij} = \lambda_i - \lambda_j. \tag{3.8}$$

3.2 Monodromy and transfer matrix

With a very standard construction, from the R-matrix one constructs the (L sites) monodromy matrix

$$\mathcal{L}_{0<1\ldots L>}(\lambda) = R_{01}(\lambda)\, R_{02}(\lambda) \cdots R_{0L}(\lambda) \tag{3.9}$$

where we tensor product one R-matrix for each site of the theory. It obeys the relation

$$R_{00'}(\lambda-\mu)\, \mathcal{L}_{0<1\ldots L>}(\lambda)\, \mathcal{L}_{0'<1\ldots L>}(\mu) = \mathcal{L}_{0'<1\ldots L>}(\mu)\, \mathcal{L}_{0<1\ldots L>}(\lambda)\, R_{00'}(\lambda-\mu)\,. \tag{3.10}$$

where 0 and $0'$ are two copies of the auxiliary space. This relation allows us to construct an (L sites) integrable XX spin chain through the transfer matrix

$$t_{1\ldots L}(\lambda) = \mathrm{str}_0 \mathcal{L}_{0<1\ldots L>}(\lambda) = \mathrm{str}_0 \Big(R_{01}(\lambda)\, R_{02}(\lambda) \cdots R_{0L}(\lambda) \Big)\,. \tag{3.11}$$

where, if $\mathcal{V}$ has infinite dimension, we assume the existence of the supertrace for the previous operator. Indeed, the relation (3.10) implies that the transfer matrices for different values of the spectral parameter commute

$$[t_{1\ldots L}(\lambda)\,,\, t_{1\ldots L}(\mu)] = 0\,. \tag{3.12}$$

Here the cyclicity of the supertrace has been used.
Since the R-matrix is regular (namely it is a permutation in $\lambda = 0$), logarithmic derivatives in $\lambda = 0$ give local operators. We choose the first one as XX-Hamiltonian

$$\begin{aligned} H &= t_{1\ldots L}(0)^{-1}\,\frac{dt_{1\ldots L}}{d\lambda}(0) \\ &= \sum_{j=1}^{L} H_{j,j+1} \quad \text{with} \quad H_{j,j+1} = P_{j,j+1}\,\Sigma_{j,j+1} \end{aligned} \tag{3.13}$$

where we have used periodic boundary conditions, i.e. identified the site $L+1$ with the first site. After (3.12), we see that any expansion of the transfer matrix in the spectral parameters λ, μ generates a set of commuting operators. In particular they commute with the Hamiltonian (3.13), so are conserved charges. This formally proves that the system is integrable.
Explicitly, the two sites Hamiltonian corresponding to the example (3.3) reads

$$H_{j,j+1} = \sum_{i\neq N,N+M} \sum_{j=N,N+M} \Big((-1)^{[j]}\, E_{ij} \otimes E_{ji} + (-1)^{[i]}\, E_{ji} \otimes E_{ij} \Big)\,. \tag{3.14}$$

3.3 Symmetries of the universal XX models

The choice of the fundamental projectors in (3.1) directly fixes the symmetries of the model.
One easily shows that an operator $\mathbb{M} \in \mathrm{End}(\mathcal{W}) \oplus \mathrm{End}(\widetilde{\mathcal{W}})$ commutes with the projectors (3.1); then it commutes with the R-matrix in the following sense

$$(\mathbb{M}_1 + \mathbb{M}_2)\, R_{12}(\lambda) = R_{12}(\lambda)\, (\mathbb{M}_1 + \mathbb{M}_2)\,. \tag{3.15}$$

Commutation does not hold if the operator mixes the two subspaces.
As a consequence of (3.15), the transfer matrix also has a symmetry (super)algebra

$$\mathcal{S} = \mathrm{End}(\mathcal{W}) \oplus \mathrm{End}(\widetilde{\mathcal{W}}) \tag{3.16}$$

with generators given by

$$\mathbb{M}_{<1\ldots L>} = \mathbb{M}_1 + \mathbb{M}_2 + \ldots + \mathbb{M}_L\,. \tag{3.17}$$

The same is true for any Hamiltonian H built from the transfer matrix so (3.17) commute with the Hamiltonian[5].
We can reverse this construction: we require a symmetry algebra $\mathcal{S}$ from which we construct the subspaces $\mathcal{W}$ and $\widetilde{\mathcal{W}}$. This uniquely fixes the fundamental projector π that immediately leads to obtain the XX model possessing $\mathcal{S}$ as symmetry.

[5] In principle, this construction cannot exclude the existence of operators that commute with the Hamiltonian but not with the R-matrix. In that case, these additional symmetries would have the strange feature of not being symmetries of at least one conserved charge (by reconstructing the R-matrix from an expansion).

The example (3.3) admits $\mathcal{S} = gl(N-1|M-1) \oplus gl(1|1)$ as symmetry superalgebra whose generators $\mathbb{M}$ have the form

$$\begin{array}{lll} E_{jk}\ ,\ j,k \neq N, N+M & \text{for} & gl(N-1|M-1) \\ E_{jk}\ ,\ j,k = N, N+M & \text{for} & gl(1|1). \end{array} \tag{3.18}$$

4 Universal Hubbard models

Starting with universal XX models, one can build universal Hubbard models, in the same way it has been done for usual and super Hubbard models [10, 17]. The logic will be to start from two possibly different universal XX models of section 3 and "glue" them with the generalisation of the construction given in section 2.

4.1 R-matrix

We start with the R-matrices of two universal XX models, $R^{\uparrow}_{12}(\lambda)$ and $R^{\downarrow}_{12}(\lambda)$, living in two different sets of spaces that we label by $\uparrow$ and $\downarrow$. Let us stress that the two XX models can be based on two different (graded) vector spaces $\mathcal{V}^{\uparrow}$ and $\mathcal{V}^{\downarrow}$, with two different projectors $\pi^{\uparrow}$ and $\pi^{\downarrow}$.
The Hubbard model is constructed from the coupling of these two XX models. Its R-matrix has two spectral parameters $\lambda_1\,, \lambda_2$ and reads:

$$R_{12}(\lambda_1,\lambda_2) = R^{\uparrow}_{12}(\lambda_{12})\, R^{\downarrow}_{12}(\lambda_{12}) + \frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})}\, \tanh(h'_{12})\, R^{\uparrow}_{12}(\lambda'_{12})\, C^{\uparrow}_1\, R^{\downarrow}_{12}(\lambda'_{12})\, C^{\downarrow}_1 \tag{4.1}$$

where $\lambda_{12} = \lambda_1 - \lambda_2$ and $\lambda'_{12} = \lambda_1 + \lambda_2$. Moreover, $h'_{12} = h(\lambda_1) + h(\lambda_2)$ and the choice of the function $h(\lambda)$ is fixed by the proof of the Yang-Baxter equation. Indeed, when the function $h(\lambda)$ is given by $\sinh(2h) = U\, \sin(2\lambda)$ for some free parameter U, the R-matrix (4.1) obeys YBE:

$$R_{12}(\lambda_1,\lambda_2)\, R_{13}(\lambda_1,\lambda_3)\, R_{23}(\lambda_2,\lambda_3) \;\; = \;\; R_{23}(\lambda_2,\lambda_3)\, R_{13}(\lambda_1,\lambda_3)\, R_{12}(\lambda_1,\lambda_2)\,. \tag{4.2}$$

As remarked in [17] the proof relies only on some intermediate properties that are not affected by the choice of the fundamental projectors (3.1). The proof follows the steps of the original proof by Shiroishi [30], in the same way it has been done for algebras in [10]. Moreover, it was already noticed in [10] that one can couple two XX models based on different $gl(M)$ algebras: this naturally extends to general (graded) vector spaces $\mathcal{V}$.
The given R-matrix is regular but non symmetric. It satisfies unitarity (we correct here an inconsequential typo that occurred in eq. 3.4 of [17]) in the form

$$R_{12}(\lambda_1,\lambda_2)\, R_{21}(\lambda_2,\lambda_1) = \left(\cos^4(\lambda_{12}) - \left(\frac{\sin(\lambda_{12})}{\sin(\lambda'_{12})}\, \tanh(h'_{12}) \cos^2(\lambda'_{12}) \right)^2 \right)\, \mathbb{I}_1 \otimes \mathbb{I}_2 \tag{4.3}$$

where $\mathbb{I}_i = \mathbb{I}^{\uparrow} \otimes \mathbb{I}^{\downarrow}$.

4.2 Monodromy and transfer matrix

We use the construction given in section 3.2 to obtain the Hamiltonian of the system, starting with the 'reduced' monodromy matrix

$$\mathcal{L}_{0<1\ldots L>}(\lambda) = R_{01}(\lambda,\mu)\ldots R_{0L}(\lambda,\mu)\Big|_{\mu=0}\,. \tag{4.4}$$

Any other choice for μ is possible but, at least in view of obtaining a local Hamiltonian, they do not give new information. Provided the supertrace exists, the transfer matrix is given by

$$t_{1\ldots L>}(\lambda) = \mathrm{str}_0\mathcal{L}_{0<1\ldots L>}(\lambda)$$

Then, one gets

$$[H, t(\lambda)] = 0\ , \quad \forall\lambda\ , \quad \text{for} \quad H = H(0) = t(0)^{-1}\,t'(0) \tag{4.5}$$

The 'reduced' R-matrices that enter in (4.4) take a particularly simple factorised form

$$R_{12}(\lambda,0) = \ R^{\uparrow}_{12}(\lambda)\,R^{\downarrow}_{12}(\lambda)\,I^{\uparrow\downarrow}_{1}(h) \tag{4.6}$$

where

$$I^{\uparrow\downarrow}_{1}(h) = \mathbb{I}\otimes\mathbb{I} + \tanh(\frac{h}{2})\,C^{\uparrow}_{1}\,C^{\downarrow}_{1} \tag{4.7}$$

and we arrive at a Hubbard-like Hamiltonian

$$H = \sum_{j=1}^{L} H_{j,j+1} = \sum_{j=1}^{L}\left[\Sigma^{\uparrow}_{j,j+1}\,P^{\uparrow}_{j,j+1} + \Sigma^{\downarrow}_{j,j+1}\,P^{\downarrow}_{j,j+1} + U\,C^{\uparrow}_{j}\,C^{\downarrow}_{j}\right] \tag{4.8}$$

where we have used periodic boundary conditions.

4.3 Symmetries

The transfer matrix of generalized Hubbard models admits as symmetry (super)algebra the direct sum of the symmetry algebras of the XX components

$$\mathcal{S} = \mathrm{End}(\mathcal{W}^{\uparrow}) \oplus \mathrm{End}(\widetilde{\mathcal{W}}^{\uparrow}) \oplus \mathrm{End}(\mathcal{W}^{\downarrow}) \oplus \mathrm{End}(\widetilde{\mathcal{W}}^{\downarrow})\,. \tag{4.9}$$

To prove this symmetry, it is useful to remark that (3.15) can be now specialised to the cases up and down. Moreover, the up R-matrix commutes with the down generators and viceversa. We also check that

$$\mathbb{M}\,C^{\sigma} = C^{\sigma}\,\mathbb{M}\,, \qquad \sigma = \uparrow,\downarrow \tag{4.10}$$

where

$$\mathbb{M} = \mathbb{M}^{\uparrow} + \mathbb{M}^{\downarrow} \qquad \text{and} \qquad \mathbb{M}^{\sigma} \in \mathrm{End}(\mathcal{W}^{\sigma}) \oplus \mathrm{End}(\widetilde{\mathcal{W}}^{\sigma})\,. \tag{4.11}$$

Thus, one gets

$$[R_{12}(\lambda,0)\,,\ \mathbb{M}_1^{\uparrow}+\mathbb{M}_2^{\uparrow}]=0=[R_{12}(\lambda,0)\,,\ \mathbb{M}_1^{\downarrow}+\mathbb{M}_2^{\downarrow}] \tag{4.12}$$

that can be easily extended to hold for the monodromy and transfer matrices and for the Hamiltonian; the generators of the symmetry have the form

$$\mathbb{M}^{\uparrow}=\sum_{j=1}^{L}\mathbb{M}_j^{\uparrow}\quad\text{and}\quad\mathbb{M}^{\downarrow}=\sum_{j=1}^{L}\mathbb{M}_j^{\downarrow} \tag{4.13}$$

The ordinary Hubbard case and all the cases where $\mathcal{V}^{\sigma}$ is two dimensional are special because, in addition to the list of generators contained in (4.9), there are new generators given by

$$V^{\pm}=\sigma_{\uparrow}^{\pm}\otimes\sigma_{\downarrow}^{\pm}\,,\qquad W^{\pm}=\sigma_{\uparrow}^{\pm}\otimes\sigma_{\downarrow}^{\pm}\,. \tag{4.14}$$

To be precise, $V^{\pm}$ commutes with the Hamiltonian if L is even while $W^{\pm}$ commutes in all cases. These additional generators do not commute with H if $\dim(\mathcal{V}^{\sigma})>2$; they are responsible for the $SU(2)\times SU(2)$ symmetry of the even Hubbard model.

5 Perturbative expansion of the Hubbard-like Hamiltonian

We expand the Hamiltonian (4.8) in the inverse coupling $\frac{1}{U}$; according to (2.2), this corresponds to the small coupling expansion of the gauge theory. Indeed, precisely that expansion has been used in [12] to match the $SU(2)$ dilatation operator with the effective Hamiltonian of the Hubbard model. The system was taken at half-filling to guarantee the required spin chain behaviour. With the form of the potential used in (2.1) the half-filled condition is enforced by the $U\to\infty$ requirement itself.

We take the set of all Hamiltonian eigenstates whose leading energy term is $-LU$, for large positive U. These states are selected by the following projector

$$\Pi_0=\prod_j\left(\pi_j^{\uparrow}-\pi_j^{\downarrow}\right)^2=\prod_j\left(\widetilde{\pi}_j^{\uparrow}-\widetilde{\pi}_j^{\downarrow}\right)^2=\Pi_0^2\,. \tag{5.1}$$

that projects on the subspace where, on each site, one and only one among $\widetilde{\pi}_j^{\uparrow}$, $\widetilde{\pi}_j^{\downarrow}$ has nonzero action.

We follow the method introduced by Klein and Seitz [18] to obtain an effective Hamiltonian for the corrections to the leading energy $-LU$:

$$H_{\text{eff}}=\frac{1}{U}H_{\text{eff}}^{(2)}+\frac{1}{U^3}H_{\text{eff}}^{(4)}+\dots \tag{5.2}$$

For $L>2$ the second order effective Hamiltonian is

$$H_{\text{eff}}^{(2)}=\sum_j H_{\text{eff}\,j,j+1}^{(2)}=2\sum_j\left(1+P_{j,j+1}^{\uparrow}P_{j,j+1}^{\downarrow}\right)\left(\pi_j^{\uparrow}\,\widetilde{\pi}_j^{\downarrow}\,\widetilde{\pi}_{j+1}^{\uparrow}\,\pi_{j+1}^{\downarrow}+\widetilde{\pi}_j^{\uparrow}\,\pi_j^{\downarrow}\,\pi_{j+1}^{\uparrow}\,\widetilde{\pi}_{j+1}^{\downarrow}\right) \tag{5.3}$$

For the ordinary Hubbard model this expression can be given in terms of Pauli matrices

$$H^{(2)}_{\text{eff}} = \sum_{i=1}^{L} (1 - \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1}) \tag{5.4}$$

where the fermionic oscillators of (2.1) have disappeared and only spin degrees of freedom are left ($\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$).
The structure of the two-sites Hamiltonian $H^{(2)}_{\text{eff}\ i,i+1}$ can be obtained explicitly. In matricial form, it has diagonal block structure, with blocks given by one of the two matrices

$$B_- = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{or} \qquad B_+ = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{5.5}$$

all other entries being zero. The number of appearances of each block depends on the actual model under examination.

6 Jordan-Wigner transformation

Let us consider p sets of fermionic oscillators $c^{(q)}_i, c^{(q)\dagger}_i$ ($i = 1, \ldots, L$ and $q = 1, \ldots, p$) that satisfy the usual anticommutation relations

$$\{c^{(q)}_i, c^{(q')\dagger}_j\} = \delta_{ij}\, \delta_{qq'} \qquad \{c^{(q)}_i, c^{(q')}_j\} = \{c^{(q)\dagger}_i, c^{(q')\dagger}_j\} = 0 \tag{6.1}$$

One defines the following matrix (where $n^{(q)}_i = c^{(q)\dagger}_i c^{(q)}_i$ is the usual number operator)

$$X^{(q)}_i = \begin{pmatrix} 1 - n^{(q)}_i & c^{(q)}_i \\ c^{(q)\dagger}_i & n^{(q)}_i \end{pmatrix} \tag{6.2}$$

The entries $X^{(q)}_{i;\alpha\beta}$ of this matrix have a natural gradation given by $[\alpha] + [\beta]$ where $[1] = 1$ and $[2] = 0$.
In the $gl(2^{p-1}|2^{p-1})$ case, one defines at each site i the generators

$$X_{i;\alpha_1 \ldots \alpha_p, \alpha'_1 \ldots \alpha'_p} = (-1)^s\, X^{(1)}_{i;\alpha_1 \alpha'_1} \ \ldots\ X^{(p)}_{i;\alpha_p \alpha'_p} \ \text{ where } \ s = \sum_{a=2}^{p} [\alpha_a] \Big(\sum_{b=1}^{a-1} \big([\alpha_b] + [\alpha'_b] \big) \Big) \tag{6.3}$$

It is easy to verify the following properties:

$$\big(X_{i;\alpha_1 \ldots \alpha_p, \alpha'_1 \ldots \alpha'_p} \big)^\dagger = X_{i;\alpha'_1 \ldots \alpha'_p, \alpha_1 \ldots \alpha_p} \tag{6.4}$$

$$X_{i;\alpha_1 \ldots \alpha_p, \alpha'_1 \ldots \alpha'_p}\ X_{i;\beta_1 \ldots \beta_p, \beta'_1 \ldots \beta'_p} = \delta_{\alpha'_1 \beta_1} \ldots \delta_{\alpha'_p \beta_p}\, X_{i;\alpha_1 \ldots \alpha_p, \beta'_1 \ldots \beta'_p} \tag{6.5}$$

$$\sum_{\alpha_1, \ldots, \alpha_p} X_{i;\alpha_1 \ldots \alpha_p, \alpha_1 \ldots \alpha_p} = 1 \tag{6.6}$$

$$X_{i;\alpha_1 \ldots \alpha_p, \alpha'_1 \ldots \alpha'_p}\ X_{j;\beta_1 \ldots \beta_p, \beta'_1 \ldots \beta'_p} = (-1)^g X_{j;\beta_1 \ldots \beta_p, \beta'_1 \ldots \beta'_p}\ X_{i;\alpha_1 \ldots \alpha_p, \alpha'_1 \ldots \alpha'_p} \qquad (i \neq j) \tag{6.7}$$

$$\text{where } \ g = \Big(\sum_{a=1}^{p} \big([\alpha_a] + [\alpha'_a] \big) \Big) \Big(\sum_{b=1}^{p} \big([\beta_b] + [\beta'_b] \big) \Big)$$

This means that the operators $X_{i;\alpha_1\dots\alpha_p,\alpha'_1\dots\alpha'_p}$ built out of fermionic oscillators are actually a realisation of the $gl(2^{p-1}|2^{p-1})$ superalgebra. A generic case $gl(N|M)$ can be understood as contained in the smallest superalgebra for which $N, M < 2^{p-1}$. The unwanted states can be consistently projected out.

7 Examples

It is possible to construct examples of both XX and Hubbard-like Hamiltonians. Clearly, the XX ones are "quasi-free models" because they do not contain external potentials and, if written with fermionic oscillators, they only contain hopping terms. In spite of this, they show curious "screening effects" namely particles that are allowed to move only if particles of other types are present (or absent, depending on the case). We will concentrate on universal Hubbard model examples.
The first example to cite is, of course, the original Hubbard model of section 2, that is described in this formalism as $gl(1|1) \oplus gl(1|1)$ with the choice $\pi^\uparrow = \pi^\downarrow = E_{11}$, $\widetilde{\pi}^\uparrow = \widetilde{\pi}^\downarrow = E_{22}$.

7.1 $gl(2|2) \oplus gl(2|2)$ Hubbard Hamiltonian

This is a more complete example of the models under examination. It precisely implements two copies (up and down) of the example (3.3) with $N = M = 2$. The kinetic term of the Hamiltonian has a factorised form

$$H_{\text{Hub}} = \sum_{i=1}^{L} \Big\{ \sum_{\sigma=\uparrow,\downarrow} \big(c^\dagger_{\sigma,i}c_{\sigma,i+1} + c^\dagger_{\sigma,i+1}c_{\sigma,i}\big)\big(c'^\dagger_{\sigma,i}c'_{\sigma,i+1} + c'^\dagger_{\sigma,i+1}c'_{\sigma,i} + 1 - n'_{\sigma,i} - n'_{\sigma,i+1}\big) + U(1-2n_{\uparrow,i})(1-2n_{\downarrow,i}) \Big\} \tag{7.1}$$

where the factor

$$\mathcal{N}'_{\sigma,i,i+1} = \big(c'^\dagger_{\sigma,i}c'_{\sigma,i+1} + c'^\dagger_{\sigma,i+1}c'_{\sigma,i} + 1 - n'_{\sigma,i} - n'_{\sigma,i+1}\big) \tag{7.2}$$

multiplies an ordinary Hubbard hopping term ; only unprimed particles enter into the potential. There are four types of fermionic particles, respectively generated by $c^\dagger_{\uparrow,i}$, $c^\dagger_{\downarrow,i}$, $c'^\dagger_{\uparrow,i}$, $c'^\dagger_{\downarrow,i}$ so that they define a 16 dimensional vector space on each site. The corresponding numbers of particles are conserved.
The factor $\mathcal{N}'_{\sigma,i,i+1}$ works on a 4×4 space and its eigenvalues are ± 1 with two-fold multiplicity. In particular this means that it cannot vanish, $\mathcal{N}'_{\sigma,i,i+1} \neq 0$. Moreover, if no primed particles are present, $\mathcal{N}'_{\sigma,i,i+1} = 1\,,\ \forall\ \sigma, i$. The same is true if the system is fully filled with primed particles in which case $\mathcal{N}'_{\sigma,i,i+1} = -1$ therefore two of the sectors described by this Hamiltonian are equivalent to the ordinary Hubbard model. A Russian doll structure is appearing: if the projectors are well chosen, a larger model contains the small ones.

If there are primed particles only, the energy vanishes (but not momentum). If the potential is interpreted as a Coulomb repulsion, then unprimed particles only carry electric charge.
The compound objects formed by $c^\dagger_{\sigma,i}\, c'^\dagger_{\sigma,i}$ are rigid: no other term in the Hamiltonian can destroy them. In this sense, we have four types of carriers, with the same charge but different behaviours: two are the elementary objects $c^\dagger_{\sigma,i}$ in two polarisations $\sigma = \uparrow, \downarrow$, two are the compound objects (in two polarisations).
The symmetry, according to (4.9), is $gl(1|1) \oplus gl(1|1) \oplus gl(1|1) \oplus gl(1|1)$.
At second order in $\frac{1}{U}$ the following effective Hamiltonian appears

$$H^{(2)}_{\text{eff}} = -\frac{1}{U}\sum_{i=1}^{L}\left[(\frac{1}{2} - 2S^z_i S^z_{i+1}) - (S^+_i S^-_{i+1} + S^-_i S^+_{i+1})\,\mathcal{N}'_{\uparrow,i,i+1}\,\mathcal{N}'_{\downarrow,i,i+1}\right] \tag{7.3}$$

that looks like a deformation of an XXX model. It has an enhancement of symmetry with respect to (7.1) in the sense that its symmetry is $gl(2|2) \oplus gl(2|2)$.
The two sites action of (7.3) is a 64×64 matrix that can be easily disentangled leading to both the blocks given in (5.5). In summary, it has eigenvalues 0 and 2, 0 with multiplicity 48, 2 with multiplicity 16.

7.2 $gl(4) \oplus gl(4)$ Hubbard Hamiltonian

We consider the model based on $gl(4) \oplus gl(4)$ and take the projectors according to the example (3.3) with $N = 4$, $M = 0$, in two copies (up and down)

$$\begin{aligned} H_{\text{Hub}} \;=\; \sum_{i=1}^{L}\Big\{ \sum_{\sigma=\uparrow,\downarrow} \big(& c^\dagger_{\sigma,i} c_{\sigma,i+1} c'^\dagger_{\sigma,i} c'_{\sigma,i+1} + c^\dagger_{\sigma,i+1} c_{\sigma,i} c'^\dagger_{\sigma,i+1} c'_{\sigma,i} + \\ & + n'_{\sigma,i} n'_{\sigma,i+1}(c^\dagger_{\sigma,i} c_{\sigma,i+1} + c^\dagger_{\sigma,i+1} c_{\sigma,i}) + n_{\sigma,i} n_{\sigma,i+1}(c'^\dagger_{\sigma,i} c'_{\sigma,i+1} + c'^\dagger_{\sigma,i+1} c'_{\sigma,i})\big) + \\ & + U(1 - 2n_{\uparrow,i} n'_{\uparrow,i})(1 - 2n_{\downarrow,i} n'_{\downarrow,i}) \Big\}. \end{aligned} \tag{7.4}$$

This model has the same vector space dimension of the $gl(2|2)$ one (7.1), $\dim(\mathcal{V}) = 16$ but the elementary projectors are different and lead to slightly different interactions. That the two Hamiltonians are different is manifest if one examines the original form (4.8) with the basis matrices $E_{\alpha\beta}$, before the Jordan-Wigner transformation.
Here there is complete symmetry between primed and non-primed particles; the effect of Coulomb repulsion only appears when both primed and unprimed particles are on the same site; if one of these types is alone, no Coulomb interaction is felt. Observe that if $n'_{\sigma,i} = 1$ everywhere and for all polarisations (or else if $n_{\sigma,i} = 1$), we re-obtain the $gl(1|1)$ Hubbard model. The kinetic term also has a strange feature: a particle is allowed to move only if it is accompanied by a particle of the same polarisation (i.e. up with up) but opposite type (i.e. primed with unprimed).

7.3 $gl(4|4) \oplus gl(4|4)$ Hamiltonian

Following the example (3.3), the following Hubbard Hamiltonian is obtained

$$\begin{aligned} H_{Hub}^{gl(4|4)} = \sum_{i=1}^{L} \Big\{ \sum_{\sigma=\uparrow,\downarrow} \big(c^\dagger_{\sigma,i}c_{\sigma,i+1} + c^\dagger_{\sigma,i+1}c_{\sigma,i} + 1 - n_{\sigma,i} - n_{\sigma,i+1}\big)\Big(c'^\dagger_{\sigma,i}c'_{\sigma,i+1}c''^\dagger_{\sigma,i}c''_{\sigma,i+1} \\ + c'^\dagger_{\sigma,i+1}c'_{\sigma,i}c''^\dagger_{\sigma,i+1}c''_{\sigma,i} - n'_{\sigma,i}n'_{\sigma,i+1}(c''^\dagger_{\sigma,i}c''_{\sigma,i+1} + c''^\dagger_{\sigma,i+1}c''_{\sigma,i}) \\ - n''_{\sigma,i}n''_{\sigma,i+1}(c'^\dagger_{\sigma,i}c'_{\sigma,i+1} + c'^\dagger_{\sigma,i+1}c'_{\sigma,i})\Big) + U(1-2n'_{\uparrow,i}n''_{\uparrow,i})(1-2n'_{\downarrow,i}n''_{\downarrow,i}) \Big\} \quad (7.5) \end{aligned}$$

Here there are six types of fermions, $c^\dagger_\sigma, c'^\dagger_\sigma, c''^\dagger_\sigma$ so the local (one site) space of states is 64×64.

One observes that this Hamiltonian exhibits a 'Russian doll' structure. Indeed, there are four sectors in the space of states where the $gl(4|4)$ Hamiltonian reduces to the $gl(2|2)$ one, that also reduces to the $gl(1|1)$ one. For example, one sector is given by $n''_{\uparrow,i} = n'_{\downarrow,i} = 1$ for $1 \leq i \leq L$.

8 Two-particles interaction

We sketch here the preliminary effects that we observed studying two particles in interaction. It is convenient to consider a reference state as being a particle "vacuum" (pseudovacuum)

$$\Omega = (e_1^\uparrow \underset{1}{\otimes} e_1^\downarrow) \otimes (e_1^\uparrow \underset{2}{\otimes} e_1^\downarrow) \otimes \ldots (e_1^\uparrow \underset{L}{\otimes} e_1^\downarrow) \quad (8.1)$$

where index under the tensor product symbol labels the lattice sites. All other states are considered excitations of this pseudovacuum. Then particles are distinguished by the type, according to the subspaces $\mathcal{W}$, $\widetilde{\mathcal{W}}$:

$$\begin{aligned} a, b, \ldots &\in \mathcal{W} \\ \tilde{a}, \tilde{b}, \ldots &\in \widetilde{\mathcal{W}} \end{aligned}$$

and an upper index $\uparrow, \downarrow$ will be added to distinguish polarisation[6].

Within the universal XX models, all particles satisfy the exclusion principle, namely they cannot appear on the same site. If two particles are both from $\mathcal{W}$ or both from $\widetilde{\mathcal{W}}$, they reflect each other; if they are one from $\mathcal{W}$, one from $\widetilde{\mathcal{W}}$, they traverse each other by tunnel effect.

In the universal Hubbard models, the coupling activates a sort of electrostatic interaction felt by particles of opposite polarisation only. Indeed, the potential term in (4.8) squares to the identity (3.6) so on one site states it has eigenvalues $\pm U$. Which sign occurs is dictated by the membership to $\mathcal{W}$ or $\widetilde{\mathcal{W}}$ according to the rule: with $U > 0$, equal type

[6] As already remarked, notice that particles of different polarisation or different type are not to be understood as conjugated: for example, $a^\uparrow$ and $a^\downarrow$ are different objects

particles $a^{\uparrow}a^{\downarrow}$ or $\tilde{a}^{\uparrow}\tilde{a}^{\downarrow}$ repel each other but different type particles $a^{\uparrow}\tilde{a}^{\downarrow}$ or $\tilde{a}^{\uparrow}a^{\downarrow}$ attract each other. Observe that the vacuum itself is in the repulsive case so actually the only "visible" effect is the attractive one.

9 Conclusions

We have constructed universal XX and Hubbard model Hamiltonians based on general properties of projectors and permutations. The underling algebraic structure could be an ordinary or graded algebra $gl(N|M)$ or possibly and infinite dimensional algebra. We have full control of the symmetries of the models and we have performed the perturbative calculation *à la* Klein and Seitz [18] in the large coupling limit.

We have emphasised that the gradation makes the Jordan–Wigner transformation a local isomorphism. Therefore, the interpretation of the graded models in terms of "electrons" is more natural.

We discussed some examples, with their phenomenology. There, it would be very nice to see if the major screening effects observed (7.1) and (7.4) in the Hamiltonians can be interpreted in some condensed matter context.

The next step in the study of our models is the determination of the spectrum and of the Bethe equations, as they were constructed for Hubbard or generalisation, using the algebraic Bethe ansatz [27, 28, 29, 31] and the coordinate Bethe Ansatz of Lieb-Wu [11]. This is an heavy calculation which we postpone for further publication, but from the analytical Bethe ansatz approach, one can guess their form. In particular, as for spin chain models, one expects as many presentations of the Bethe equations as there are inequivalent Dynkin diagrams. All these presentations should lead to the same spectrum. For more informations, we refer to [32, 33] where similar calculations were performed in the case of XXX super spin chains.

Our models are graded by construction so they naturally contain bosonic as well fermionic degrees of freedom. We are working on examples with bosonic particles, that necessarily will be on infinite dimensional algebras.

Finally, the Bethe equations will allow us to keep in touch with super-symmetric gauge theories, where integrability appears precisely in relation to the Hubbard model.

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Integrable models for Bose-Einstein condensates

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Abstract

In this work we investigate integrable models for Bose-Einstein condensates (BEC). To illustrate the main procedures in the analysis of the physical properties of integrable BEC-models in general, we present a complete classical and quantum investigation for a simple, yet non-trivial model, which describes Josephson tunneling between two Bose-Einstein condensates. Using a classical analysis we study the fixed point bifurcations and the classical dynamics of this model. The quantum dynamics is also presented and the quantum phase transitions of the Hamiltonian are investigated using the concepts of entanglement, energy gap and fidelity. Finally we discuss some other integrable models in the BEC context.

1 Introduction

The phenomenon of Bose-Einstein condensation, while predicted long ago [1], is currently one of the most active fields in Physics due to its contribution to the comprehension of physical phenomena emerging from mesoscopic systems. Since the early experimental realizations of BECs using ultracold dilute alkali gases [2], intense efforts have been devoted to the study of new properties of BEC. In recent years the creation of a molecular BEC form an atomic BEC has been obtained by different techniques [3], such as photoassociation and Feshbach ressonance. The field was further broadened by the achievement of quantum degeneracy in ultracold fermionic gases [4].
>From the theoretical point of view, one approach that continues to attract considerable interest in this context is that of the exact solution of a physical model [5, 6]. At the nanoscale level, the quantum fluctuations are sufficiently large, such that exact solutions are highly welcome and desired [7]. Our main goal here is to discuss some integrable models in the BEC scenario, such as a model for two Josephson-coupled BEC [8], an atom-molecule BEC model [9] and a three-coupled BEC model [10]. To illustrate the main procedures in the analysis of the physical properties of integrable BEC-models in general, we present a complete classical and quantum investigation for a simple, yet non-trivial model, the model of two coupled Bose-Einstein condensates.
The Hamiltonian is given by [8]

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1). \tag{1.1}$$

where $a_1^\dagger, a_2^\dagger$ denote the single-particle creation operators in the two wells and $N_1 = a_1^\dagger a_1, N_2 = a_2^\dagger a_2$ are the corresponding boson number operators. The total boson number $N_1 + N_2$ is conserved and set to the fixed value of N. The coupling k provides the strength of the scattering interaction between bosons, $\Delta\mu$ is the external potential and $\mathcal{E}_J$ is the coupling for the tunneling. Despite this apparent simplicity, this Hamiltonian captures the essence of competing linear and non-linear interactions, leading to interesting, non-trivial behaviour and ground-state properties. In particular, the model predicts tunneling, self-trapping and collapse and revival oscillation, observed experimentally [11]. This model is integrable and its exact solution can be derived through Bethe ansatz methods. Details of these derivations may be found in [12].
In the next section we present a classical analysis of this model. In section 3 the quantum dynamics and the quantum phase transitions of this model are investigated. These techniques may be adapted to investigate other integrable generalised BEC models, such as those discussed in section 4.

2 Classical analysis

To get an insight in the physical properties we study first a classical analogue of the model. Let N_j, θ_j, $j = 1, 2$ be quantum variables satisfying the canonical commutation

relations. We make a change of variables from the operators a_j, $a_j^\dagger$, $j = 1, 2$ via

$$a_j = \exp(i\theta_j)\sqrt{N_j}, \quad a_j^\dagger = \sqrt{N_j}\exp(-i\theta_j)$$

such that the Heisenberg canonical commutation relations are preserved. Now define the variables

$$z = (N_1 - N_2)/N$$

$$\phi = N(\theta_1 - \theta_2)/2$$

where z represents the fractional occupation difference (or the *imbalance*) and ϕ the phase difference. In the classical limit where N is large, but still finite, we may equivalently consider the Hamiltonian [13]

$$H(z,\phi) = \frac{\mathcal{E}_J N}{2}\left(\frac{\lambda}{2}z^2 - \beta z - \sqrt{1-z^2}\cos(2\phi/N)\right) \tag{2.1}$$

where

$$\lambda = \frac{kN}{2\mathcal{E}_J}, \qquad \beta = \frac{\Delta\mu}{\mathcal{E}_J}$$

and (z, ϕ) are canonically conjugate variables. >From this Hamiltonian we can derive the equations of motion in the usual way. We find

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial z} = \frac{\mathcal{E}_J N}{2}\left(\lambda z - \beta + \frac{z}{\sqrt{1-z^2}}\cos(2\phi/N)\right) \\ \dot{z} &= -\frac{\partial H}{\partial \phi} = -\mathcal{E}_J\left(\sqrt{1-z^2}\sin(2\phi/N)\right). \end{aligned} \tag{2.2}$$

With these equations we can study the fixed point bifurcations as well as the classical dynamics of the system.

2.1 Fixed points

Now we study the fixed points of the Hamiltonian (2.1), determined by the condition

$$\dot{z} = \dot{\phi} = 0. \tag{2.3}$$

By performing a numerical analysis we find that these equations may have one, two or three solutions, depending on the values of the coupling parameters (see [13] for details). This allow us to construct a diagram of parameters identifying the different types of solutions, depicted in Fig. 6.1 We remark that in the absence of the external potential ($\Delta\mu = \beta = 0$), we have two fixed point bifurcations given by $\lambda_0 = \pm 1$.

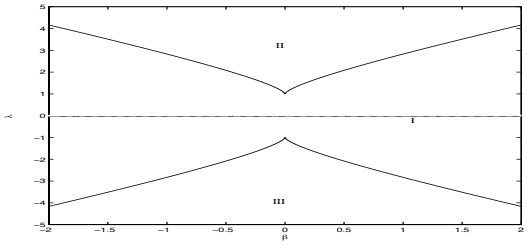

Figure 6.1: Coupling parameter space diagram identifying the different types of solutions for the equations (2.3). In region I there is one solution for $\phi = N\pi/2$ and one solution for $\phi = 0$. In region II there are three solutions for $\phi = N\pi/2$ and one solution for $\phi = 0$ and in region III there is one solution for $\phi = N\pi/2$ and three solutions for $\phi = 0$.

2.2 Classical dynamics

Next we look at the dynamical evolution. We will consider the equations (2.2) in the absence of the external field ($\Delta\mu = 0$ or, equivalently, $\beta = 0$). We integrate (2.2) to find the time evolution for the imbalance z, using the initial condition $z(0) = 1$, $\phi(0) = 0$. By plotting z against the time, it is evident that there is a threshold coupling $\lambda_c = 2$ separating two different behaviours in the classical dynamics, as can be seen in Fig. 6.2:

(i) For $\lambda < 2$ the system oscillates between $z = -1$ and $z = 1$. Here the evolution is delocalised;

(ii) For $\lambda > 2$ the system oscillates between $z = 0$ and $z = 1$. Here the evolution is localised.

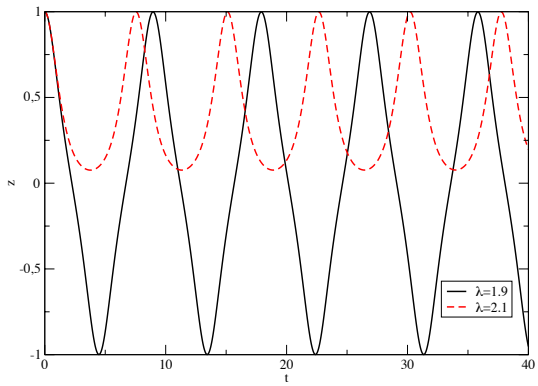

Figure 6.2: Time evolution for the imbalance z. The solid line is for $\lambda = 1.9$, while the dashed curve is for $\lambda = 2.1$. Here we are using $N = 100$, $\mathcal{E}_J = 1$ and the initial conditions $z(0) \approx 1$, $\phi(0) = 0$. The threshold coupling occurs at $\lambda_t = 2$.

The threshold coupling $\lambda_t = 2$, first observed in [14] (or $k/\mathcal{E}_J = 4/N$, in terms of the original variables) separates two distinct dynamical behaviours. This value for the threshold

between delocalisation and self-trapping also occurs for the quantum dynamics, as we will show in the next section.

3 Quantum analysis

We now turn our attention to a quantum mechanical treatment of the model. In particular, we analyse the quantum dynamics as well as the quantum phase transitions of the Hamiltonian (1.1).

3.1 Quantum dynamics

We will investigate the quantum dynamics of the Hamiltonian (1.1) in the absence of the external potential ($\Delta\mu = 0$) using the exact diagonalisation method (see [15] for more details). The time evolution of any state is determined by $|\Psi(t)\rangle = U(t)|\phi_0\rangle$, where U is the temporal evolution operator given by $U(t) = \sum_{m=0}^{M} |m\rangle\langle m| \exp(-iE_m t)$, $|m\rangle$ is an eigenstate with energy E_m and $|\phi_0\rangle$ represents the initial state. Using these expressions we can compute the expectation value of the relative number of particles

$$\langle (N_1 - N_2)(t)\rangle = \langle \Psi(t)|N_1 - N_2|\Psi(t)\rangle. \tag{3.1}$$

for all ratios of the coupling $k/\mathcal{E}_J$. In particular, in Fig. 6.3 we consider the interval $k/\mathcal{E}_J \in [1/N, 1]$ for the case $N = 100$: we observe the evolution of the dynamics from a collapse and revival sequence for $k/\mathcal{E}_J < 4/N$, through the self-trapping threshold at $k/\mathcal{E}_J = 4/N$, and toward small amplitude harmonic oscillations in the imbalance of the localised state when $k/\mathcal{E}_J = 1$. A more detailed investigation can be found in ref. [15]. From the above picture it is clear that the threshold coupling $k/\mathcal{E}_J = 4/N$ predicted by the classical analysis, representing the boundary between a delocalised evolution ($k/\mathcal{E}_J < 4/N$) and self-trapped evolution ($k/\mathcal{E}_J > 4/N$), also holds for the quantum dynamics.

3.2 Quantum phase transitions

A Quantum Phase Transition (QPT) is usually defined as a phase transition in the ground-state of the system under the variation of some parameter. Basically, there is a sudden change in the structure of the ground state at the QPT, and the properties such as entanglement, correlations, etc reflect this sudden change [16]. There are different methods to determine a QPT. In particular, we will study the behaviour of the energy gap, the entanglement and fidelity of the system to identify a QPT. We observe that there is also an analytical method based on the Bethe ansatz solution developed by Dunning et al [17]. Here we mention that a QPT is rigorously defined in the thermodynamical limit $N \to \infty$. For large but finite N the system does display an increasing sharp distinction between ground state regions, called Quantum Pre-Phase Transitions (QPPT). The occurrence of a QPPT in a finite system is a precursor for a QPT in the thermodynamic limit. Let us now study the QPPT of the Hamiltonian (1.1) when the external potential is zero.

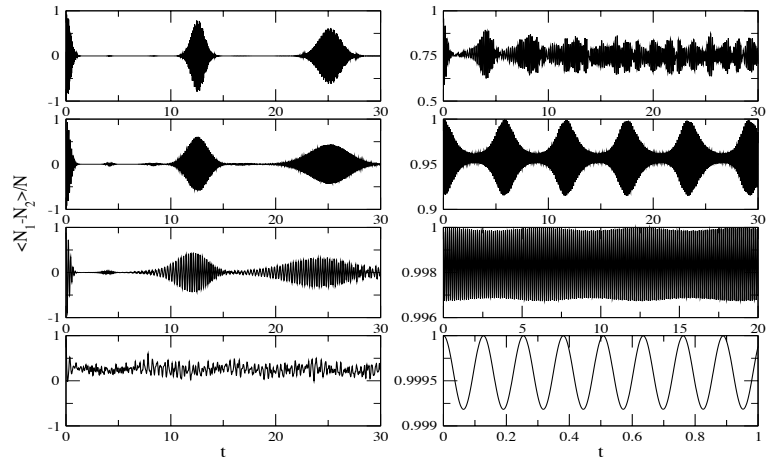

Figure 6.3: Time evolution of the expectation value between $k/\mathcal{E}_J = 1/N$ and $k/\mathcal{E}_J = 1$. On the left, from the top to the bottom $k/\mathcal{E}_J = 1/N, 2/N, 3/N, 4/N$ and on the right, from the top to the bottom $k/\mathcal{E}_J = 5/N, 10/N, 50/N, 1$, where $N = 100$ and the initial state is $|N, 0\rangle$.

3.2.1 Energy gap

We consider first the energy gap, which is defined as the difference between the first excited state and the ground-state of the system

$$\Delta E = E^{(1)} - E^{(0)}.$$

In Fig. 6.4 we plot the gap against the ratio of the parameters $(-\frac{1}{\lambda} = -\frac{2\mathcal{E}_J}{kN})$ for $k = -1$ and different values of N. We observe that as long as N increases, the value where the gap opens[1] corresponds to $\lambda = -1$. Similar discussions in the context of creating a macroscopic "Schrödinger-cat" can be found in [18].

3.2.2 Entanglement

We can consider the pair of coupled Bose-Einstein condensates as a bipartite system of two modes, "1" and "2". In this case, the standard measure of entanglement is the von Neumann entropy of the reduced density operator of either of the modes [14]. The state of each mode is characterized by its occupation number. Using the fact that the total number of atoms N is constant, a general state of the system can be written in terms of the Fock states by

$$|\Psi\rangle = \sum_{n=0}^{N} c_n |n\rangle |N-n\rangle \tag{3.2}$$

[1] rigorously, the gap is almost closed below this value

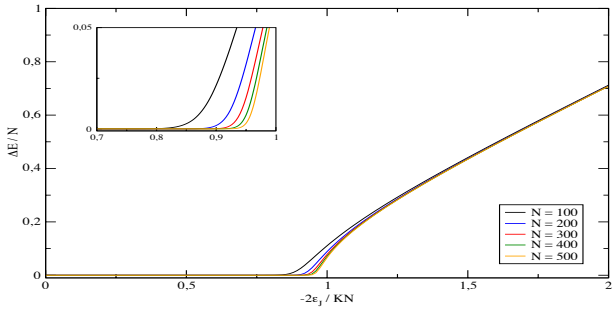

Figure 6.4: Energy gap between the first excited state and the ground state as a function of $-\frac{1}{\lambda} = -\frac{2\mathcal{E}_J}{kN}$ for $\Delta\mu = 0$, $k = -1$ and different values of N.

where c_n are complex numbers. The density operator of the system is given by

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{m,n=0}^{N} c_m^* c_n |m\rangle|N-m\rangle\langle n|\langle N-n|. \tag{3.3}$$

Taking the partial trace with respect to mode "2" yields the reduced density operator for mode "1",

$$\rho_1 = \mathrm{Tr}_2(\rho) = \sum_{n=0}^{N} |c_n|^2 |n\rangle\langle n|. \tag{3.4}$$

Thus the entropy of entanglement of the ground-state of the system is given by

$$E(\rho_1) = -\mathrm{Tr}[\rho_1 \log(\rho_1)] = -\sum_{n=0}^{N} |c_n|^2 \log(|c_n|^2). \tag{3.5}$$

In Fig. 6.5 we plot the von Neuman entropy of the entanglement of the ground-state against the same ratio of the parameters of the previous case for different values of N. It

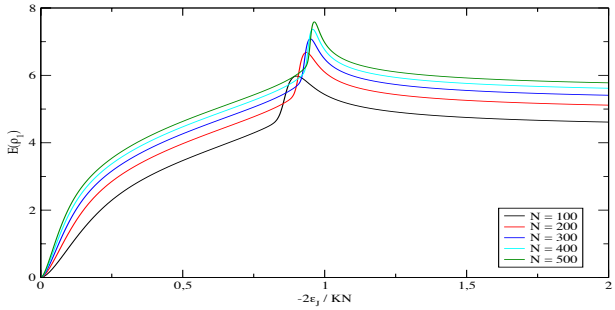

Figure 6.5: Entropy of entanglement of the ground-state as a function of $-\frac{1}{\lambda} = -\frac{2\mathcal{E}_J}{kN}$, for $k = -1$, $\Delta\mu \approx 0$ and different values of N.

is clear that there is a peak in the entanglement which tends to occur at $\lambda = -1$ as long as N increases. Similar results have been discussed in [19].

3.2.3 Fidelity

Another possibility to investigate the QPPT is through the behaviour of the fidelity, which is a concept widely used in the Quantum Information Theory [20, 21, 22]. The fidelity is basically defined as the modulus of the wavefunction overlap between two states

$$\mathcal{F}(\psi, \phi) = |\langle \psi | \phi \rangle|.$$

In Fig. 6.6 we present the wavefunction overlap between two ground-states corresponding to two slightly different and small values of the external potential $\Delta\mu$ versus the same ratio of the parameters. It is clear that as long as N increases, the value where the fidelity

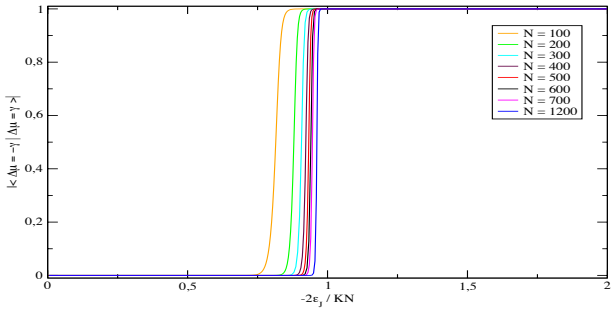

Figure 6.6: Ground-state wavefunction overlaps as a function of $-\frac{1}{\lambda} = -\frac{2\mathcal{E}_J}{kN}$, for $k = -1$, $\gamma = \pm 0.001$ and different values of N.

exhibits an abrupt decay corresponds to $\lambda = -1$.

Therefore, we have identified the QPPT point at $\lambda = -1$ by

(i) the peak in the entanglement of the ground-state,

(ii) the point where the gap opens,

(iii) a sudden drop in the fidelity.

We emphasize that the point where the QPPT occurs ($\lambda = -1$) corresponds to a fixed point bifurcation obtained in the classical analysis. This result coincides with that obtained in[17] to higher orders in N.

4 Integrable generalised BEC models

Let us now discuss other exactly solvable models in the BEC scenario. The basic idea to construct integrable generalised BEC models is to explore different representations of some algebras or superalgebras. Below we present some representative examples.

4.1 A model for atom-molecule BEC

After the experimental realisation of Bose-Einstein condensation in dilute alkali gases, many physicists started to consider the possibility of producing a molecular Bose-Einstein condensate from photoassociation and/or the Feshbach resonance of an atomic Bose-Einstein condensate of a weakly interacting dilute alkali gas [3]. A Bose-Einstein condensate of rubidium has been achieved comprised of a coherent superposition of atomic and molecular states in 2002. In this context, an integrable Hamiltonian that describes the interconversion of atoms and molecules is given by

$$H = U_a N_a^2 + U_b N_b^2 + U_{ab} N_a N_b + \mu_a N_a + \mu_b N_b + \Omega(a^\dagger a^\dagger b + b^\dagger a a)$$

where $a^\dagger$ and $b^\dagger$ denote the creation operators for atomic and molecular modes respectively, the U_i's are interaction strengths, μ_i's are external potentials and Ω is the amplitude for interconversion of atoms and molecules. It was shown in [12] that this model is exactly solvable by Bethe ansatz methods.

4.2 A three-coupled BEC model

Another interesting model that can be constructed is a three-coupled BEC model, given by the Hamiltonian [10]

$$\mathcal{H} = \Omega_2 \, (a_2^\dagger a_1 + a_1^\dagger a_2 + a_2^\dagger a_3 + a_3^\dagger a_2) + \Omega \, (a_1^\dagger a_3 + a_3^\dagger a_1) + \mu \, n_1 + \mu \, n_3 + \mu_2 \, n_2,$$

which describes the tunneling between three coupled wells, referred to as the left (1), middle (2) and right (3) wells, respectively. Above, Ω (resp. Ω_2) denote the tunneling of atoms between the left and the right wells (resp. the left-middle tunneling and the middle-right tunneling), while μ_2 and μ are the external potentials. It was shown in [10] that this model is exactly solvable by Bethe ansatz methods.
We observe here that there has been recently an increasing interest in the study of three-well systems (trimers) for a variety of reasons, such as

(i) its possible application in the construction of a BEC-transistor [23];

(ii) it is the simplest model which provides a bridge between the double-well and the multi-well systems [24];

(iii) recent achievements in the experimental field, in particular the control promised by microtraps suggest the realization of the trimer to be at hand [25].

4.3 Other integrable BEC-models

There are many other integrable models that can be constructed exploring different representations of some algebras/superalgebras, such as

- Two-coupled BEC model with a field
- Two-coupled BEC model with different types of atoms
- N-coupled BEC model
- Heteroatomic-molecular BEC model
- Triatomic-molecular BEC model
- Fermionic BEC models

Some of these models have been derived and discussed in [10].

5 Conclusion

To summarize, we have presented integrable models for Bose-Einstein condensates. To illustrate the main techniques that may be employed to study the algebraic and physical properties of these models we have presented a complete classical and quantum analysis for a particular simple model, the model of two coupled Bose-Einstein condensates. These techniques may be adapted to investigate other integrable models, such as an integrable BEC-transistor.

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Finite size effects and 2-string deviations in the spin-1 XXZ chains

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Abstract

We present and study the nonlinear integral equations (NLIE) governing the finite size effects in the spin-1 XXZ chains in the regime $0 < \gamma < \frac{\pi}{2}$. With the help of the NLIEs we discuss the typical root configurations of the thermodynamic limit, the 2-string deviations for excited states and we calculate analytically the conformal spectrum of the spin chain.

1 Introduction

It is well known that in the thermodynamic limit the anti-ferromagnetic ground state of the spin-1 XXZ chain is composed of quasi 2-strings, namely of pairs of complex roots having imaginary parts close to $\pm\frac{\pi}{2}$. The deviations of their imaginary parts from the values $\pm\frac{\pi}{2}$ are called 2-string deviations. According to the string hypothesis these deviations should be exponentially small in N [4, 5, 6], but it turns out that these deviations are much larger; they are of order $1/N$ [7, 8]. This is why these 2-string deviations are not negligible even in the large N limit calculations of the physical quantities. To treat correctly the technical difficulties coming from 2-string deviations one needs to use the so-called nonlinear integral equation technique (NLIE). The NLIE technique was originally introduced in [8] where with the help of this technique the finite size effects of the ground states of the spin-1/2 and spin-1 XXZ chains were studied calculating analytically their central charges and in the spin-1 case the 2-string deviations in the ground state for the first time in a unified NLIE approach. Then the NLIE technique was successfully applied for describing finite size effects in various integrable spin chains and quantum field theories.

The NLIEs being the best for analytical calculations in the spin-1 XXZ chain were derived by J. Suzuki [1] for the repulsive regime (i.e. $0 < \gamma < \frac{\pi}{3}$). In this work we present his NLIEs in a form being valid for the attractive regime of the model as well (i.e. $\frac{\pi}{3} < \gamma < \frac{\pi}{2}$). With the help of the NLIEs we discuss the typical root configurations of the thermodynamic limit, the 2-string deviations for excited states and we calculate analytically the conformal spectrum of the spin chain.

The two motivations for such a study can be mentioned: The first is that spin-1 XXX chains emerged in 1-loop anomalous dimensions of certain operators in large N_c QCD [3]. Secondly the NLIEs can help in the numerical evaluation of form factors and correlation functions in integrable spin-1 chains.

2 Some generalities

The integrable spin-1 XXZ chain is defined by the Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{N} \left(\sigma_i^{\perp} - (\sigma_i^{\perp})^2 + \cos 2\gamma\,(\sigma_i^z - (\sigma_i^z)^2) - (2\,\cos\gamma - 1)(\sigma_i^{\perp}\sigma_i^z + \sigma_i^z\sigma_i^{\perp}) - 4\sin^2\gamma (S_i^z)^2\right), \tag{2.1}$$

$$\text{where} \qquad \sigma_i = S_i \cdot S_{i+1} = \sigma_i^{\perp} + \sigma_i^z, \qquad \sigma_i^z = S_i^z \cdot S_{i+1}^z.$$

Henceforth we consider N to be even and we impose periodic boundary conditions on the spins $S_{N+1}^a = S_1^a$, $a \in \{x, y, z\}$.

It is well known that the above spin Hamiltonian can be obtained from the transfer matrix of the 19-vertex model [2, 4, 5, 6] in the following way: let $V_i \simeq \mathbb{C}^{l_i+1}$ be the irreducible

$SU(2)$ representation with spin $l_i/2$ and let $R_{ij}^{(l_i,l_j)}(\theta)$ the R-matrices acting on $V_i \otimes V_j$ satisfying the Yang-Baxter equations

$$R_{12}^{(l_1,l_2)}(\theta)R_{13}^{(l_1,l_3)}(\theta+\theta')R_{23}^{(l_2,l_3)}(\theta') = R_{23}^{(l_2,l_3)}(\theta')R_{13}^{(l_1,l_3)}(\theta+\theta')R_{12}^{(l_1,l_2)}(\theta). \tag{2.2}$$

These R-matrices can be obtained by fusion [10] from the well known R-matrix of the six-vertex model. From these R-matrices one can define a family of transfer matrices:

$$T_k(\theta) = \mathrm{Tr}_a \left(R_{a1}^{(k,2)}(\theta - i\frac{\pi}{2}(1+k)) \dots R_{aN}^{(k,2)}(\theta - i\frac{\pi}{2}(1+k)) \right). \tag{2.3}$$

The Hamiltonian of the spin-1 XXZ chain is expressed as logarithmic derivative of the second transfer matrix of the fusion hierarchy:

$$\mathcal{H} = \frac{\pi}{\gamma}\frac{d}{d\theta} \log T_2(\theta)|_{\theta=-i\frac{\pi}{2}}. \tag{2.4}$$

Due to the Yang-Baxter relation (2.2) the model is integrable and the transfer matrices (2.3) form a commutative family of operators (i.e. $\left[T_k(\theta), T_{k'}(\theta')\right] = 0,$) and so they can be diagonalized by algebraic Bethe Ansatz [13]. The eigenvalues of the transfer matrices can be characterized by the solutions of the Bethe Ansatz equations (BAE):

$$\frac{\Phi(\theta_j + i\pi)}{\Phi(\theta_j - i\pi)} = -\frac{Q(\theta_j + i\pi)}{Q(\theta_j - i\pi)}, \qquad j = 1, \dots, M, \tag{2.5}$$

where

$$\Phi(\theta) = \left(\sinh\frac{\gamma}{\pi}(\theta)\right)^N, \qquad Q(\theta) = \prod_{j=1}^{M} \sinh\frac{\gamma}{\pi}(\theta - \theta_j), \tag{2.6}$$

γ is the anisotropy of the model, M is the number of Bethe roots. We recall that $S = N - M$ is the third component of the total spin of the spin chain.

Let us parametrize the anisotropy parameter as $\gamma = \frac{\pi}{p+2}$, then the $0 < p < 1$ and $1 < p$ regimes correspond to the attractive and repulsive regimes of the model respectively. All the functions entering the BAE are periodic with respect to $\pi(p+2)$ so the positions of the Bethe roots can be restricted into the fundamental strip: $-\frac{\pi}{2}(p+2) < \mathrm{Im}\theta_j \leq \frac{\pi}{2}(p+2)$. We recall that the Bethe roots can be real or self-conjugate ($|\mathrm{Im}\,\theta_j| = \frac{\pi(p+2)}{2}$,) or they appear in complex conjugate pairs.

For later convenience it is useful to classify the Bethe roots according to their positions on the fundamental strip. The classification is as follows: (See figure 1.)

1. *Inner roots*: $|\mathrm{Im}\,\theta_j| < \frac{\pi}{2}$,
2. *Close roots*: $\frac{\pi}{2} < |\mathrm{Im}\,\theta_j| < \frac{\pi}{2} + \min(1,p)\,\pi$,
3. *Wide roots*: $\frac{\pi}{2} + \min(1,p)\,\pi < |\mathrm{Im}\,\theta_j| < \frac{\pi(p+2)}{2}$,
4. *Self-conjugated roots*: $|\mathrm{Im}\,\theta_j| = \frac{\pi(p+2)}{2}$,

From the point of view of the formulation of the NLIEs it is convenient to introduce the *effective roots* by the following definition: To each root θ_j with $\frac{\pi}{2} < |\mathrm{Im}\theta_j| \leq \frac{\pi(p+2)}{2}$ we associate an *"effective root"* $\tilde{\theta}_j$ by the following definition:

$$\tilde{\theta}_j = \theta_j - i\frac{\pi}{2}\mathrm{sign}(\mathrm{Im}\,\theta_j). \tag{2.7}$$

This transformation is equivalent to the removal of the middle strip of inner roots from the fundamental domain of the original Bethe roots. (See figure 2.) The fundamental domain for effective roots is: $-\frac{\pi}{2}(p+1) < \mathrm{Im}\tilde{\theta}_j \leq \frac{\pi}{2}(p+1)$, and their classification is analogous to that of the original Bethe roots:

1. *Close effective roots*: $0 < |\mathrm{Im}\,\tilde{\theta}_j| < \min(1,p)\,\pi$,

2. *Wide effective roots*: $\min(1,p)\,\pi < |\mathrm{Im}\,\tilde{\theta}_j| < \frac{\pi(p+1)}{2}$,

3. *Self-conjugated effective roots*: $|\mathrm{Im}\,\tilde{\theta}_j| = \frac{\pi(p+1)}{2}$,

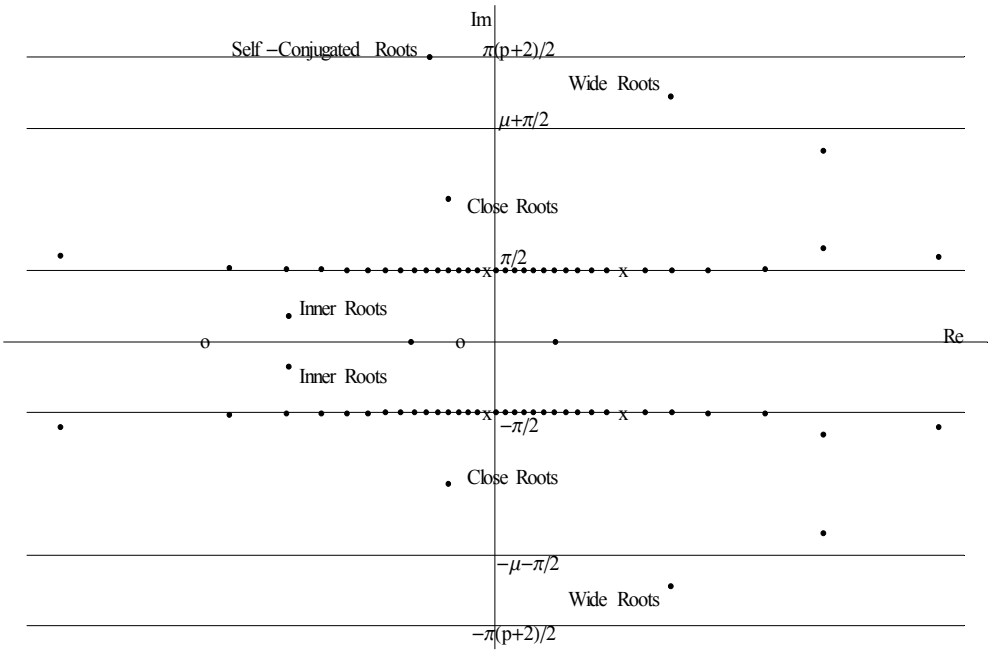

Figure 7.1: Classification of Bethe roots. Crosses stand for the 2-string holes, empty circles denote the type I holes, black dots represent the Bethe roots and μ denotes $\pi\min(1,p)$.

The antiferromagnetic ground state of the model is formed by quasi 2-strings, namely pairs of Bethe roots with $|\mathrm{Im}\theta_j|$ being close to $\frac{\pi}{2}$. Since the 2-strings are not exact, for large N it is hard to solve the BAE directly, this is why it is useful to transform them into NLIEs. In the NLIE description the contribution of 2-strings is summed up by the integral terms of the NLIEs.

In order to get NLIEs we have to create proper auxiliary functions, we must have sufficient number of functional relations among them and we must know their analytical properties.

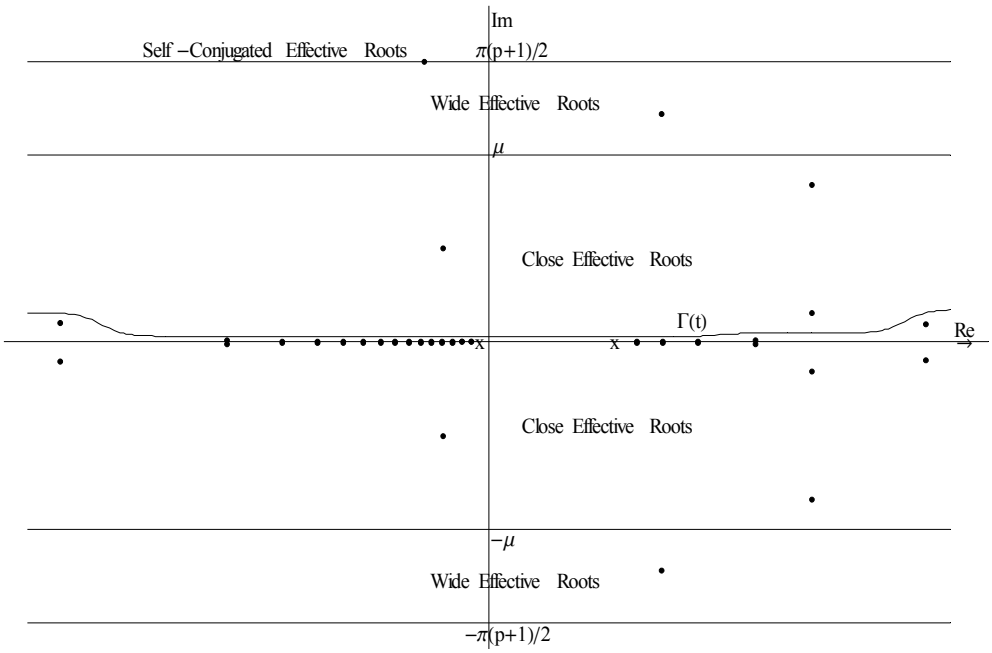

Figure 7.2: Classification of effective roots. Crosses stand for holes, effective roots are represented by black dots and μ denotes $\pi\min(1,p)$.

3 Auxiliary functions

For the spin-1 XXZ model the proper auxiliary functions were introduced in [1] by J. Suzuki. The most natural auxiliary function $a(\theta)$ is defined by

$$a(\theta)=\frac{\Phi(\theta+i\pi)}{\Phi(\theta-i\pi)}\frac{Q(\theta-i\pi)}{Q(\theta+i\pi)},\qquad 1+a(\theta)=\frac{1}{\Phi(\theta-i\pi)}\frac{Q(\theta)}{Q(\theta+i\pi)}T_1(\theta). \tag{3.1}$$

Introducing the counting function: $Z_a(\theta)=i\,\log a(\theta)$ the BAE (2.5) can be recasted in the form:

$$Z_a(\theta_j)=2\pi\,I_j \qquad I_j=\text{half-integer} \qquad j=1,...,M,$$
$$Z_a(h_k^{(1)})=2\pi\,I_k^{(1)} \qquad I_k^{(1)}=\text{half-integer} \qquad T_1(h_k^{(1)})=0.$$

$Z_a(\theta)$ is the counting function of real roots, since it can be shown that the integer part of $\frac{Z_a(\theta')-Z_a(\theta)}{2\pi}+1$ gives the number of real roots plus the number of real zeroes of $T_1(\theta)$ lying in the $[\theta,\theta']$ interval. So real zeroes of $T_1(\theta)$ correspond to holes in the distribution of real roots and we call them *Type I holes* (See figure 1.).
Another important auxiliary function is defined as follows:

$$b(\theta)=\frac{\Phi(\theta-i\frac{\pi}{2})}{\Phi(\theta+i\frac{\pi}{2})\Phi(\theta+i\frac{3\pi}{2})}\frac{Q(\theta+i\frac{3\pi}{2})}{Q(\theta-i\frac{3\pi}{2})}\,T_1\left(\theta-i\frac{\pi}{2}\right), \tag{3.2}$$

$$B(\theta)=1+b(\theta)=\frac{1}{\Phi(\theta+i\frac{\pi}{2})\Phi(\theta+i\frac{3\pi}{2})}\frac{Q(\theta+i\frac{\pi}{2})}{Q(\theta-i\frac{3\pi}{2})}\,T_2(\theta). \tag{3.3}$$

Let us introduce the counting function: $Z_b(\theta) = \frac{1}{i}\log b(\theta)$. Then the BAE can be reformulated as:

$$Z_b(\theta_j - i\frac{\pi}{2}) = 2\pi\, I_j, \qquad I_j = \text{half-integer} \qquad j = 1, ..., M \tag{3.4}$$

$$Z_b(h_j) = 2\pi\, I_{h_j}, \qquad I_{h_j} = \text{half-integer} \qquad T_2(h_j) = 0. \tag{3.5}$$

In the large N limit the 2-strings tend to be exact, and thus following from (3.4,3.5), in this limit considering $Z_b(\theta)$ on the real axis one can recognize that the integer part of the real part of $\frac{Z_b(\theta')-Z_b(\theta)}{2\pi} + 1$ provides the sum of the number of real zeros of $T_2(\theta)$ lying in the interval $[\theta, \theta']$ and the number of 2-strings with $\theta < \mathrm{Re}\,\theta_j < \theta'$. In this way real zeroes of the transfer matrix eigenvalue $T_2(\theta)$ can be considered as holes in the distribution of 2-strings hence we will call them simply *holes*. The holes can be depicted in the plane of effective roots, so that they lie on the real axis (See figure 2.). To formulate the NLIE one needs to define one more TBA type auxiliary function by the definition as follows:

$$y(\theta) = \frac{T_2(\theta)}{\Phi(\theta - i\frac{3\pi}{2})\Phi(\theta + i\frac{3\pi}{2})}, \qquad Y(\theta) = 1 + y(\theta).$$

The knowledge of the auxiliary functions, their functional relations and analytical properties lead to the NLIEs. We skip the details of the derivation (they can be found in [1, 9, 14]), and we simply present the final form of the NLIEs in the next section.

4 Nonlinear integral equation

The NLIEs read as follows:

$$\begin{aligned}
\log b(\theta) &= C_b + iD(\theta) + ig_1(\theta) + ig_b(\theta) + (G *_{\Gamma} \ln B)(\theta) - (G *_{\bar\Gamma} \ln \bar B)(\theta) \\
&+ \lim_{\varepsilon\to 0^+} (K^{+\frac{\pi}{2}-\varepsilon} * \ln Y)(\theta) \\
\log y(\theta) &= ig_y(\theta) + (K^{+\frac{\pi}{2}} *_{\Gamma} \ln B)(\theta) + (K^{-\frac{\pi}{2}} *_{\bar\Gamma} \ln \bar B)(\theta), \\
\log b_{II}(\theta) &= C_b^{II} + iD_{II}(\theta) + ig_{1II}(\theta) + ig_{bII}(\theta) + (G_{II} *_{\Gamma} \ln B)(\theta) \\
&- (G_{II} *_{\bar\Gamma} \ln \bar B)(\theta) + \lim_{\varepsilon\to 0^+} (K_{II}^{+\frac{\pi}{2}-\varepsilon} * \ln Y)(\theta),
\end{aligned} \tag{4.1}$$

where C_b and C_y are constants whose values can be found in [14] furthermore $B(\theta) = 1 + b(\theta)$, $Y(\theta) = 1 + y(\theta)$, $\bar B(\theta)$ stands for the complex conjugate of $B(\theta)$, and we introduced the notation for any function f: $f^{\pm\eta}(\theta) = f(\theta \pm i\eta)$. The function $b_{II}(\theta)$ is simply related to the auxiliary function $a(\theta)$ and serves for the determination of wide and self conjugate effective roots. The equations contain two types of convolutions; one goes on the real axis: $(f*g)(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta-\theta')g(\theta')$ and the other ones runs along the integration contours $\Gamma(t)$ and its complex conjugate $\bar\Gamma(t)$ so as to avoid the effective 2-strings from above and from

below respectively: $(f *_\Gamma g)(x) = \int_\Gamma dz\, f(x-z)g(z)$, $(f *_{\bar\Gamma} g)(x) = \int_{\bar\Gamma} dz\, f(x-z)g(z)$. There is one of order N bulk source term in the equations:

$$D(\theta) = N \arctan\sinh\theta. \tag{4.2}$$

The kernel functions G and K of (4.1) read as

$$G(\theta) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{iq\theta}\, \frac{\sinh\frac{\pi(p-1)q}{2}}{2\sinh\frac{\pi pq}{2}\cosh\frac{\pi q}{2}}, \qquad K(\theta) = \frac{1}{2\pi\cosh(\theta)}. \tag{4.3}$$

The non-bulk source terms are expressed by the odd primitives of the kernels denoted by $\chi(\theta)$ and $\chi_K(\theta)$ respectively. The explicit expressions for these sources read as:

$$\begin{aligned} g_b(\theta) &= \sum_{j=1}^{N_H}\chi(\theta-h_j) + \sum_{j=1}^{N_V^S}\left(\chi(\theta-v_j)+\chi(\theta-\bar v_j)\right) - \sum_{j=1}^{N_S}\left(\chi(\theta-s_j)+\chi(\theta-\bar s_j)\right) \\ &- \sum_{j=1}^{M_C}\chi(\theta-c_j) - \sum_{j=1}^{M_W}\chi_{II}(\theta-w_j) - \sum_{j=1}^{M_{sc}}\chi_{II}(\theta-w_{sc}^{(j)}), \end{aligned} \tag{4.4}$$

$$g_1(\theta) = \sum_{j=1}^{N_1}\chi_K(\theta-h_j^{(1)}), \qquad g_y(\theta) = \lim_{\eta\to 0^+}\tilde g_y\left(\theta+i\frac{\pi}{2}-i\eta\right)+C_y, \tag{4.5}$$

$$\begin{aligned} \tilde g_y(\theta) &= \sum_{j=1}^{N_H}\chi_K(\theta-h_j) + \sum_{j=1}^{N_V^S}\left(\chi_K(\theta-v_j)+\chi_K(\theta-\bar v_j)\right) \\ &- \sum_{j=1}^{M_S}\left(\chi_K(\theta-s_j)+\chi_K(\theta-\bar s_j)\right) - \sum_{j=1}^{M_C}\chi_K(\theta-c_j) \\ &- \sum_{j=1}^{M_W}\chi_{KII}(\theta-w_j) - \sum_{j=1}^{M_{sc}}\chi_{KII}(\theta-w_{sc}^{(j)}), \end{aligned} \tag{4.6}$$

where the second determination of any function $f(\theta)$:

$$f_{II}(\theta) = \begin{cases} f(\theta) + f(\theta - i\,\pi\,\mathrm{sign}(\mathrm{Im}\,\theta)) & 1 < p \\ f(\theta) - f(\theta - i\,p\,\pi\,\mathrm{sign}(\mathrm{Im}\,\theta)) & 0 < p < 1. \end{cases} \tag{4.7}$$

The objects appearing in the source terms are close- $\{c_j\}$ wide- $\{w_j\}$ and self-conjugate $\{w_j^{(sc)}\}$ effective roots, furthermore holes $\{h_j\}$, type I holes $\{h_j^{(1)}\}$, ordinary- $\{s_j\}$ and virtual- $\{v_j\}$ special objects. It is important to note that the second determinations are different in the attractive and repulsive regimes of the model causing differences in the typical root configurations of the two regimes. Only those effective roots and their complex conjugates appear in the source terms of the NLIE which have lie above the integration contour $\Gamma(t)$ and under the upper boundary line of the fundamental domain of the effective roots.

The objects appearing in the source terms are subjected to quantization conditions. They are as follows:

$$\frac{1}{i}\log b(\tilde{\theta}_j) = 2\pi I_j, \qquad I_j \in \mathbb{Z} + \frac{1}{2} \tag{4.8}$$

where $\tilde{\theta}_j$ may denote holes $\{h_j\}$ and close effective roots $\{c_j\}$.

$$\frac{1}{i}\log b_{II}(\tilde{\theta}_j) = 2\pi I_j, \qquad I_j \in \mathbb{Z} + \frac{1}{2} \tag{4.9}$$

$\tilde{\theta}_j$s can be wide effective roots: $\{w_j\}$ or self-conjugate effective roots: $\{w_j^{(sc)}\}$. The determination of type I holes $\{h_j^{(1)}\}$ can be achieved from the equation

$$\frac{1}{i}\log y(h_j^{(1)} - i\frac{\pi}{2}) = 2\pi I_j^{(1)}, \qquad I_j^{(1)} \in \mathbb{Z} + \frac{1}{2}. \tag{4.10}$$

The special objects $\{s_j\}$ and $\{v_j\}$ satisfy similar quantization equations, for details see ref. [14]. The energy of the spin-1 XXZ chain can be expressed by the solutions of the NLIEs as follows:

$$\begin{aligned} E &= \frac{2\pi^2}{\gamma}\Bigg\{\sum_{j=1}^{N_H} K(h_j) + \sum_{j=1}^{N_V^S}\left(K(v_j)+K(\bar{v}_j)\right) - \sum_{j=1}^{M_C} K(c_j) - \sum_{j=1}^{N_S}\left(K(s_j)+K(\bar{s}_j)\right) \\ &- \sum_{j=1}^{M_W} K_{II}(w_j) - \sum_{j=1}^{M_{sc}} K_{II}(w_{sc}^{(j)}) + \frac{i}{2\pi}\int_\Gamma d\theta\, K'(\theta)\,\ln B(\theta) - \frac{i}{2\pi}\int_{\bar{\Gamma}} d\theta\, K'(\theta)\ln\bar{B}(\theta)\Bigg\}. \end{aligned} \tag{4.11}$$

5 Typical root configurations over the 2-strings

With the help of the NLIEs we can discuss the typical root configurations other than 2-strings of the model in the large N limit, namely in what kind of configurations the large N limit forces the roots? The typical root configurations are different in the attractive and repulsive regimes of the model. It will turn out that on the effective plane the typical root configurations are the same as in the spin-1/2 case. This is due to the fact that the source terms of the NLIEs in the spin-1/2 and spin-1 chains are the same. It can be said that the of order N bulk source term $D(\theta)$ of the NLIEs forces the roots to fall into certain configurations. To sketch the mechanism first let us consider the repulsive regime. In this regime the second determination of the bulk source term is zero, so there are no restrictions on the (relative) positions of the wide effective roots. On the other hand there are bulk source terms in the quantization equations for close effective roots imposing restrictions on their relative positions as follows: Let $c_j^{\uparrow}$ a close effective root with $0 < \mathrm{Im} c_j^{\uparrow}$, then it satisfies the quantization:

$$Z_b(c_j^{\uparrow}) = 2\pi I_j, \quad Z_b(c_j^{\uparrow}) = N\,\mathrm{arctan\,sinh}(c_j^{\uparrow}) + g_b(c_j^{\uparrow}) + ..., \quad g_b(c_j^{\uparrow}) = \sum_k \chi(c_j^{\uparrow} - c_k) + \dots \tag{5.1}$$

where only the relevant part of $Z_b(\theta)$ and $g_b(\theta)$ have been written. It can be seen that the right hand side of the quantization equation is real so its imaginary part is zero. On the other hand the bulk source term of the left hand side has imaginary part of order N diverging in the large N limit ($\mathrm{Im}\left\{N \arctan\sinh(c_j^{\uparrow})\right\} \simeq O(N)$). This diverging part must be compensated by $g_b(\theta)$ as follows: the function $\chi(\theta)$ has logarithmic singularities at $\pm i\pi$, so the singularity coming from the bulk source term can be compensated by $g_b(\theta)$ if to the close effective root $c_j^{\uparrow}$ there exist another one at a distance of $i\pi$. Thus effective close roots must form pairs of the form:

$$c^{+} = c_0 + i\left(\frac{\pi}{2} + \mu\right), \quad c^{-} = c_0 - i\left(\frac{\pi}{2} - \mu\right), \quad 0 < \mu < \frac{\pi}{2}. \tag{5.2}$$

On the other hand the Bethe roots appear in complex conjugate pairs, so the close roots are forced to form quartets on the effective plane: $\{c^{+}, c^{-}, (c^{+})^{*}, (c^{-})^{*}\}$. They can degenerate ($\mu \to 0$) and form 2-strings on the effective plane ($(c^{+})^{*} = c^{-}$). To summarize the typical effective root configurations of the spin-1 BAE are 2-strings and quartets, exactly the same as those of the spin-1/2 case [12]. Then one can transform everything back to the language of original Bethe roots and gets that in the large N limit the roots fall into configurations as follows:

1. *Real roots:* $\mathrm{Im}\,\theta_j = 0$,
2. *Inner roots:* $0 < |\mathrm{Im}\,\theta_j| < \frac{\pi}{2}$,
3. *Quasi 2-strings:* $\mathrm{Im}\,\theta_j \simeq \pm\frac{\pi}{2}$,
4. *Close roots with:* $\mathrm{Im}\,\theta_j \simeq \pm\pi$,
5. *Close quartets:* $\theta^{\pm\uparrow\downarrow} = \theta_0 \pm i\,(\pi \pm \mu)\,, \quad 0 < \mu < \frac{\pi}{2},\ \theta_0 \in \mathbb{R}$,
6. *Wide roots* with arbitrary imaginary parts,
7. *Self-conjugated roots* with: $\mathrm{Im}\,\theta_j = \frac{\pi(p+2)}{2}$.

It is worth to mention that according to our results at best only 2- and 3-strings can appear in the repulsive regime of the spin- XXZ chain.

Similar argument leads to the typical rot configurations of the attractive regime. The main difference is that $D_{II}(\theta)$ is not zero anymore giving constraints on the imaginary parts of the wide roots as well. The typical root configurations on the effective plane are the same those of the spin-1/2 XXZ chain. For more details see [14].

6 2-string deviations

Having the NLIEs in hand the leading order deviations of the 2-strings can be calculated analytically with ease. To do this let us introduce the counting function:

$$\tilde{Z}_b(\theta) = \frac{1}{i\,N}\log b(\theta) \tag{6.1}$$

and parametrize the positions of quasi 2-strings as $\theta_j = x_j \pm i(\frac{\pi}{2} + \delta_j)$, where x_j and δ_j are real parameters denoting the 2-string centers and 2-string deviations respectively. Let us

assume that the deviations are small: $|\delta_j| \ll 1$. Then the quantization equations for the 2-strings take the form:

$$\tilde{Z}_b(x_j + i\delta_j) = \frac{2\pi I_j}{N}, \tag{6.2}$$

Introducing the following functions:

$$\mathcal{K}(\theta) = \lim_{\varepsilon\to 0^+}(K^{+\frac{\pi}{2}-\varepsilon} * \ln Y)(\theta), \qquad \mathcal{K}_b(\theta) = \frac{1}{i}(G *_{\Gamma} \ln B)(\theta) - \frac{1}{i}(G *_{\bar{\Gamma}} \ln \bar{B})(\theta) \tag{6.3}$$

the quantization equation (6.2) takes the form:

$$\tilde{Z}_b(\theta) = Z_0(\theta) + \frac{1}{i\,N}\mathrm{Re}\,\mathcal{K}(\theta) \tag{6.4}$$

with

$$Z_0(\theta) = \mathrm{arctan\,sinh}\,\theta + \frac{\pi\delta_b + g_1(\theta) + g_b(\theta) + \mathcal{K}_b(\theta) + \mathrm{Im}\,\mathcal{K}(\theta)}{N}\,. \tag{6.5}$$

It can be seen that $\tilde{Z}_b(\theta)$ can be written as a sum of two terms. The first $Z_0(\theta)$ is real on the real axis, so according to the quantization equations it would give exact 2-strings. The second term is purely imaginary and of order $1/N$ giving account for the 2-string deviations. Assuming that the 2-string deviations are small the quantization equations can be Taylor expanded around the string centers:

$$\frac{2\pi I_j}{N} = Z_0(x_j) + i\,\delta_j\,Z_0'(x_j) + \frac{1}{i\,N}\mathrm{Re}\,\mathcal{K}(x_j) + \frac{\delta_j}{N}(\mathrm{Re}\,\mathcal{K})'(x_j) + O(\delta^2)\,. \tag{6.6}$$

Assuming that δ_js are of order $1/N$ we get two equations:

$$\frac{2\pi I_j}{N} = Z_0(x_j), \qquad \delta_j = \frac{1}{N}\frac{\mathrm{Re}\,\mathcal{K}(x_j)}{Z_0'(x_j)}\,. \tag{6.7}$$

The first one says that $Z_0(x_j)$ is the counting function of 2-string centers and so its derivative $Z_0'(x_j)$ gives their density. The second equation expresses the 2-string deviations. It can be seen that the deviations are inversely proportional to the system size N and the density of 2-string centers, in addition the numerator of the formula given by the convolution term of the NLIE containing the TBA type auxiliary functions. Having the general formula in hand let us consider some examples.

The simplest example is the ground state. For this state in the large N limit $y(\theta) = 1$, $\mathcal{K}(\theta) = \frac{1}{2}\ln 2$ and $\mathcal{K}_b(\theta)$ is exponentially small in N. Then in accordance with the results of [7] the 2-string deviations for the ground state are given by the formula:

$$\delta_j = \frac{\ln 2}{2N}\,\cosh x_j\,. \tag{6.8}$$

From this formula it can be seen that all the 2-string deviations are positive in this state and they are of order $1/N$ in accordance with our assumptions as long as $|x_j| \lesssim \ln N$. Thus it can be seen that our approximation is valid in the middle of the root distribution and fails at the edges of it ($|x_j| > \ln N$) where the deviations are getting to be of order 1.

Let us consider as a second example a 2-hole state in the repulsive regime. Let the positions of the holes: h_1 and h_2 with $h_1 < h_2$. Than the numerator in (6.7) takes the form:

$$\mathrm{Re}\mathcal{K}(\theta) = \frac{1}{2}\ln\cosh\left(\theta - \frac{h_1+h_2}{2}\right) - \frac{1}{2}\ln\cosh\left(\frac{\theta-h_1}{2}\right) - \frac{1}{2}\ln\cosh\left(\frac{\theta-h_2}{2}\right)$$

This is not a purely positive function on the real axis, but it changes sign exactly at the positions of the two holes. It is negative between the two holes and positive elsewhere. So from this example one can see that the 2-string deviations turn from negative to positive and vice versa at the positions of the holes.

7 Conformal limit

The NLIE technique being an exact method can be used to calculate the $1/N$ corrections to the energy levels. In the large N limit the energy levels form a conformal spectrum given by the formula:

$$E \simeq \frac{2\pi^2}{\gamma}\left(-\frac{c}{12} + \Delta^+ + \Delta^-\right)\frac{1}{N}\,, \tag{7.1}$$

where c is the central charge and $\Delta^\pm$ are the left and right conformal dimensions of the underlying conformal field theory. Carrying out the lengthy calculations finally it turns out that for N even the large N limit of the spin-1 XXZ chain corresponds to a modular invariant $c = 3/2$ conformal field theory whose operator content is encoded into the partition function of Di Francesco et al. [11]:

$$\begin{aligned} Z(R) &= \frac{1}{|\eta(q)|^2}\Bigg\{(\chi_0(q)\bar{\chi}_{1/2}(q) + \chi_{1/2}(q)\bar{\chi}_0(q)) \sum_{n\in\mathbb{Z}+\frac{1}{2},\, m\in 2\mathbb{Z}+1} \\ &+ (|\chi_0(q)|^2 + |\chi_{1/2}(q)|^2) \sum_{n\in\mathbb{Z},\, m\in 2\mathbb{Z}} + |\chi_{1/16}(q)|^2 \sum_{2n-m\in 2\mathbb{Z}+1}\Bigg\} q^{\Delta^+_{n,m}}\bar{q}^{\Delta^-_{n,m}} \end{aligned} \tag{7.2}$$

where

$$\Delta^\pm_{n,m} = \frac{1}{2}\left(\frac{n}{R} \pm \frac{m}{2}R\right)^2 \tag{7.3}$$

are the conformal weights of the Gaussian part of the CFT, $\eta(q)$ is the Dedekind function, and $q = e^{2\pi i\tau}$, τ being the modular parameter. Finally the compactification radius R is related to the anisotropy parameter of the spin chain as: $R = \sqrt{\frac{p}{p+2}}$.

8 Summary

In this work we demonstrated that the NLIE technique is a useful and effective tool in the analytical study of the spin-1 XXZ chains. In the large N limit this technique allows

one to determine the typical Bethe root configurations, the deviations of 2-strings, and the $1/N$ corrections to the energy spectrum. Furthermore it also enables one to perform accurate numerical calculations.

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PT-invariance and representations of the Temperley-Lieb algebra on the unit circle

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Abstract

We present in detail a recent conjecture on self-adjoint representations of the Temperley-Lieb algebra for particular values on the unit circle. The formulation in terms of graphical calculus is emphasized and discussed for several examples. The role of PT (parity and time reversal) invariance is highlighted as it might prove important for generalizing the construction to other cases.

1 Introduction

We summarize recent novel results [1, 2] on the PT-invariant construction of self-adjoint representations of the Temperley-Lieb algebra [3]. The latter is defined as follows.

Definition 1.1 *Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q. The Temperley-Lieb algebra $TL_N(q)$ is the associative algebra over $\mathbb{C}(q)$ generated by $\{e_1, ..., e_{N-1}\}$ subject to the relations*

$$e_i^2 = -(q+q^{-1})e_i, \qquad e_i e_{i\pm 1} e_i = e_i, \qquad e_i e_j = e_j e_i \quad \textit{for} \quad |i-j| > 1 \ . \tag{1.1}$$

It is well-kown that if the indeterminate is evaluated in the real numbers, $q \in \mathbb{R}$, or takes the special root-of-unity values

$$q = \exp(i\pi/r), \qquad r = 3, 4, 5, ... \ , \tag{1.2}$$

that there exist self-adjoint representations of $TL_N(q)$ [4, 6, 7, 5].

The novel aspect we wish to highlight here are recent results [1] of the construction of self-adjoint representations for the above values on the unit circle using ideas from quasi-Hermitian quantum mechanics and PT-invariance, e.g. [8, 9, 10, 11]. This approach is physically motivated: it is a *preparatory* step to explicitly construct for *arbitrary* values $q \in \mathbb{S}^1$ representations of $TL_N(q)$ for which the following quantum Hamiltonian

$$H = \sum_{i=1}^{N-1} e_i \tag{1.3}$$

is Hermitian or self-adjoint. The latter requirement is necessary in order to allow for a sound physical interpretation of the associated quantum model. Clearly, the case when each of the Temperley-Lieb generators e_i is self-adjoint is a particular subset of this class of representations.

The case when q is evaluated on the unit circle is of special physical interest. For instance, at the aforementioned values (1.2) the model is related to the Q-state Potts model with $Q = (q+q^{-1})^2$. For the value $q = \exp(i\pi/2)$ the Hamiltonian (1.3) describes critical dense polymers on the square lattice, while for $q = \exp(2\pi i/3)$ it is related to the problem of percolation. More generally, it has been argued [12, 13] that the Hamiltonian (1.3) can be viewed as a discrete system which either in the thermodynamic limit ($N \to \infty$) or through its algebraic properties can be effectively described by logarithmic conformal field theories. While these applications are beyond the scope of this article, it needs to be stressed that they provide some of the basic motivation for the present construction using quasi-Hermiticity and PT-invariance.

As we wish to outline the basic principles of the approach we shall focus on the special case

$$q = \exp(i\pi/r), \qquad r > N, \tag{1.4}$$

where N is the number of strands and r can take *any real* values greater than N (not only integer values). This section of the unit circle - while not of immediate physical interest as it shrinks to $q = 1$ as $N \to \infty$ - is distinguished mathematically as it not only allows for a self-adjoint representation of the Temperley-Lieb algebra but also for the application of graphical calculus in terms of Kauffman diagrams [14]. This graphical formulation of the Temperley-Lieb algebra is very elegant and greatly facilitates computations. For these reasons we wish to maintain it when $q \in \mathbb{S}^1$. In this article we shall describe in more detail a construction of an inner product [2] which achieves this for the values (1.4). Before we can start our discussion we need to recall some previous results on PT-invariance and self-adjoint representations of the Temperley-Lieb algebra.

2 Review of previous results

We will concentrate on the $U_q(sl_2)$-invariant XXZ quantum spin-chain model [15, 16]. The latter model corresponds to a realisation of the Hamiltonian (1.3) in terms of the fundamental two-dimensional $U_q(sl_2)$-module. Prior to introducing it, we recall the following definition.

Definition 2.1 *The q-deformed enveloping algebra (or quantum group) $U_q(sl_2)$ is the associative algebra over $\mathbb{C}(q)$ generated by $\{E, F, K, K^{-1}\}$ subject to the relations*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ . \tag{2.1}$$

$U_q(sl_2)$ can be endowed with structure of an Hopf algebra with co-multiplication

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \tag{2.2}$$

and co-unit

$$\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K^{\pm 1}) = 1 \ . \tag{2.3}$$

There is also an antipode but we will not use it in the following.

Setting

$$V = \mathbb{C}v_+ \oplus \mathbb{C}v_- \tag{2.4}$$

we define the two-dimensional fundamental $U_q(sl_2)$-module by

$$Ev_+ = 0, \qquad Ev_- = v_+, \qquad Fv_- = 0, \qquad Fv_+ = v_-, \qquad Kv_\pm = q^{\pm 1}v_\pm \ . \tag{2.5}$$

There is a natural inner product on V given by

$$\langle v_\sigma, v_{\sigma'} \rangle = \delta_{\sigma,\sigma'}, \qquad \sigma, \sigma' = \pm 1 \ . \tag{2.6}$$

We choose the inner product to be antilinear in the first factor. Consider now the N-fold tensor product $V^{\otimes N}$ of the fundamental representation with the inner product

$$\langle v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N}, v_{\sigma'_1} \otimes \cdots \otimes v_{\sigma'_N} \rangle = \delta_{\sigma_1,\sigma'_1} \cdots \delta_{\sigma_N,\sigma'_N} \ . \tag{2.7}$$

Then the Temperley-Lieb algebra has the following matrix representation over $V^{\otimes N}$,

$$e_i \mapsto \underbrace{1 \otimes \cdots 1}_{i-1} \otimes \boldsymbol{e} \otimes \underbrace{1 \cdots \otimes 1}_{N-i-1} \ , \tag{2.8}$$

where the matrix elements $\boldsymbol{e}_{\sigma,\sigma'} := \langle v_\sigma, \boldsymbol{e} v_{\sigma'} \rangle$ of the operator $\boldsymbol{e} : V \otimes V \to V \otimes V$ are

$$(\boldsymbol{e}_{\sigma,\sigma'}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{2.9}$$

Note that with respect to the Hamiltonian (1.3) the quantum group provides a symmetry,

$$[H, U_q(sl_2)] = 0 \ .$$

This is a direct consequence of the quantum analogue of Schur-Weyl duality [18].

In the following we will now evaluate q to be a complex number. If q is real then one finds that the above product is invariant with respect to the action of the Temperley-Lieb algebra,

$$q \in \mathbb{R}: \qquad \langle v, e_i w \rangle = \langle e_i v, w \rangle, \quad v, w \in V^{\otimes N} \ . \tag{2.10}$$

For the quantum group generators one finds

$$q \in \mathbb{R}: \qquad \langle v, \Delta^{(N)}(x) w \rangle = \langle \Delta^{(N)}_{op}(x) v, w \rangle, \quad v, w \in V^{\otimes N}, \ x \in U_q(sl_2), \tag{2.11}$$

where $\Delta_{op} = \tau \circ \Delta$ is the opposite coproduct with τ being the "flip"-operator, $\tau(x \otimes y) = y \otimes x$, and

$$\Delta^{(N)} = (1 \otimes \Delta)\Delta^{(N-1)} = (\Delta \otimes 1)\Delta^{(N-1)}, \qquad \Delta^{(2)} \equiv \Delta \ . \tag{2.12}$$

The opposite coproduct $\Delta^{(N)}_{op}$ is defined analogously.

In contrast, if $q \neq \pm 1$ lies on the unit circle $\mathbb{S}^1$ the inner product is no longer invariant,

$$q \in \mathbb{S}^1, \ q \neq \pm 1: \qquad \langle v, e_i w \rangle \neq \langle e_i v, w \rangle, \quad v, w \in V^{\otimes N} \tag{2.13}$$

and the Hamiltonian (1.3) ceases to be Hermitian. On physical grounds we therefore need to introduce a new inner product which renders H Hermitian. For the values (1.2) and (1.4) it turns out that this new inner product can also be chosen to be invariant with respect to the Temperley-Lieb action [1]. The language which we are going to employ in the construction of the invariant product is physically motivated, but as we will see the associated concepts have a clear mathematical interpretation and can be generalized beyond the section (1.4) of the unit circle.

2.1 Quasi-Hermiticity and PT-invariance

We wish to construct a map $\eta : V^{\otimes N} \to V^{\otimes N}$ which has the following properties:

1. It is Hermitian, $\langle v, \eta w\rangle = \langle \eta v, w\rangle$, invertible, $\det \eta \neq 0$, and positive, $\eta > 0$.

2. It intertwines the Hamiltonian and its Hermitian adjoint with respect to the inner product (2.7),
$$\eta H = H^* \eta \, . \tag{2.14}$$

The properties listed uner (1) guarantee that the *new* inner product $\langle \cdot \, , \cdot \, \rangle_\eta : V^{\otimes N} \times V^{\otimes N} \to \mathbb{C}$ given by
$$\langle v, w\rangle_\eta := \langle v, \eta w\rangle, \qquad v, w \in V^{\otimes N} \tag{2.15}$$
is well-defined, while property (2) ensures that the Hamiltonian becomes Hermitian,
$$\langle Hv, w\rangle_\eta = \langle v, Hw\rangle_\eta, \qquad v, w \in V^{\otimes N} \, . \tag{2.16}$$

Clearly, the existence of such a map η is not guaranteed but needs to be proved and - for all practical purposes - we wish to obtain η explicitly. This has been achieved [1] so far for (1.2) and (1.4) using ideas from quantum group reduction [17]. We omit the details here and instead will explicitly state the new inner product (and with it η) for (1.4) below, after introducing a convenient graphical formalism.

Besides the above requirements, which render the Hamiltonian H quasi-Hermitian, one can impose further constraints based on certain transformation properties in connection with parity, time and spin-reversal.

Definition 2.2 *Let P (parity-reversal), T (time-reversal) and R (spin-reversal) be the involutions $V^{\otimes N} \to V^{\otimes N}$ defined in terms of the following action on the basis elements*

$$\begin{aligned}
P &: \; v_{\sigma_1} \otimes v_{\sigma_2} \cdots \otimes v_{\sigma_N} \mapsto v_{\sigma_N} \otimes v_{\sigma_{N-1}} \cdots \otimes v_{\sigma_1}, \\
T &: \; \alpha \, v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mapsto \bar{\alpha} \, v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N}, \quad \alpha \in \mathbb{C}, \\
R &: \; v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mapsto v_{-\sigma_1} \otimes \cdots \otimes v_{-\sigma_N} \, .
\end{aligned}$$

We define P, R to be linear but T to be anti-linear.

Remark. While the definition of the parity and spin-reversal operators is conceptually clear, the identification of the antilinear map T with time-reversal warrants an additional comment. One important method to construct eigenvectors of the quantum Hamiltonian (1.3) is the coordinate Bethe ansatz. The latter involves the definition of a discrete quantum mechanical wave function ψ. The map T is defined such that this wave function is transformed into its complex conjugate, $\psi \to \bar{\psi}$. This transformation of the wave function which ought to obey a discrete version of the Schrödinger equation, $i\partial_t \psi = H\psi$, corresponds to time-reversal.

A straightforward computation exploiting (2.8), (2.9) shows that the Hamiltonian (1.3) is PT and RT-invariant. Namely, we have that

$$\langle Hv, w\rangle = \langle v, PHPw\rangle = \langle v, RHRw\rangle = \langle v, THTw\rangle, \qquad v, w \in V^{\otimes N} \ . \tag{2.17}$$

These identities motivate the following additioanl requirements on the map η,

$$P\eta P = R\eta R = T\eta T = \eta^{-1} \ . \tag{2.18}$$

We shall refer to these transformations as PT and RT-invariance of the map η and its associated inner product, respectively. An immediate consequence of these relations is that

$$\det \eta = \det P\eta^{-1}P = \frac{1}{\det \eta} \ \Rightarrow \ \det \eta = 1 \ . \tag{2.19}$$

As already mentioned such a map η ensuring quasi-Hermiticity of the Hamiltonian and satisfying PT and RT-invariance does indeed exist and, moreover, can be explicitly constructed [1]. We summarize the previous results [1] in the following theorem.

Theorem 2.3 *Evaluate q in the section (1.4) of the unit circle. Then there exists a map $\eta : V^{\otimes N} \to V^{\otimes N}$ possessing the properties mentioned above and in addition enjoys the more restrictive constraints*

$$\langle e_i v, w\rangle_\eta = \langle v, e_i w\rangle_\eta, \qquad i = 1, 2, ..., N-1 \tag{2.20}$$

and

$$\langle \Delta_{op}^{(N)}(\varphi(x))v, w\rangle_\eta = \langle v, \Delta^{(N)}(x)w\rangle_\eta, \qquad x \in U_q(sl_2) \tag{2.21}$$

where $v, w \in V^{\otimes N}$ and φ is the $U_q(sl_2)$ -automorphism

$$\varphi(K^{\pm 1}) = K^{\mp 1}, \quad \varphi(E) = F, \quad \varphi(F) = E, \quad \varphi(xy) = \varphi(y)\varphi(x), \ x, y \in U_q(sl_2) \ . \tag{2.22}$$

If one explicitly computes η using the previous results in the literature [1], one quickly realizes the importance of the choice of basis. For example, if η is to be computed with respect to the basis vectors

$$\{v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mid \sigma_i = \pm 1\} \subset V^{\otimes N} \tag{2.23}$$

one finds in general none of its matrix element are nonzero within a fixed spin sector, i.e. in a subspace with $\sum_i \sigma_i =$ const. This makes an evaluation of the inner product (2.15) and computations of physically relevant quantities extremely cumbersome. We therefore shall introduce another basis which is algebraically motivated: it transforms very simply under the action of the Temperley-Lieb algebra and the quantum group. Most importantly, in this basis the action of the latter two algebras can be described graphically.

3 Change of basis and graphical calculus

The new basis, we denote it by $\{t_i\}$, is not *orthonormal* and leads to a shift in emphasis from the map $\eta : V^{\otimes N} \to V^{\otimes N}$ discussed above to the Gram matrix

$$G_{ij} = \langle t_i, \eta t_j \rangle \ . \tag{3.1}$$

The latter nicely reflects the algebraic properties of the new basis and can be evaluated by graphical means. Crucial for this graphical evaluation is the fact that there is a correspondence between basis vectors t_i and elements a_i in the Temperley-Lieb algebra. Using this correspondence one can directly define the Gram matrix in terms of a real functional

$$\omega : TL_N(q) \to \mathbb{R} \tag{3.2}$$

by setting

$$G_{ij} := \omega(a_i^* a_j) \tag{3.3}$$

where $*$ denotes a conjugation in the Temperley–Lieb algebra which corresponds to taking the Hermitian adjoint in the associated representation. The values of the functional can be computed using Kauffman diagrams. This suggests to circumvent the construction of the map η in the spin basis (2.23) entirely and instead to focus on the algebraically distinguished Gram matrix G. Obviously, all the properties of η can be translated to properties of the matrix G and it is convenient to do so because many matrix elements of G turn out to be vanishing. We shall list the properties of the Gram matrix G below. First we introduce the new basis vectors.

3.1 The new basis in terms of Young tableaux

The new basis $\{t_i\}$ which are going to define is closely related to the dual canonical basis discussed by Frenkel and Khovanov [19], see also [20] and [21] and references therein. The alert reader will notice, however, that there are important differences in our conventions from the ones used by the latter authors, since we need to accommodate that q lies on the unit circle. In particular our definition of the inner product, respectively the Gram matrix G, differs from the one for q real where the construction of Frenkel and Khovanov applies.

We start by decomposing the representation space $V^{\otimes N}$ with respect to the number of "down spins",

$$V^{\otimes N} = \bigoplus_{n=0}^{N} W_n, \qquad W_n = \operatorname*{span}_{\mathbb{C}} \left\{ v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N} \mid \sum\nolimits_i \sigma_i = N - 2n \right\} \ . \tag{3.4}$$

Note that the action of the Temperley-Lieb algebra respects this decomposition,

$$TL_N(q) W_n = W_n \ . \tag{3.5}$$

In each subspace W_n we now introduce the following basis [22]. Let λ_n be the rectangular Young diagram with n rows of $N-n$ boxes,

$$\lambda_n = \underbrace{\left.\begin{array}{|c|c|c|c|c|}\hline &&&&\\\hline &&&&\\\hline &&&&\\\hline &&&&\\\hline\end{array}\right\} n}_{N-n}$$

Then we assign to each subdiagram $\lambda' \subset \lambda_n$ a vector in W_n as follows. Let t be the unique standard tableau (column and row strict) of shape λ' whose entries are consecutive integers with entry n in the upper left corner. For example,

$$t = \begin{array}{|c|c|c|c|c|}\hline n & n+1 & n+2 & \cdots & s\\\hline n-1 & n & \cdots & s-2 & \\\cline{1-4} \vdots & & & & \\\cline{1-3} s' & & & & \\\cline{1-1}\end{array} \quad , \quad n < s < N, \quad 1 \leq s' < n \ . \tag{3.6}$$

Reading the entries of the tableau from left to right and top to bottom we set

$$t \mapsto e_{s'} e_{s'-1} \cdots e_{s-2} \cdots e_{n-1} e_s \cdots e_{n+1} e_n \Omega_n , \tag{3.7}$$

where

$$\Omega_n = \underbrace{v_- \otimes v_- \cdots \otimes v_-}_{n} \otimes v_+ \otimes v_+ \cdots \otimes v_+ \tag{3.8}$$

is the vector corresponding to $\lambda' = \varnothing$. Note that for fixed n there are as many of these tableaux as the dimension of the subspace W_n, namely $\dim W_n = \binom{N}{n}$.

Example. Let $N = 5$ and $n = 2$ then we have the following Young diagrams and tableaux:

$$t = \varnothing, \quad \begin{array}{|c|}\hline 2\\\hline\end{array}, \quad \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&3\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2&3\\\hline\end{array} .$$

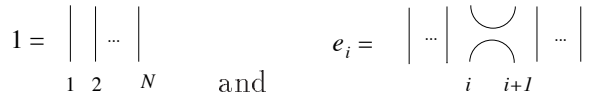
The corresponding algebra elements $a \in TL_N(q)$ are

$$a = 1,\ e_2,\ e_1e_2,\ e_3e_2,\ e_1e_3e_2,\ e_2e_1e_3e_2,\ e_4e_3e_2,\ e_1e_4e_3e_2,\ e_2e_1e_4e_3e_2,\ e_3e_2e_1e_4e_3e_2 \ .$$

Note that we do not distinguish in our notation between Young tableaux and the associated basis vectors. Henceforth, we will also often identify the basis vectors respectively tableaux with the corresponding algebra elements.

3.2 Kauffman and oriented cup diagrams

There is an elegant graphical calculus connected with the new basis. Represent each algebra element in $TL_N(q)$ in terms of Kauffman diagrams, see the graphical depiction below,



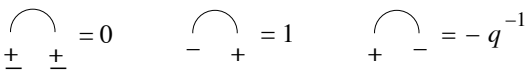
and realize the multiplication through concatenation from above. To compute the action of the algebra on the basis $\{t_i\}$ we can identify each basis vector with a half or cup-diagram which carries an orientation. Inspired by the formalism of Frenkel and Khovanov [19] we introduce the following graphical rules. Define a "cap" to be the map $\cap : V \otimes V \to \mathbb{C}$ with

$$v_+ \otimes v_+ \mapsto 0, \qquad v_+ \otimes v_- \mapsto -q^{-1}, \qquad v_- \otimes v_+ \mapsto 1, \qquad v_- \otimes v_- \mapsto 0 \tag{3.9}$$

and a "cup" $\cup$ to be the map $\cup : \mathbb{C} \to V \otimes V$ such that

$$1 \mapsto v_+ \otimes v_- - q\, v_- \otimes v_+ \ . \tag{3.10}$$

Graphically these maps are represented as follows

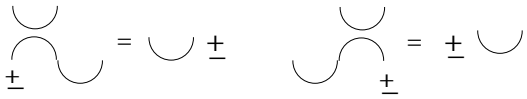


and

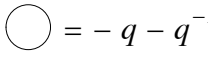
Each Temperley-Lieb generator $e_i : V_i \otimes V_{i+1} \to V_i \otimes V_{i+1}$ then corresponds to the composition

$$e_i = \cup_{i,i+1} \circ \cap_{i,i+1}, \tag{3.11}$$

where $V_i \otimes V_{i+1}$ are the i^{th} and $(i+1)^{\text{th}}$ copy in the tensor product $V^{\otimes N}$. Employing the graphical rules



and



one can now easily generate the basis $\{t_i\}$ by acting with the corresponding algebra elements $\{a_i\}$ on the cyclic vector Ω_n . Applying the same graphical rules one also easily deduces that the action of $TL_N(q)$ simply permutes the basis elements $\{t_i\}$ up to factors of $-(q + q^{-1})$.

Besides a simple action of the Temperley-Lieb algebra the new basis vectors display also a nice transformation behaviour under the action of the quantum group [19]. Notice that the spin sectors are not preserved under the $U_q(sl_2)$-action, instead we have for the quantum group generators E, F that

$$E : W_n \to W_{n-1} \qquad \text{and} \qquad F : W_n \to W_{n+1} \ . \tag{3.12}$$

We now describe the action of E. Suppose we are given a cup diagram/Young tableaux t with k_- down spins (minus signs) then

$$E = \sum_{m=1}^{k_-} [m]_q E_m, \qquad [m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}, \tag{3.13}$$

where E_m connects the m^{th} and $(m+1)^{\text{th}}$ down spin with a cup (here any intermediate cups are ignored in the counting). For $m = k_-$ the map E_k simply flips the rightmost down-spin (minus sign) to an up-spin (plus sign). Note that by construction of the basis $\{t_i\}$ all down-spins are to the left of all up-spins. Similarly, the action of F on a cup diagram with k_+ up-spins can be described in terms of the sum

$$F = \sum_{m=1}^{k_+} [m]_q F_m, \tag{3.14}$$

where F_m connects the m^{th} and $(m+1)^{\text{th}}$ up-spin and one starts counting from the right. Again, any intermediate cups are ignored and F_{k_+} simply flips the leftmost up-spin to a down spin.

Example. Let $N = 5$ and take the tableau

$$t = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} = + \cup + + \ .$$

Then we have $k_+ = 3$ and

$$F \ + \cup + + = (+ \cup \cup) + [2]_q(\Cup +) + [3]_q(- \cup ++) \ .$$

Using the above graphical calculus one can now easily verify that the matrices defined through

$$e_i t_j = \sum_k t_k (\varepsilon_i)_{kj} \tag{3.15}$$

and

$$E t_j = \sum_k t_k \mathcal{E}_{kj}, \qquad F t_j = \sum_k t_k \mathcal{F}_{kj} \tag{3.16}$$

are all real valued, but not symmetric. Because of (3.15) this particularly applies also to the matrix of the Hamiltonian (1.3),

$$H t_j = \sum_k t_k \mathcal{H}_{kj}, \qquad \mathcal{H}_{kj} \in \mathbb{R} \qquad \text{and} \qquad \mathcal{H} \neq \mathcal{H}^t \ . \tag{3.17}$$

In contrast, the Hamiltonian matrix with respect to the spin basis (2.23) is symmetric but not real valued. Thus, as before we need to introduce a new inner product in terms of the Gram matrix (3.1) with respect to which the Hamiltonian becomes Hermitian.

4 Graphical definition of the Gram matrix

We are now ready to introduce the Gram matrix with respect to the new basis defined in the previous section. Since we wish to generalize the construction described here to other values of q on the unit circle in future work (see the comments in the introduction), it is worthwhile to first formulate its general properties before specializing to the section (1.4).

The analogues of property (1) and (2) for η are:

1. G is Hermitian, $G_{ij} = \bar{G}_{ji}$, invertible, $\det G \neq 0$, and positive, $G > 0$.
2. G intertwines the matrix $\mathcal{H}$ with its transpose,

$$G\mathcal{H} = \mathcal{H}^t G \, . \tag{4.1}$$

Besides these "minimal" requirements on G we can impose the additional constraints originating from PT and RT-invariance. Namely, from the equalities (2.18) we deduce that

$$G_{ij} = \langle Tt_i, \eta^{-1} Tt_j \rangle = \langle PTt_i, \eta PTt_j \rangle = \langle RTt_i, \eta RTt_j \rangle$$

and hence

$$\pi^* G \pi = G \qquad \text{and} \qquad \rho^* G \rho = G,$$

where

$$PT\ t_i = \sum_j t_j \pi_{ji} \qquad \text{and} \qquad RT\ t_i = \sum_j t_j \rho_{ji} \, .$$

Employing Hermiticity in conjunction with time-reversal, we find that the matrix

$$\eta_{\boldsymbol{\sigma},\boldsymbol{\sigma}'} = \langle v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_N}, \eta v_{\sigma'_1} \otimes \cdots \otimes v_{\sigma'_N} \rangle$$

obeys the identities $\eta^{-1} = \eta^t = \bar{\eta} = T\eta T$ from which we conclude that

$$G_{ij} = G_{ji} \in \mathbb{R} \, . \tag{4.2}$$

Finally, G also inherits the following properties from η: G is block-diagonal with respect to the decomposition (3.4) and in addition $\det G = 1$.

We now introduce for (1.4) a Gram matrix which satisfies all of these requirements and, furthermore, is invariant under the action of the Temperley-Lieb algebra. As already hinted at previously we define G in terms of a functional over the Temperley-Lieb algebra which can be computed in terms of Kauffman diagrams. This functional is the subject of the next definition.

Definition 4.1 *Identify each $a \in TL_N(q)$ with its Kauffman diagram and fix an integer $0 \leq n \leq N$. Assign to the top and bottom of the diagram the orientation*

$$\boldsymbol{\sigma}_n = \{\underbrace{-,...,-}_{n},\underbrace{+,...,+}_{N-n}\} .$$

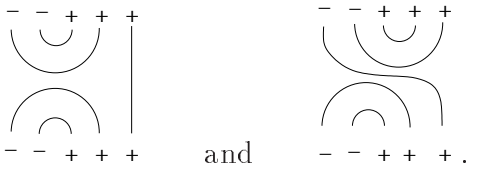
Let x be the number of anti-clockwise oriented cups

y the number of closed loops, and z the number of unoriented cups, caps or through lines,

Then we define the following functional $\omega_n : TL_N(q) \to \mathbb{R}$ by setting[1]

$$a \mapsto \omega_n(a) = \begin{cases} (-)^{x+y}(q+q^{-1})^y \dfrac{q^{\frac{N}{2}-n}+q^{n-\frac{N}{2}}}{q^{\frac{N}{2}-x}+q^{x-\frac{N}{2}}}, & \text{if } z=0 \\ 0, & \text{else} \end{cases} .$$

Example. We illustrate the above definition for two examples. Let $N = 5$, $n = 2$ and set $a = e_1e_2e_3e_2$ and $b = e_3e_2e_1e_4e_3e_2$. Then the associated oriented Kauffman diagrams are



We thus obtain

$$\omega_{n=2}(a) = 1 \qquad \text{and} \qquad \omega_{n=2}(b) = 0 .$$

Conjecture 4.2 *Let $\{t_i\}$ denote the basis of $V^{\otimes N}$ described above in terms of Young tableaux and $\{a_i\} \subset TL_N(q)$ be the corresponding algebra elements. For each $n = 0, 1, 2, ..., N$ we set*

$$G_{ij} = \langle t_i, \eta t_j \rangle := \omega_n(a_i^* a_j), \qquad \forall t_i,\ t_j \in W_n . \tag{4.3}$$

Here $* : TL_N(q) \to TL_N(q)$ is the antilinear automorphism defined by

$$(e_{i_1}e_{i_2}\cdots e_{i_k})^* = e_{i_k}e_{i_{k-1}}\cdots e_{i_1} \tag{4.4}$$

[1] Previously, we distinguished the case N odd and even [2]. However, simplifying the expression for N odd one can see that both cases are described by the same formula.

and in terms of Kauffman diagrams is realized by flipping at the horizontal axis. The matrix G satisfies all of the above identities, in particular those arising from PT-invariance, and in addition obeys the relations

$$G\mathcal{E} = \mathcal{F}^t G \qquad \textit{and} \qquad G\varepsilon_i = \varepsilon_i^t G, \qquad i = 1, ..., N-1, \tag{4.5}$$

where $\mathcal{E}, \mathcal{F}$ and ε_i are the matrix expressions for the quantum group and Temperley-Lieb generators in the basis $\{t_i\}$ as introduced earlier, see (3.16) and (3.15).

Remark. The implicit definition of the map η contained in the above expression for the Gram matrix G coincides with the earlier construction [1]. In fact, this is one way of checking the above conjecture [2]. Alternatively, one can verify the various identities and properties independently of the map η and this the point of view which we have taken here. Numerical checks of the above conjecture have been carried out for $N = 2, 3, 4, 5, 6, 7, 8$.

Note that there are simplifications in the computation of the inner product if we restrict to certain subspaces W' which are left invariant under the Temperley-Lieb action within a fixed sector W_n. Namely, set $n = \lfloor N/2 \rfloor$ (the integer part of $N/2$) and consider the subspace $W_{\max} \subset W_n$ of cup diagrams with a maximal number of cups. From the graphical calculus reviewed earlier, it is clear that

$$TL_N(q) W_{\max} = W_{\max}$$

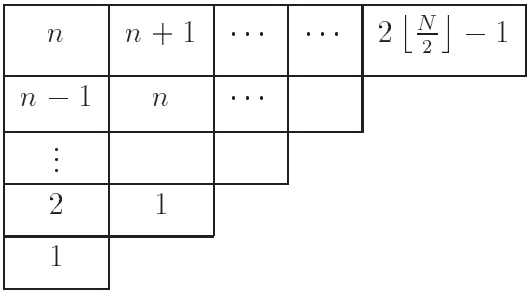
and that for N even

$$EW_{\max} = FW_{\max} = \{0\} \ .$$

If N is odd we have obviously two subspaces $W^{\pm}_{\max}$ due to one unpaired vector or spin with

$$EW^{-}_{\max} = W^{+}_{\max} \qquad \text{and} \qquad FW^{+}_{\max} = W^{-}_{\max} \ .$$

In terms of Young tableaux these subspaces are spanned by all t which contain the "staircase" tableaux



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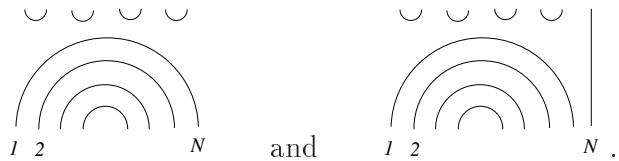
The corresponding Kauffman diagrams for N even and odd look as follows,



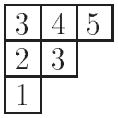
.

From this graphical representation it is apparent that for each Kauffman diagram associated with the algebra element $a_i^* a_j,\ a_i, a_j \in W_{\max}$ the number x of anti-clockwise oriented cups equals the number n of down spins (minus signs). Thus, we may for this special case compute the Gram matrix purely in terms of the upper half diagrams (= cup diagrams) as we only need to count the number of closed loops, i.e.

$$(G|_{W_{\max}})_{ij} = \omega_n(a_i^* a_j) = (-q - q^{-1})^{y_{ij}},$$

where $y_{ij} = y_{ji}$ is the number of closed loops in $a_i^* a_j$. We illustrate the comments just made for a simple example.

Example. Set $N = 7$ and $n = 3$. Then the subspace $W_{\max}$ of all half-diagrams containing 3 cups is spanned by the tableaux t which contain

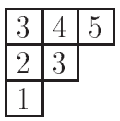


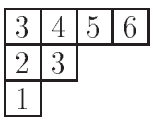
$\equiv \cup\cup\cup +$.

Here we have expressed the corresponding basis vector

$$e_1 e_3 e_2 e_5 e_4 e_3\ v_- \otimes v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_+ \otimes v_+$$

in terms of a cup diagram by omitting the lower half diagram. If we now wish to compute the scalar product between the two diagrams



$\equiv \cup\cup\cup +$ and 

$\equiv \cup\cup +\cup$

we simply need to combine them by flipping one of them at the horizontal axis and count the number of closed loops. We find

$$\langle \cup\cup\cup +, \cup\cup +\cup \rangle = (q + q^{-1})^2 .$$

Note, that this simplification of computing the Gram matrix purely in terms of half or cup-diagrams is not possible in general when computing scalar products between diagrams which differ in their number of cups. For instance, to determine the scalar product between a vector in W_n and the cyclic vector Ω_n requires the full diagram. The computation of these scalar products is necessary to ensure quasi-Hermiticity of the Hamiltonian on the entire state space $V^{\otimes N}$.

5 Discussion

The Temperley-Lieb algebra arises in the context of quantum integrable models whose dynamics and quantum statistics is described by the Hamiltonian (1.3) and we have

discussed for a special example how to render it Hermitian by constructing an appropriate inner product. These quantum integrable models are closely related to classical two-dimensional statistical mechanics models which are defined in terms of the following solution of the quantum Yang-Baxter equation,

$$R_{i,i+1}(x) = \mathbf{1} + \frac{x - x^{-1}}{xq - x^{-1}q^{-1}}\, e_i \ . \tag{5.1}$$

There are various integrable boundary conditions one can impose on a square-lattice and here we only concentrated on those which turn the model quantum group invariant. In context of the six-vertex model solution the statistical transfer matrix then reads

$$t(x) = \operatorname*{Tr}_0 K_0 R_{M,0}(x) R_{M-1,M}(x) \cdots R_{1,2}(x)^2 R_{2,3}(x) \cdots R_{M-1,M}(x) R_{M,0}(x),$$

where the boundary conditions are encoded in the only non-trivial boundary matrix

$$K = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \ .$$

Obviously, one can widen the discussion to include the transfer matrix and in the cases (1.2) and (1.4) the latter turns out to be Hermitian as well.

In comparison, different choice of products have been made in the literature, e.g. in the context of applications to logarithmic conformal field theory the choice $G_{ij} = \delta_{ij}$ leads to non-trivial Jordan blocks in the Hamiltonian and transfer matrix [12]. The results presented here show that other choices might be possible where the Hamiltonian or transfer matrix are Hermitian.

A natural extension of our discussion is to include more complicated boundary conditions or other representations of the Temperley-Lieb algebra [23]. Additional open problems are the formulation of the graphical calculus for the values (1.2) as this would provide a more convenient formalism for computations. The main hurdle to overcome is to find a graphical rule for the reduction of the state space which has to be carried out first in order to remove non-trivial Jordan blocks in the Hamiltonian. One may also wish to extend the discussion to roots of unity other than (1.2). We hope to address these questions in future work [24].

Acknowledgements

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Quantum Field Theory on Star Graphs

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Abstract

We discuss some basic aspects of quantum fields on star graphs, focusing on boundary conditions, symmetries and scale invariance in particular. We investigate the four-fermion bulk interaction in detail. Using bosonization and vertex operators, we solve the model exactly for scale invariant boundary conditions formulated in terms of the fermion current. This result is applied for deriving the charge and spin transport, which have interesting physical features.

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The QCD spin chain S matrix

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Abstract

Beisert *et al.* have identified an integrable $SU(2,2)$ quantum spin chain which gives the one-loop anomalous dimensions of certain operators in large N_c QCD. We derive a set of nonlinear integral equations (NLIEs) for this model, and compute the scattering matrix of the various (in particular, magnon) excitations.

[arXiv: 0711.2415 and Nucl. Phys. B (in press)]

Polymers, Percolation and Fusion

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Abstract

We review recent progress in obtaining the chiral fusion algebras of the logarithmic minimal models $\mathcal{LM}(p,p')$ empirically from a lattice approach. Our conjectured fusion rules are based on extensive numerical studies of the associated integrable lattice models. Although our results are general, we focus here on the fusion algebras of critical dense polymers and critical percolation. The generators of fusion are countably infinite in number but the fusion is quasi-rational in the sense that the fusion of a finite number of representations decomposes into a finite direct sum of representations. The fusion rules are commutative, associative and exhibit an $s\ell(2)$ structure but require so-called Kac representations which can be reducible yet indecomposable representations of rank 1. Our fusion rules agree with Gaberdiel and Kausch for polymers $\mathcal{LM}(1,2)$ and with Eberle and Flohr for percolation $\mathcal{LM}(2,3)$ including the appearance of indecomposable representations of rank 3.

1 Introduction

The statistical lattice models of polymers and percolation have a long history. Percolation was introduced [1] by Broadbent and Hammersley in the late fifties and polymers were modelled by de Gennes [2] and des Cloizeaux [3] in the early seventies. However, the conformal theory of polymers and percolation only emerged in the mid-eighties with the seminal work of Saleur [4] and Duplantier [5]. As statistical lattice models, these theories are very interesting because they do not have local degrees of freedom only extended degrees of freedom in the form of polymers or connectivities. As conformal field theories, these models are interesting because they admit logarithmic operators [6]. They are not rational and not unitary. They possess an infinite number of scaling operators and exhibit many unusual properties. Perhaps the most characteristic property of such logarithmic theories is that they admit indecomposable representations of the Virasoro algebra.

In recent years, a number of approaches have evolved to study logarithmic Conformal Field Theories (CFTs), namely, the supergroup approach [7], the algebraic approach [8] and the lattice approach [9]. Here we review some recent results [10] on the chiral fusion algebras of polymers and percolation coming from the lattice approach.

Statistical systems with local degrees of freedom are associated with rational CFTs not logarithmic CFTs. Starting with a lattice model, such as a Q-state Potts model or RSOS model, with local degrees of freedom typically implies that the transfer matrices are normal and diagonalizable. In turn, this implies that the Virasoro dilatation generator L_0 is diagonalizable and does not exhibit the non-trivial Jordan cells associated with indecomposability:

$$\begin{array}{ccccccccc} \text{local} & & \text{symmetric} & & \text{diagonalizable} & & \text{no rank} \geq 2 & & \text{not} \\ \text{degrees of} & \Rightarrow & \text{transfer} & \Rightarrow & \text{transfer} & \Rightarrow & \text{indecomposable} & \Rightarrow & \text{logarithmic} \\ \text{freedom} & & \text{matrices} & & \text{matrices} & & \text{representations} & & \text{theory} \end{array} \tag{1.1}$$

Since polymers and percolation are associated with logarithmic CFTs, this requires a paradigm shift to lattice models with *nonlocal* degrees of freedom:

$$\begin{array}{ccc} \text{logarithmic} & & \text{nonlocal} \\ \text{theory} & \Rightarrow & \text{degrees of} \\ & & \text{freedom} \end{array} \tag{1.2}$$

Planar algebras, such as the planar Temperley-Lieb (TL) algebra (Jones 1990 [11]), are the key to constructing Yang-Baxter integrable lattice models on the square lattice with nonlocal degrees of freedom. The family of models we consider here are called logarithmic minimal models because of parallels with the rational minimal models. In the continuum scaling limit, they yield logarithmic CFTs as in Figure 1.

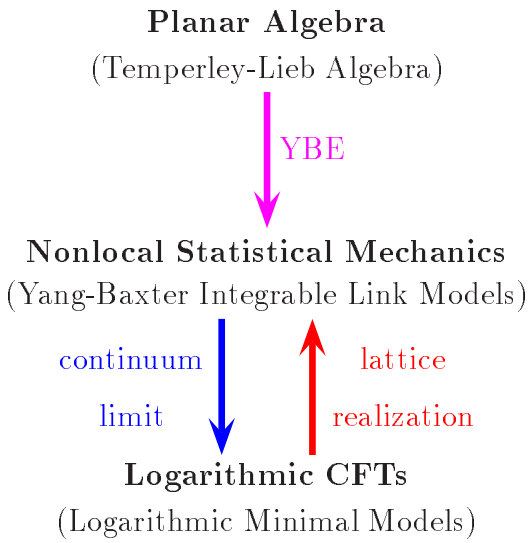


Figure 11.1: Schematic relation between the planar algebra, nonlocal statistical mechanics models and logarithmic CFTs. The relations in the planar algebra ensure integrability through the Yang-Baxter equations. The continuum scaling limit of the lattice model yields a logarithmic CFT. Alternatively, the lattice model provides a lattice realization of the logarithmic theory.

2 Logarithmic Minimal Models

Let p, p' be positive coprime integers with $p < p'$ and define the crossing parameter

$$\lambda \;=\; \frac{(p'-p)\pi}{p'} \tag{2.1}$$

The logarithmic minimal models $\mathcal{LM}(p,p')$ [9] are defined on the square lattice by face operators in the planar TL algebra

$$X(u) \;=\; \boxed{u} \;=\; \frac{\sin(\lambda-u)}{\sin\lambda}$$  $$+\; \frac{\sin u}{\sin\lambda}$$  (2.2)

where u is a spectral parameter. An elementary face is thus in one of the two possible configurations with the indicated weights (a priori probabilities). The arcs represent segments of a polymer or connectivities. The two diagrams are generators of the planar TL algebra. The relations of the planar TL algebra ensure that each closed loop gets a weight given by the nonlocal loop fugacity

$$\beta \;=\; 2\cos\lambda \tag{2.3}$$

The first two members of the logarithmic minimal models are critical dense polymers $\mathcal{LM}(1,2)$ with $\lambda = \pi/2$, $\beta = 0$ and critical percolation $\mathcal{LM}(2,3)$ with $\lambda = \pi/3$, $\beta = 1$. Typical configurations on a strip for polymers and percolation are shown in Figures 2 and 3 respectively.

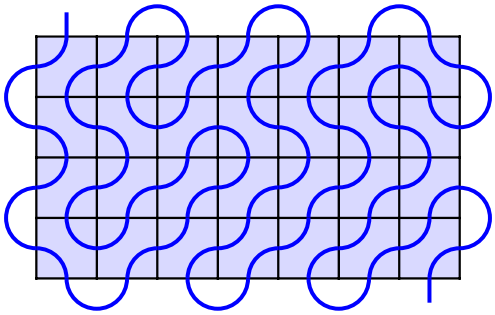


Figure 11.2: Typical configuration of dense polymers $\mathcal{LM}(1,2)$ on a strip. Since $\beta = 0$, no closed loops are allowed. Here the boundary conditions are chosen such that the single polymer enters at the top left and exits at the bottom right. Since the polymer passes uniformly through every face, the fractal dimension of the limiting curve in the continuum scaling limit is $d = 2$ so the polymer is space filling (dense).

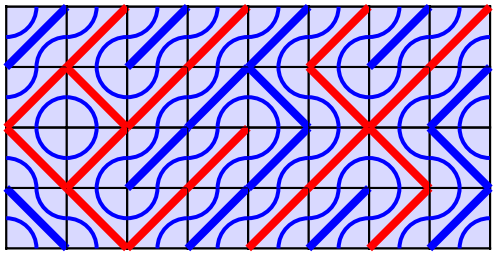


Figure 11.3: Typical configuration of critical percolation $\mathcal{LM}(2,3)$ on a strip. Since $\beta = 1$, there are no nonlocal loop weights and the model reduces to a local stochastic process. The model is isotropic if $u = \lambda/2 = \pi/6$. In this case, the appropriately normalized weights $\sin(\lambda - u) = \sin u = \frac{1}{2} = p_c$ coincide with the critical bond probability of bond percolation on the square lattice. The two sublattices of the square lattice are associated with blue and red nodes which are joined respectively by blue or red bonds according to the configuration of the loops. Ignoring the red bonds, we see that the model is equivalent (in the sense of a 1-1 mapping between configurations) to bond percolation on the blue square lattice which is rotated 45 degrees from the original square lattice.

In the continuum scaling limit, the central charges of the associated CFTs are given by

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'} \tag{2.4}$$

The conformal weights are

$$\Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'}, \qquad r,s = 1,2,3,\ldots \tag{2.5}$$

with associated characters

$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)} \tag{2.6}$$

where q is the modular parameter. The Kac tables of critical dense polymers with $c = -2$ and critical percolation with $c = 0$ are shown in Figure 4.

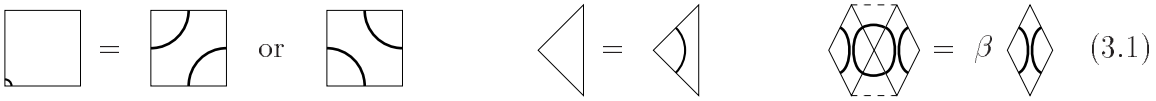
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Figure 11.4: Infinitely extended Kac tables of conformal weights for critical dense polymers with $c = -2$, $\Delta_{r,s} = \frac{(2r-s)^2-1}{8}$ and critical percolation with $c = 0$, $\Delta_{r,s} = \frac{(3r-2s)^2-1}{24}$. The Kac representations which happen to be irreducible are marked with a red quadrant.

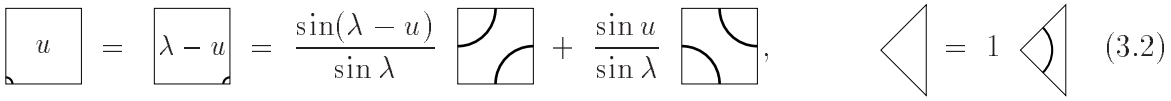
3 Temperley-Lieb Algebra and Yang-Baxter Integrable Boundaries

The planar Temperley-Lieb algebra is a *diagrammatic algebra* generated by elementary 2-boxes (oriented monoids) and elementary 1-triangles which are needed to apply boundary conditions on a strip



(3.1)

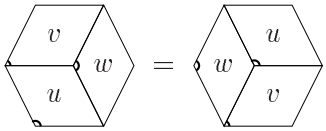
The 2-boxes and 1-triangles occur with “probabilities" or “weights" given by



(3.2)

Two N-tangles are equal if they have the same connectivities with the same weights. An example 3-tangle is shown in Figure 5.

The Yang-Baxter Equations (YBE) express the equality of two planar 3-tangles



$$w = v - u \tag{3.3}$$

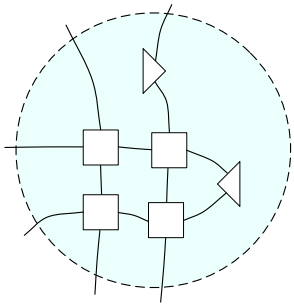
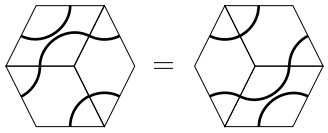
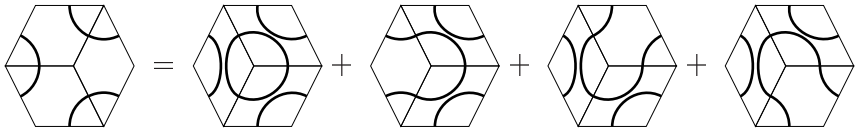


Figure 11.5: Example of a 3-tangle in the planar TL algebra. Any 3 consecutive strings can be taken as "in-states", the other 3 are then "out-states". As a planar operator, the 3-tangle can act in "6 different directions".

The five possible connectivities of the external nodes give the diagrammatic equations



$\times$ 3 (120° rotations) (3.4)



$\times$ 2 (180° rotations) (3.5)

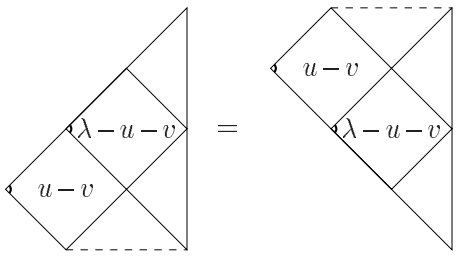
The first equation is trivial. The second equation follows from the identity

$$\begin{aligned} s_1(-u)s_0(v)s_1(-w) \;&=\; \beta\, s_0(u)s_1(-v)s_0(w) \;+\; s_0(u)s_1(-v)s_1(-w) \\ &\quad +\; s_1(-u)s_1(-v)s_0(w) \;+\; s_0(u)s_0(v)s_0(w) \end{aligned} \tag{3.6}$$

where

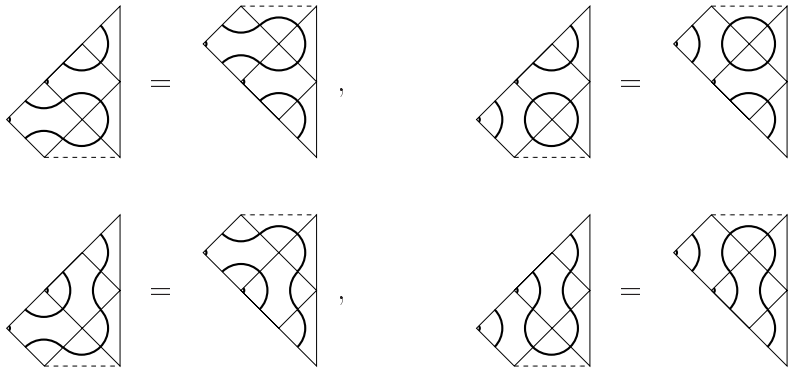
$$s_r(u) = \frac{\sin(u + r\lambda)}{\sin\lambda}, \qquad \beta \;=\; 2\cos\lambda \;=\; \text{loop fugacity} \tag{3.7}$$

The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary 2-tangles



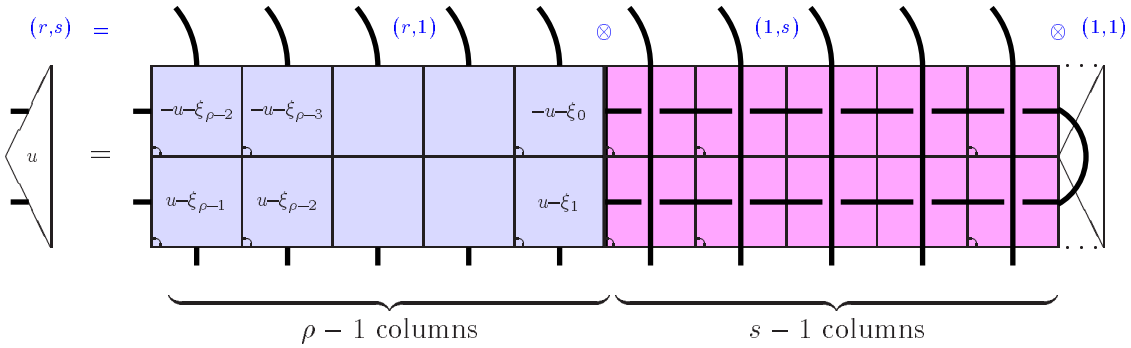
(3.8)

For the elementary 1-triangle, this follows from four identities among the weights



(3.9)

For $r, s = 1, 2, 3, \ldots$, the $(r, s) = (r, 1) \otimes (1, s)$ BYBE solution is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ triangle:

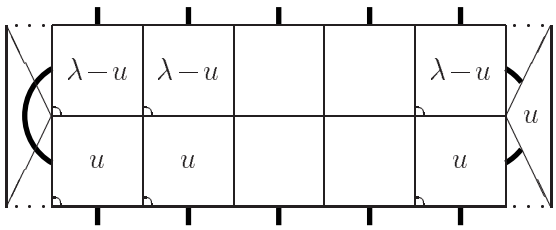


(3.10)

Here the column inhomogeneities are $\xi_k = (k + k_0 + \frac{1}{2})\lambda$ and there is at least one choice of the integers ρ and k_0 for each r. The $\rho + s - 2$ columns are considered part of the right boundary. The arches at the top close to the left with up to $\rho + s - 2$ defects propagating in the bulk. Some of the s-arches can close with some of the r-arches. Left boundary solutions (r', s') are constructed similarly.

For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$\vec{D}(u) =$



(3.11)

(3.12)

Using the Yang-Baxter (YBE) and Boundary Yang-Baxter Equations (BYBE) in the planar Temperley-Lieb (TL) algebra, it can be shown that, for any (r, s), these commute

$$\vec{D}(u)\vec{D}(v) = \vec{D}(v)\vec{D}(u) \tag{3.13}$$

Multiplication is vertical concatenation of diagrams, equality is the equality of N-tangles. Crossing symmetry can also be shown in the planar TL algebra

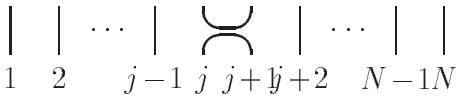
$$\vec{D}(u) = \vec{D}(\lambda - u) \tag{3.14}$$

In the case of only one non-trivial boundary condition, the transfer matrices are found to be diagonalizable. For fusion, we take non-trivial boundary conditions on the left and right $(r',s')\otimes(r,s)$. In this case, the transfer matrices can exhibit non-trivial Jordan cells and are not in general diagonalizable.

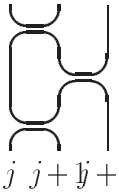
The linear TL algebra is obtained by fixing the in- and out-states, that is a distinguished direction of transfer, in the planar TL algebra. It is generated by $e_1,\ldots,e_{N-1}$ and the identity I acting on a vector space of link states on N strings

$$\begin{cases} e_j^2 \;=\; \beta\, e_j, \\ e_j\, e_k\, e_j \;=\; e_j, \qquad |j-k|=1, \qquad j,k=1,2,\ldots,N-1; \qquad \beta=2\cos\lambda \\ e_j\, e_k \;=\; e_k\, e_j, \qquad |j-k|>1 \end{cases} \tag{3.15}$$

The TL generators e_j are represented graphically by *monoids*

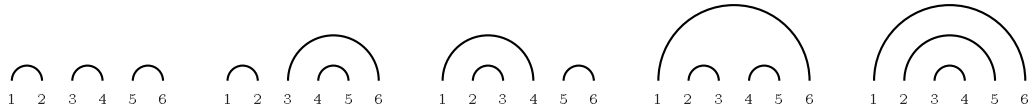
$e_j \;=$ 

(3.16)

$e_j^2 \;=$  $=\;\beta$  $=\;\beta\, e_j,$ $\qquad e_je_{j+1}e_j \;=$  $=$  $=\; e_j$ (3.17)

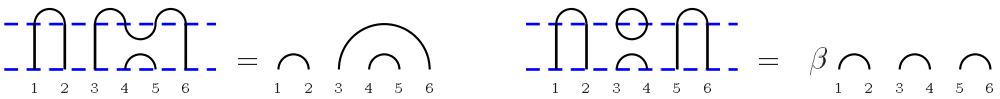
It is necessary to act on a vector space of states to obtain *matrix representatives* of the operators in the planar algebra and their associated *spectra*. For this purpose, the double-row transfer matrices can all be expressed in terms of the linear TL algebra.

The planar N-tangles act on a vector space $\mathcal{V}_N$ of *planar link diagrams*. The dimension of $\mathcal{V}_N$ is given by Catalan numbers. For $N=6$, there is a basis of 5 link diagrams:



(3.18)

The first link diagram is the vacuum state. The excited states are generated by the action of the TL generators by concatenation from below



etc. (3.19)

The action of the TL generators on the states is inherently nonlocal. It leads to *non-symmetric matrices* with entries $0,1,\beta$ that represent the TL generators. For $N=6$, the action of e_1 and e_2 on $\mathcal{V}_6$ is

$$e_1=\begin{pmatrix} \beta & 0 & 1 & 0 & 1 \\ 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.20}$$

Despite the symmetry of the monoid diagrams, the matrices e_j break time-reversal symmetry! They are not transpose symmetric.
More generally, an element of the vector space of states $\mathcal{V}_N^{(\ell)}$ contains $\ell = \rho + s - 2$ *defects*:

$N = 4,\ \ell = 2:$ 

(3.21)

The ℓ defects can be closed on the right or the left. In this way, the number of defects propagating in the bulk is controlled by the boundary conditions. In particular, for $(1, s)$ boundary conditions, the defects simply propagate along the right boundary as shown in Figure 6.

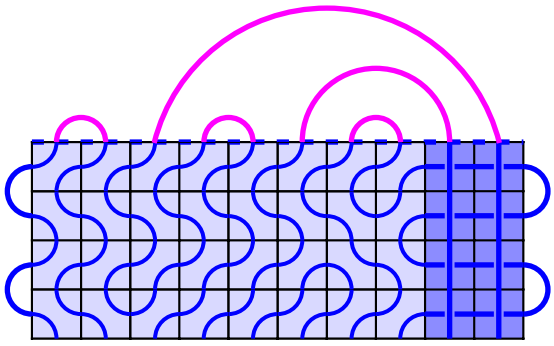
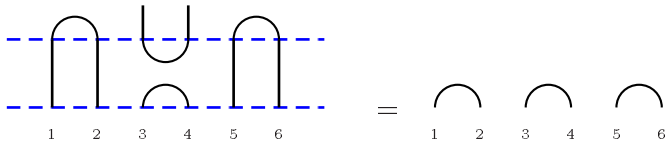


Figure 11.6: Typical strip configuration with 2 defects propogating in the bulk. The defects are closed on the right and propogate along the right boundary.

Defects in the bulk can be annihilated in pairs but not created under the action of TL



(3.22)

The transfer matrices are thus block-triangular with respect to the number of defects.

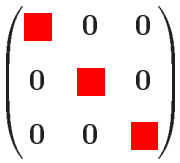
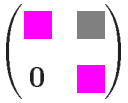
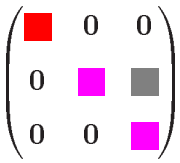
4 Virasoro Representations and Fusion

In the continuum scaling limit, each logarithmic minimal model gives rise to a CFT

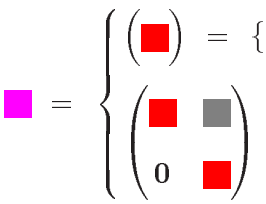
$$\vec{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \ \mapsto\ L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \operatorname{Tr} \vec{D}(u)^P \mapsto q^{-c/24} \operatorname{Tr} q^{L_0} = \chi_{(r,s)}(q) \tag{4.1}$$

An important step in understanding these CFTs is to describe their representation content. For the usual rational theories, irreducible representations are the physically relevant representations and are the building blocks for fusion. For these theories, fusion closes on the irreducible representations and there are no higher-rank indecomposable representations. In stark contrast, for logarithmic theories, Kac representations are the physically relevant representations and building blocks for fusion. Higher-rank indecomposable

representations, which are a characteristic of logarithmic theories, arise from fusing Kac representations. More explicitly, the fusion products $(r',s')\otimes(r,s)$ of Kac representations corresponding to non-trivial boundaries on the left and right of the strip lead to higher-rank indecomposable representations of the Virasoro algebra.

	Rational Theories		**Logarithmic Theories**	
Type	Irreducible	Fully Reducible	(Reducible but) Indecomposable	Decomposable
				
L_0	Diagonalizable	Diagonalizable	Jordan Cells of Rank 1, 2 or 3	Jordan Cells

The Kac representations can also be viewed as subrepresentations of the indecomposable representations generated by fusion. The Kac representations are either irreducible, as indicated in the Kac table in Figure 4, or a rank-1 indecomposable representation:


$$= \{\text{Irreducible}\} \quad = \{\text{Indecomposable}\} \qquad (L_0 \text{ Diagonalizable} = \text{Rank } 1) \tag{4.2}$$

A rank-1 indecomposable representation is an indecomposable representation in which L_0 is diagonalizable.

Let us discuss a simple example of fusion on the lattice. For *critical dense polymers*, the $(1,2)\otimes(1,2)$ fusion yields an indecomposable representation. For $N=4$, the finitized partition function is (q = modular parameter)

$$Z^{(N)}_{(1,2)|(1,2)}(q) = \underbrace{\chi^{(N)}_{(1,1)}(q)}_{0 \text{ defects}} + \underbrace{\chi^{(N)}_{(1,3)}(q)}_{2 \text{ defects}} = q^{-c/24}[(1+q^2)+(1+q+q^2)] = q^{-c/24}(2+q+2q^2) \tag{4.3}$$

The Hamiltonian $\mathcal{H}$, given by

$$\vec{D}(u) \sim e^{-2u\mathcal{H}/\sin\lambda} \qquad -\mathcal{H} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) + \sqrt{2}\, I \qquad -\mathcal{H} \mapsto L_0 - \tfrac{c}{24} \tag{4.4}$$

acts on the five states with $\ell = 0$ or $\ell = 2$ defects

(4.5)

The Jordan canonical form for $\mathcal{H}$ has rank-2 Jordan cells

$$-\mathcal{H} \sim \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & \sqrt{8} & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right) \sim \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{8} & 1 \\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right) \sim \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) = L_0^{(4)}$$

The eigenvalues of $-\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$ as $N \to \infty$.

5 Critical Dense Polymer Fusion Algebra

The fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\big\langle (2,1),(1,2) \big\rangle \;=\; \big\langle (r,1),(1,2k),\mathcal{R}_{1,2k};\; r,k \in \mathbb{N} \big\rangle \tag{5.1}$$

With the identifications $(k,2k') \equiv (k',2k)$, the fusion rules obtained empirically from the lattice are commutative, associative and agree with Gaberdiel and Kausch (1996) [12]

$$\begin{aligned}
(r,1)\otimes(r',1) &= \bigoplus_{j=|r-r'|+1,\text{ by }2}^{r+r'-1} (j,1) \\
\hline
(1,2k)\otimes(1,2k') &= \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_{1,2j} \\
(1,2k)\otimes\mathcal{R}_{1,2k'} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j) \\
\mathcal{R}_{1,2k}\otimes\mathcal{R}_{1,2k'} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\,\mathcal{R}_{1,2j} \\
\hline
(r,1)\otimes(1,2k) &= \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} (1,2j) = (r,2k) \\
(r,1)\otimes\mathcal{R}_{1,2k} &= \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} \mathcal{R}_{1,2j}
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&\mathcal{R}_{1,2k} = \text{rank-2 indecomposable} = (1,2k-1) \oplus_i (1,2k+1), \\
&\delta^{(2)}_{j,\{k,k'\}} = 2 - \delta_{j,|k-k'|} - \delta_{j,k+k'}
\end{aligned} \tag{5.3}$$

6 Critical Percolation Fusion Algebra

We build up the fusion algebra of critical percolation in three steps:

$$\big\langle (2,1) \big\rangle, \quad \big\langle (1,2) \big\rangle, \quad \big\langle (2,1),(1,2) \big\rangle \tag{6.1}$$

We call these algebras the *horizontal*, *vertical* and *fundamental* fusion algebras. These fusion algebras are all commutative and associative. Since the spectra and characters are blind to indecomposability, the fusion rules are compatiable with the underlying $s\ell(2)$ structure at the level of spectra and characters. We use the generalized Kronecker delta functions

$$\begin{aligned} \delta^{(2)}_{j,\{k,k'\}} &= 2 - \delta_{j,|k-k'|} - \delta_{j,k+k'} \\ \delta^{(4)}_{j,\{k,k'\}} &= 4 - 3\delta_{j,|k-k'|-1} - 2\delta_{j,|k-k'|} - \delta_{j,|k-k'|+1} - \delta_{j,k+k'-1} - 2\delta_{j,k+k'} - 3\delta_{j,k+k'+1} \end{aligned} \tag{6.2}$$

We identify the following representations:

$$\text{Kac:} \quad (r,-s) \equiv (-r,s) \equiv -(r,s), \qquad\qquad \text{Indecomposable:} \quad \mathcal{R}^{a,b}_{-r,s} \equiv \mathcal{R}^{a,b}_{r,-s} \equiv -\mathcal{R}^{a,b}_{r,s} \tag{6.3}$$

so that

$$(0,s) \equiv (r,0) \equiv \mathcal{R}^{a,b}_{0,s} \equiv \mathcal{R}^{a,b}_{r,0} \equiv 0 \tag{6.4}$$

The *horizontal* fusion algebra is generated by the Kac representation $(2,1)$

$$\big\langle (2,1) \big\rangle = \big\langle (2k,1),\ \mathcal{R}^{1,0}_{2k,1};\ k \in \mathbb{N} \big\rangle \tag{6.5}$$

Its closure requires the Kac representations $(2k,1)$ and the indecomposable representations

$$\mathcal{R}^{1,0}_{2k,1} = \text{rank-2 indecomposable} = (2k-1,1) \oplus_i (2k+1,1) \tag{6.6}$$

The fusion rules are

$$\begin{aligned} (2k,1) \otimes (2k',1) &= \bigoplus_{j=|k-k'|+1,\ \text{by } 2}^{k+k'-1} \mathcal{R}^{1,0}_{2j,1} \\ (2k,1) \otimes \mathcal{R}^{1,0}_{2k',1} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} (2j,1) \\ \mathcal{R}^{1,0}_{2k,1} \otimes \mathcal{R}^{1,0}_{2k',1} &= \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \mathcal{R}^{1,0}_{2j,1} \end{aligned} \tag{6.7}$$

This fusion algebra does not contain an identity.
The *vertical* fusion algebra is generated by the Kac representation $(1,2)$

$$\big\langle (1,2) \big\rangle = \big\langle (1,1),\ (1,2),\ (1,3k),\ \mathcal{R}^{0,b}_{1,3k};\ \ b=1,2;\ k \in \mathbb{N} \big\rangle \tag{6.8}$$

Its closure requires the Kac representations $(1,1)$ and $(1,3k)$ and the indecomposable representations

$$\mathcal{R}^{1,b}_{1,3k} \;=\; \text{rank-2 indecomposable} \;=\; (1,3k-b)\oplus_i (1,3k+b) \tag{6.9}$$

With $(1,1)$ = identity, the fusion rules are

$$\begin{aligned}
(1,2)\otimes(1,2) &= (1,1)\oplus(1,3)\\
(1,2)\otimes(1,3k) &= \mathcal{R}^{0,1}_{1,3k}\\
\hline
(1,2)\otimes\mathcal{R}^{0,1}_{1,3k} &= \mathcal{R}^{0,2}_{1,3k}\oplus 2(1,3k)\\
(1,2)\otimes\mathcal{R}^{0,2}_{1,3k} &= \mathcal{R}^{0,1}_{1,3k}\oplus(1,3k-3)\oplus(1,3k+3)\\
\hline
(1,3k)\otimes(1,3k') &= \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}\big(\mathcal{R}^{0,2}_{1,3j}\oplus(1,3j)\big)
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
(1,3k)\otimes\mathcal{R}^{0,1}_{1,3k'} &= \Big(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} 2\mathcal{R}^{0,1}_{1,3j}\Big)\oplus\Big(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'}\delta^{(2)}_{j,\{k,k'\}}(1,3j)\Big)\\
(1,3k)\otimes\mathcal{R}^{0,2}_{1,3k'} &= \Big(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'}\delta^{(2)}_{j,\{k,k'\}}\mathcal{R}^{0,1}_{1,3j}\Big)\oplus\Big(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}2(1,3j)\Big)
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}^{0,1}_{1,3k}\otimes\mathcal{R}^{0,1}_{1,3k'} &= \Big(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'}\delta^{(2)}_{j,\{k,k'\}}\mathcal{R}^{0,1}_{1,3j}\Big)\oplus\Big(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}\big(2\mathcal{R}^{0,2}_{1,3j}\oplus 4(1,3j)\big)\Big)\\
\mathcal{R}^{0,1}_{1,3k}\otimes\mathcal{R}^{0,2}_{1,3k'} &= \Big(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}2\mathcal{R}^{0,1}_{1,3j}\Big)\oplus\Big(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'}\delta^{(2)}_{j,\{k,k'\}}\big(\mathcal{R}^{0,2}_{1,3j}\oplus 2(1,3j)\big)\Big)\\
\mathcal{R}^{0,2}_{1,3k}\otimes R^{0,2}_{1,3k'} &= \Big(\bigoplus_{j=|k-k'|,\text{ by }2}^{k+k'}\delta^{(2)}_{j,\{k,k'\}}\mathcal{R}^{0,1}_{1,3j}\Big)\oplus\Big(\bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}2\mathcal{R}^{0,2}_{1,3j}\Big)\\
&\qquad\oplus\Big(\bigoplus_{j=|k-k'|-1,\text{ by }2}^{k+k'+1}\delta^{(4)}_{j,\{k,k'\}}(1,3j)\Big)
\end{aligned} \tag{6.11}$$

The subalgebra $\big\langle(1,1),(1,6k-3),\mathcal{R}^{0,1}_{1,6k},\mathcal{R}^{0,2}_{1,6k-3};\ k\in\mathbb{N}\big\rangle$ agrees with Read and Saleur (2007) [13].

The *fundamental* fusion algebra is generated by $(1,2)$ and $(2,1)$

$$\big\langle (1,2),(2,1)\big\rangle = \big\langle (1,1),(1,2),(2k,1),(1,3k),(2k,3),\mathcal{R}^{1,0}_{2k,1},\mathcal{R}^{0,b}_{1,3k},\mathcal{R}^{1,0}_{2k,3},\mathcal{R}^{0,b}_{2k,3},\mathcal{R}^{1,b}_{2k,3}\big\rangle$$
$$b=1,2;\quad k\in\mathbb{N} \tag{6.12}$$

In addition to the Kac and rank-2 indecomposable representations, its closure requires rank-3 indecomposable representations (dim = 4 = rank-3 + single state)

$$\mathcal{R}^{1,b}_{2k,3} \;=\; (2k-1,3-b)\oplus_i(2k-1,3+b)\oplus_i(2k+1,3-b)\oplus_i(2k+1,3+b) \tag{6.13}$$

The fundamental fusion algebra factors over the horizontal and vertical fusion algebras:

$$\begin{aligned}
(2k,3k') &= (2k,1)\otimes(1,3k') = \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}(2j,3)\\
\mathcal{R}^{1,0}_{2k,3k'} &= \mathcal{R}^{1,0}_{2k,1}\otimes(1,3k') = \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}\mathcal{R}^{1,0}_{2j,3}\\
\mathcal{R}^{0,b}_{2k,3k'} &= (2k,1)\otimes\mathcal{R}^{0,b}_{1,3k'} = \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}\mathcal{R}^{0,b}_{2j,3}\\
\mathcal{R}^{1,b}_{2k,3k'} &= \mathcal{R}^{1,0}_{2k,1}\otimes\mathcal{R}^{0,b}_{1,3k'} = \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1}\mathcal{R}^{1,b}_{2j,3}
\end{aligned} \tag{6.14}$$

Consequently, we have the further identifications

$$(2k,3k')\equiv(2k',3k)\quad \mathcal{R}^{1,0}_{2k,3k'}\equiv\mathcal{R}^{1,0}_{2k',3k}\quad \mathcal{R}^{0,b}_{2k,3k'}\equiv\mathcal{R}^{0,b}_{2k',3k}\quad \mathcal{R}^{1,b}_{2k,3k'}\equiv\mathcal{R}^{1,b}_{2k',3k} \tag{6.15}$$

The fundamental fusion is completely determined by commutativity, associativity and the factorization into horizontal and vertical fusions. It also agrees with Eberle and Flohr (2006) [14]. As an illustrative example, consider the fusion of two rank-2 indecomposable

representations:

$$\begin{aligned}
\mathcal{R}^{1,0}_{2k,2} \otimes \mathcal{R}^{0,2}_{2,3k'} &= \big(\mathcal{R}^{1,0}_{2k,1} \otimes (1,2)\big) \otimes \big((2,1) \otimes \mathcal{R}^{0,2}_{1,3k'}\big) \\
&= \big(\mathcal{R}^{1,0}_{2k,1} \otimes (2,1)\big) \otimes \big((1,2) \otimes \mathcal{R}^{0,2}_{1,3k'}\big) \\
&= \big((2k-2,1) \oplus 2(2k,1) \oplus (2k+2,1)\big) \otimes \big(\mathcal{R}^{0,1}_{1,3k'} \oplus (1,3k'-3) \oplus (1,3k'+3)\big) \\
&= \big(\mathcal{R}^{0,1}_{2k-2,3k'} \oplus 2\,\mathcal{R}^{0,1}_{2k,3k'} \oplus \mathcal{R}^{0,1}_{2k+2,3k'}\big) \\
&\quad \oplus \big((2k-2,3k'-3) \oplus 2\,(2k,3k'-3) \oplus (2k+2,3k'-3)\big) \\
&\quad \oplus \big((2k-2,3k'+3) \oplus 2\,(2k,3k'+3) \oplus (2k+2,3k'+3)\big) \\
&= \Big(\bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}} \mathcal{R}^{0,1}_{2j,3}\Big) \oplus \Big(\bigoplus_{j=|k-k'|-1}^{k+k'+1} \delta^{(4)}_{j,\{k,k'\}} (2j,3)\Big)
\end{aligned} \tag{6.16}$$

7 Conclusions

Critical dense polymers and critical percolation are the first two members of the Yang-Baxter integrable series of logarithmic minimal models. On the strip, an infinity of integrable boundary conditions labelled by $r, s = 1, 2, 3, \ldots$ are conjugate to scaling fields in an infinitely extended Kac table. In the continuum scaling limit, these give rise to so-called Kac representations (r, s) of the Virasoro algebra which are not in general irreducible. Fusion of two representations is implemented on the lattice by taking the integrable boundary conditions associated to the representations on the left and right boundaries of the strip. Fusion of the Kac representations (r, s) gives rise to indecomposable representations with rank-2 or rank-3 Jordan cells. The fundamental fusion rules for critical dense polymers and critical percolation have been obtained empirically by studying fusion on the lattice. These fusion rules are closed, commutative, associative and compatible (at the level of spectra and characters) with an underlying $s\ell(2)$ structure. The fusion rules are *quasi-rational* in the sense of Nahm (1994) [15], that is, fusion of any two representations decomposes into a finite sum of representations. Our fusion rules agree with and generalize the fusion rules of Gaberdiel-Kausch (1996) [12], Read-Saleur (2007) [13] and Eberle-Flohr (2006) [14]. Our empirical fundamental fusion rules have also been generalized [10] to all of the logarithmic minimal models. In all cases, only indecomposable representations (Jordan cells) of rank 1, 2 or 3 occur.

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Non-linear integral equations in $\mathcal{N} = 4$ SYM

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Abstract

We survey and discuss the applications of the non-linear integral equation in the framework of the Bethe Ansatz type equations which are conjectured to give the eigenvalues of the dilatation operator in $\mathcal{N} = 4$ SYM. Moreover, an original idea (different from that of [1]) to derive a non-linear integral equation is briefly depicted in Section 4.

Keywords: Integrability; counting function; SYM theories; non-linear integral equation.

1 Introduction

The AdS/CFT correspondence [2] states the equivalence between a string theory on the curved space-time $AdS_5 \times S^5$ and a conformal quantum field theory on its boundary. In particular, type IIB superstring theory should be dual to $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) in four dimensions: energies of string states correspond to anomalous dimensions of local gauge invariant operators of the quantum field theory.

One of the most important recent development in this context was the discovery of integrability in both planar field theory [3] and string theory [4]. In a nutshell, integrable models appear as spin chain type Bethe equations to be satisfied by 'rapidities' which parametrise on the one side [5, 6, 7] composite operators and their anomalous dimensions in planar $\mathcal{N} = 4$ SYM and on the other side [4, 8, 9, 10, 7] the corresponding dual objects in string theory, i.e. states and their energies, respectively.

Thanks to this discovery, one could start to use the powerful technique of the Bethe Ansatz in order to compute anomalous dimensions of long operators, giving incredible boost towards a proof of the AdS/CFT correspondence. However, the majority of the results (with the exception, to our knowledge, of [11]) concerns the calculation of only the leading term of anomalous dimensions of arrays of L operators in the limit $L \to \infty$. In order to study the (physically relevant) operators with a finite number L of components, in a series of papers [12, 13, 14] we proposed to interpret the Bethe equations in terms of the non-linear integral equation (NLIE). The NLIE [1] allows to write exact expressions for the eigenvalues of the observables for arbitrary values of the length of the chain (L in this context) and the number of Bethe roots. This allows to perform numerical evaluations for such eigenvalues as well as, in some cases, to give explicit analytic expressions for them, at least in some particular limits.

In this contribution, we want to discuss our main results on the application of the NLIE technique in the context of $\mathcal{N} = 4$ SYM and sketch at the end (Section 4) a new approach to and kind of NLIE. We start (Section 2) from the so-called BDS model, which historically [5] was the first proposal for the many loops asymptotic (i.e. up to the order λ^{L-1}, λ being the 't Hooft coupling) description of anomalous dimensions in the $SU(2)$ sector. In this case, the NLIE and the exact expressions for the eigenvalues of the observables [12] have the same structure as in models studied in the past. This is a consequence of two facts. First, the scattering matrix between magnons, which appears on the right hand side of the Bethe equations, depends only on the difference of their rapidities (Bethe roots). Second, for the state at exam the Bethe roots completely fill the real axis and allow only the presence of a finite number of holes. However, at least one of these properties fails when considering general states, or other sectors of $\mathcal{N} = 4$ SYM or when higher loops corrections are improved by introducing the so-called dressing factor [7]. In Section 3 we show how the NLIE and the corresponding expressions for the eigenvalues of the observables [14] in the $SU(2)$ sector assume a new and more complicated form as a consequence of the failure of the first of previous properties, due to the presence of the dressing factor. In order

to cope with the intriguingly intricated structure of the Bethe equations and solutions appearing in $\mathcal{N}=4$ SYM, in Section 4 we propose a new path to a NLIE substantially different from the idea of [1]. This procedure is effective when the magnon scattering matrix has a general dependence on the rapidities and the Bethe roots are concentrated on intervals of the real axis or complex lines.

2 The NLIE for the BDS model

We start by considering the $SU(2)$ scalar sector of planar $\mathcal{N}=4$ Super Yang-Mills. In a first proposal [5], composite operators (i.e. arrays of L scalar fields) were parametrised by rapidities $\{u_k\}_{k=1,\dots,M}$ satisfying the asymptotic [1] Bethe-type equations

$$\left[\frac{X(u_j+\frac{i}{2})}{X(u_j-\frac{i}{2})}\right]^L = \prod_{\substack{k=1\\k\neq j}}^{M} \frac{u_j-u_k+i}{u_j-u_k-i}\,, \tag{2.1}$$

where we introduced the function

$$X(u) = \frac{u}{2}\left(1+\sqrt{1-\frac{\lambda}{4\pi^2u^2}}\right)\,, \tag{2.2}$$

$\lambda = Ng_{YM}^2 = 16\pi^2g^2$ being the 't Hooft coupling of planar theory ($N\to\infty, g_{YM}\to 0$). Anomalous dimensions (i.e. eigenvalues of the dilatation operator) are given by

$$\Delta = L + \sum_{k=1}^{M}\left[\left(\frac{1}{2}-iu_k\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}-iu_k\right)^2}} + \left(\frac{1}{2}+iu_k\right)\sqrt{1+\frac{4g^2}{\left(\frac{1}{2}+iu_k\right)^2}} - 1\right]\,. \tag{2.3}$$

In [12] we wrote, following the standard route pioneered by [1], the NLIE describing the highest anomalous dimension operator and its excitations. We now go briefly through its derivation (for details, see [12]). We first introduce two functions

$$\phi(u) = i\ln\frac{i+u}{i-u} = 2\arctan u\,,\ \mathrm{Im}u<1\,,\quad \Phi(u) = i\ln\frac{X\left(\frac{i}{2}+u\right)}{X\left(\frac{i}{2}-u\right)}\,,\ \mathrm{Im}u<1/2\,, \tag{2.4}$$

which allows to define the counting function as

$$Z(u) = L\,\Phi(u) - \sum_{k=1}^{M}\phi(u-u_k)\,. \tag{2.5}$$

We now consider a state with M real roots u_k and a number $H=L-2M$ of real holes x_h. Using the fact that

$$e^{iZ(u_k)} = e^{iZ(x_h)} = (-1)^{L-M-1}\,, \tag{2.6}$$

[1] The term asymptotic means exactly that this Ansatz is believed to give the exact loop expansion to the anomalous dimension up to *wrapping corrections*, starting at order λ^L.

the sum over all the real roots of this state of a function f analytic in a strip around the real axis can be written as [1]

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) &= -\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,f'(v)\,Z(v)+ \\ &+ \int_{-\infty}^{\infty}\frac{dv}{\pi}\,f'(v)\,\mathrm{Im}\ln\left[1+(-1)^{L-M}e^{iZ(v+i0)}\right]-\sum_{h=1}^{H}f(x_h)\,.\end{aligned} \tag{2.7}$$

This may be applied to the sum in the counting function (2.5) bringing

$$\begin{aligned}Z(u) &= L\,\Phi(u)-\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\phi'(u-v)\,Z(v)+ \\ &+ \int_{-\infty}^{\infty}\frac{dv}{\pi}\,\phi'(u-v)\,\mathrm{Im}\ln\left[1+(-1)^{L-M}e^{iZ(v+i0)}\right] \\ &+ \sum_{h=1}^{H}\phi(u-x_h)\,.\end{aligned} \tag{2.8}$$

It is convenient to introduce the usual synthetic notation

$$L(u)=\ \mathrm{Im}\ln\left[1+(-1)^{L-M}e^{iZ(u+i0)}\right]\ .$$

After u Fourier transforming all the terms and moving the first convolution to the l.h.s., we obtain

$$\hat{Z}(k)=\hat{F}(k)+2\hat{G}(k)\hat{L}(k)+\sum_{h=1}^{H}e^{-ikx_h}\,\hat{H}(k)\,, \tag{2.9}$$

where (P is the principal value distribution)

$$\hat{F}(k)=L\frac{\pi}{i}P\left(\frac{1}{k}\right)\frac{J_0\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{\cosh\frac{k}{2}}\,,\quad \hat{G}(k)=\frac{1}{1+e^{|k|}}\,,\quad \hat{H}(k)=\frac{2\pi}{i}P\left(\frac{1}{k}\right)\hat{G}(k)\,. \tag{2.10}$$

Inverting the Fourier transforms of (2.9) leads to the NLIE valid for this multi-loop Bethe equations (and for the aforementioned states)

$$\begin{aligned}Z(u) &= F(u)+\sum_{h=1}^{H}H(u-x_h)+ \\ &+ 2\int_{-\infty}^{\infty}dv\;G(u-v)\,\mathrm{Im}\ln\left[1+(-1)^{L-M}e^{iZ(v+i0)}\right]\ .\end{aligned} \tag{2.11}$$

Let us observe that in this case the structure of this NLIE is quite the same as in many other models. And, as usual, plugging the equation (2.11) in relation (2.7), we can

disentangle the bulk term (proportional to the size L) to the finite size corrections in the expression for the eigenvalues of the observables:

$$\begin{aligned}\sum_{k=1}^{M} f(u_k) &= -\int_{-\infty}^{\infty}\frac{dv}{2\pi}f'(v)F(v)+\\ &+\int_{-\infty}^{\infty}\frac{dv}{\pi}f'(v)\int_{-\infty}^{\infty}dw\,[\delta(v-w)-G(v-w)]\,\mathrm{Im}\ln\left[1+(-1)^{L-M}e^{iZ(w+i0)}\right]\\ &-\sum_{h=1}^{H}\left\{\int_{-\infty}^{\infty}\frac{dv}{2\pi}f'(v)\,H(v-x_h)+f(x_h)\right\}\,. \end{aligned}\tag{2.12}$$

It is of interest to apply this formula to the conserved charges of the model, i.e.

$$\mathcal{Q}_r=\sum_{k=1}^{M}q_r(u_k)\,,\quad q_r(u)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{iku}\;2^{r-1}\frac{(2\pi)^r}{(\sqrt{\lambda})^{r-1}}\frac{1}{i^{r-2}}\frac{J_r\left(\frac{\sqrt{\lambda}}{2\pi}k\right)}{ke^{\frac{|k|}{2}}}\,.\tag{2.13}$$

In particular the anomalous dimension is given by $\Delta=L+2g^2\mathcal{Q}_2$. We get

$$\begin{aligned}\mathcal{Q}_r &= L\frac{i^{r+2}}{g^{r-1}}\int_{-\infty}^{\infty}dk\frac{J_{r-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)}-\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{i^{3+r}}{g^{r-1}}\frac{J_{r-1}(2gk)}{\cosh\frac{k}{2}}\hat{L}(k)\\ &-\sum_{h=1}^{H}\int_{-\infty}^{\infty}\frac{dk}{2k}e^{-ikx_h}\frac{i^{2+r}}{g^{r-1}}\frac{J_{r-1}(2gk)}{\cosh\frac{k}{2}}\,.\end{aligned}\tag{2.14}$$

This relation is exact and allows the (numerical and analytical) study of the dependence of the eigenvalues of the charges on the size of the system. In particular, when $L\to\infty$, such eigenvalues behave as [12]

$$\begin{aligned}\mathcal{Q}_r &= L\frac{i^{r+2}}{g^{r-1}}\int_{-\infty}^{\infty}dk\frac{J_{r-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)}-H\sum_{l=0}^{\infty}\frac{i^{2l+2-r}\pi^{2l+r-1}g^{2l}}{l!(r+l-1)!}|E_{2l+r-2}|\\ &-\frac{i^{r-1}}{g^{r-1}}\frac{J_{r-1}(2gi\pi)}{J_0(2gi\pi)}\frac{\pi}{12L}[1+(-1)^r]+o\left(\frac{1}{L}\right)\,,\end{aligned}\tag{2.15}$$

where E_l are the Euler numbers.

3 The NLIE in the $SU(2)$ sector with dressing factor

In order to match with results in string theory, the asymptotic Bethe Ansatz type equations describing planar $\mathcal{N}=4$ SYM should contain – with respect to the first proposals – a universal dressing phase [9, 10, 7]. Consequently, in the $SU(2)$ scalar sector the BDS equations (2.1) ought to be modified into

$$\left[\frac{X(u_j+\frac{i}{2})}{X(u_j-\frac{i}{2})}\right]^L=\prod_{\substack{k=1\\k\neq j}}^{M}\frac{u_j-u_k+i}{u_j-u_k-i}\exp[2i\theta(u_j,u_k)]\,.\tag{3.1}$$

On the other hand, the (renormalised) dimension corresponding to the operator/solution $\{u_k\}_{k=1,\dots,M}$ of (3.1) is still given by (2.3). The complete dressing phase has been conjectured to be [7]

$$\theta(u_j,u_k)=\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)[q_r(u_j)q_{r+1+2\nu}(u_k)-q_r(u_k)q_{r+1+2\nu}(u_j)]\,. \tag{3.2}$$

In order to fix the notations in (3.2), we remind that $q_r(u)$ is the magnon r-th charge, which is given by formula (2.13) and the functions $\beta_{r,r+1+2\nu}(g)$ are meromorphic functions of g, introduced and studied in [7]. In that paper their weak coupling expansion was proposed as

$$\beta_{r,r+1+2\nu}(g)=\sum_{\mu=\nu}^{\infty}g^{2r+2\nu+2\mu}\,\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}\,, \tag{3.3}$$

the coefficients $\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$ being

$$\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}=2(-1)^{r+\mu+1}\frac{(r-1)(r+2\nu)}{2\mu+1}\left(\begin{array}{c}2\mu+1\\ \mu-r-\nu+1\end{array}\right)\left(\begin{array}{c}2\mu+1\\ \mu-\nu\end{array}\right)\zeta(2\mu+1)\,. \tag{3.4}$$

In [14] we wrote a NLIE equivalent to the Bethe equations (3.1) for the states[2] characterised by M real Bethe roots, u_k, and $H=L-2M$ real holes, x_h. We now go briefly through its derivation. We start by defining the counting function as

$$Z(u)=L\,\Phi(u)-\sum_{k=1}^{M}\phi(u-u_k)+2\sum_{k=1}^{M}\theta(u,u_k)\,. \tag{3.5}$$

Then, as a consequence of (2.6), it is simple to express a sum on the Bethe roots for a function $f(u)$ as

$$\sum_{k=1}^{M}f(u_k)=-\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\frac{d}{dv}f(v)\,[Z(v)-2L(v)]-\sum_{h=1}^{H}f(x_h)\,. \tag{3.6}$$

In particular, we will be interested in the eigenvalues of the conserved charges

$$\mathcal{Q}_r=\sum_{k=1}^{M}q_r(u_k)=-\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\frac{d}{dv}q_r(v)\,[Z(v)-2L(v)]-\sum_{h=1}^{H}q_r(x_h)\,. \tag{3.7}$$

Applying (3.6) to (3.5) we get

$$\begin{aligned}Z(u)\;&=\;L\Phi(u)-\int_{-\infty}^{\infty}\frac{dv}{2\pi}\frac{2}{(u-v)^2+1}[Z(v)-2L(v)]+\sum_{h=1}^{H}\phi(u-x_h)\\ &-\;2\int_{-\infty}^{\infty}\frac{dv}{2\pi}\,\frac{d}{dv}\theta(u,v)\,[Z(v)-2L(v)]-2\sum_{h=1}^{H}\theta(u,x_h)\,.\end{aligned} \tag{3.8}$$

[2] As in the BDS model, these states are the highest anomalous dimension state and its excitations.

Going now to the Fourier space, after grouping the terms containing $\hat{Z}(k)$, the following equation shall hold

$$\begin{aligned}\hat{Z}(k) &= L\frac{2\pi}{i}P\left(\frac{1}{k}\right)\frac{J_0(2gk)}{2\cosh\frac{k}{2}}+\frac{2}{1+e^{|k|}}\hat{L}(k)+\sum_{h=1}^{H}e^{-ikx_h}\frac{2\pi}{i}P\left(\frac{1}{k}\right)\frac{1}{1+e^{|k|}}\\ &+ \frac{1}{\cosh\frac{k}{2}}\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)\Big[\frac{2\pi}{g^{r-1}}\frac{1}{i^{r-2}}\frac{J_{r-1}(2gk)}{k}\mathcal{Q}_{r+1+2\nu}\\ &- \frac{2\pi}{g^{r+2\nu}}\frac{1}{i^{r+2\nu-1}}\frac{J_{r+2\nu}(2gk)}{k}\mathcal{Q}_r\Big]\,, \end{aligned}\tag{3.9}$$

where we have introduced the conserved charges (3.7) and the explicit form (2.13) of the Fourier transform of the charge densities, $\hat{q}_r(k)$, has been used. A very crucial difference of this non-linear integral equation from the others in the literature may be stated in the presence of $Z(u)$ in infinite many places, i.e. all the charges $\mathcal{Q}_r$ (3.7).
Concerning the charges, we can write for them a system of equations. In fact, we first rewrite the expressions (3.7) in terms of Fourier transforms

$$\begin{aligned}\mathcal{Q}_s &= -\int_{-\infty}^{\infty}\frac{dk}{4\pi^2}\hat{q}_s'(-k)[\hat{Z}(k)-2\hat{L}(k)]-\sum_{h=1}^{H}q_s(x_h)\\ &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{i^{3+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{e^{\frac{|k|}{2}}}[\hat{Z}(k)-2\hat{L}(k)]-\sum_{h=1}^{H}q_s(x_h)\,.\end{aligned}\tag{3.10}$$

Then we insert relation (3.9) for $\hat{Z}(k)$ into this expression to obtain,

$$\begin{aligned}\mathcal{Q}_s &= \frac{i^{s+2}}{g^{s-1}}\Big[L\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_0(2gk)}{k(e^{|k|}+1)}\\ &+ 2\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_{r-1}(2gk)}{k(e^{|k|}+1)}\frac{g^{1-r}}{i^{r+2\nu-1}}\mathcal{Q}_{r+2\nu+1}\\ &+ 2\sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty}\beta_{r,r+1+2\nu}(g)(-1)^{1+\nu}\int_{-\infty}^{\infty}dk\frac{J_{s-1}(2gk)J_{r+2\nu}(2gk)}{k(e^{|k|}+1)}\frac{g^{-r-2\nu}}{i^{r-2}}\mathcal{Q}_r\Big]\\ &- \int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{i^{3+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{\cosh\frac{k}{2}}\hat{L}(k)-\sum_{h=1}^{H}\int_{-\infty}^{\infty}\frac{dk}{2k}e^{-ikx_h}\frac{i^{2+s}}{g^{s-1}}\frac{J_{s-1}(2gk)}{\cosh\frac{k}{2}}\,.\end{aligned}\tag{3.11}$$

This relation is exact and, at least in principle, may be efficient in the analysis of the conserved charges, though now $Z(u)$ appears.

4 A new approach to a NLIE

In almost all the cases considered up to now, the NLIE was written for counting functions defined as

$$Z(u)=\Phi(u)-\sum_{k=1}^{M}\phi(u-u_k)\,,\tag{4.1}$$

and when the Bethe roots distribute on the real axis, allowing the presence of only a finite number of holes and possibly complex roots. Even if this case is relevant for the study of the fundamental state and the first excitations of many models, it does not cover many of the Bethe Ansatz systems proposed in the context of $\mathcal{N}=4$ SYM.
For this reason we want to write the NLIE (and the expression for the eigenvalues of the observables in terms of its solution) for the more general case in which the counting function is defined as

$$Z(u) = \Phi(u) - \sum_{k=1}^{M} \phi(u, u_k)\,, \tag{4.2}$$

(i.e. the function $\phi(x,y)$ does not depend only on the difference $x-y$: this happens, for instance, when the dressing factor is present). We suppose also that the M Bethe roots are concentrated in an interval $[A,B]$ of the real axis [3] and that holes are present only outside this interval. This second property is peculiar, for instance, of some states in the $sl(2)$ sector of $\mathcal{N}=4$ SYM.
On this state we consider a sum over the Bethe roots $\{u_k\}_{k=1,\dots,M}$ of a function (observable) $O(u)$ analytic in a strip around the real axis. If the condition $e^{iZ(u_k)} = -1$ holds, this sum can be written as

$$\begin{aligned} 2\pi i \sum_{k=1}^{M} O(u_k) &= \lim_{\epsilon\to 0^+} \left[\int_A^B du O(u-i\epsilon) \frac{e^{iZ(u-i\epsilon)} i Z'(u-i\epsilon)}{1+e^{iZ(u-i\epsilon)}} \right. \\ &+ \left. \int_B^A du O(u+i\epsilon) \frac{e^{iZ(u+i\epsilon)} i Z'(u+i\epsilon)}{1+e^{iZ(u+i\epsilon)}} \right]\,. \end{aligned} \tag{4.3}$$

Supposing $Z'(u) < 0$, we can rearrange this expression as follows,

$$\begin{aligned} \sum_{k=1}^{M} O(u_k) &= -\int_A^B \frac{dv}{2\pi} O(v) Z'(v) + \int_A^B \frac{dv}{\pi} O(v) \frac{d}{dv} \mathrm{Im}\ln\left[1+e^{iZ(v-i0)}\right] \\ &= -\frac{1}{2\pi}\left[O(B)Z(B) - O(A)Z(A)\right] \\ &+ \frac{1}{\pi}\left\{O(B)\mathrm{Im}\ln\left[1+e^{iZ(B)}\right] - O(A)\mathrm{Im}\ln\left[1+e^{iZ(A)}\right]\right\} \\ &+ \int_A^B \frac{dv}{2\pi} O'(v) Z(v) - 2\int_A^B \frac{dv}{2\pi} O'(v) \mathrm{Im}\ln\left[1+e^{iZ(v-i0)}\right]\,. \end{aligned} \tag{4.4}$$

In brief, what we are doing is to evaluate a sum on the Bethe roots by integrating just on the interval containing them. Therefore, this method is alternative and complementary to the idea proposed in the first of [1] which consists in first integrating on all the real axis and then subtracting the contributions coming from the real holes.

[3] The case in which the Bethe roots are concentrated on a finite number of intervals on the real axis follows straightforwardly from the results of this Section. Moreover, even the case when the roots lie on complex lines can be treated as follows.

We now apply (4.4) to the sum over the Bethe roots appearing in the definition (4.2). We get the following equation

$$Z(u) = f(u) - \int_A^B \frac{dv}{2\pi}\frac{d}{dv}\phi(u,v)\, Z(v) + 2\int_A^B \frac{dv}{2\pi}\frac{d}{dv}\phi(u,v)\,\mathrm{Im}\ln\left[1+e^{iZ(v-i0)}\right] \quad (4.5)$$

where

$$\begin{aligned} f(u) &= \Phi(u) + \frac{1}{2\pi}\left[\phi(u,B)Z(B) - \phi(u,A)Z(A)\right] \\ &- \frac{1}{\pi}\left\{\phi(u,B)\mathrm{Im}\ln\left[1+e^{iZ(B)}\right] - \phi(u,A)\mathrm{Im}\ln\left[1+e^{iZ(A)}\right]\right\} . \end{aligned} \quad (4.6)$$

We can now write a NLIE for the counting function by inserting in an iterative way (4.5) for Z in the right hand side of the same equation. Using the notation

$$(\varphi \star f)(u) = \int_A^B dv\, \varphi(u,v) f(v)\,, \quad (4.7)$$

eventually we get the NLIE in the form

$$Z(u) = F(u) + 2(G\star L)(u)\,, \quad (4.8)$$

where

$$F(u) = f(u) + \sum_{k=1}^{\infty}(-1)^k((\varphi^{\star k})\star f)(u)\,, \quad G(u,v) = \varphi(u,v) + \sum_{k=2}^{\infty}(-1)^{k-1}(\varphi^{\star k})(u,v)\,. \quad (4.9)$$

We used the simplified notations

$$L(u) = \mathrm{Im}\ln\left[1+e^{iZ(u+i0)}\right]\,, \quad \varphi(u,v) = \frac{1}{2\pi}\frac{d}{dv}\phi(u,v)\,. \quad (4.10)$$

More explicitly,

$$\begin{aligned} F(u) &= f(u) + \sum_{k=1}^{\infty}(-1)^k \int_A^B dv_1\,\varphi(u,v_1)\int_A^B dv_2\,\varphi(v_1,v_2)\ldots \\ &\ldots \int_A^B dv_k\,\varphi(v_{k-1},v_k) f(v_k)\,, \end{aligned} \quad (4.11)$$

$$\begin{aligned} G(u,v) &= \varphi(u,v) + \sum_{k=1}^{\infty}(-1)^k \int_A^B dv_0\,\varphi(u,v_0)\int_A^B dv_1\,\varphi(v_0,v_1)\ldots \\ &\ldots \int_A^B dv_{k-1}\,\varphi(v_{k-2}-v_{k-1})\,\varphi(v_{k-1}-v)\,. \end{aligned} \quad (4.12)$$

Eventually, inserting (4.8) in (4.4) we get an expression for the eigenvalues of an observable

as

$$\begin{aligned}\sum_{k=1}^{M} O(x_k) &= -\frac{1}{2\pi}\left[O(B)Z(B) - O(A)Z(A)\right] \\ &+ \frac{1}{\pi}\left\{O(B)\mathrm{Im}\ln\left[1+e^{iZ(B)}\right] - O(A)\mathrm{Im}\ln\left[1+e^{iZ(A)}\right]\right\} \\ &+ \int_A^B \frac{dv}{2\pi} O^{'}(v) F(v) + \\ &+ 2\int_A^B \frac{dv}{2\pi} O^{'}(v) \int_A^B dw [G(v,w) - \delta(v-w)] \mathrm{Im}\ln\left[1+e^{iZ(w-i0)}\right] \ .\end{aligned} \tag{4.13}$$

We remark that all the already known NLIEs can be reproduced in this way, without Fourier transforming. Moreover, formulæ (4.8, 4.9) and (4.13) can be used in order to write, respectively, the NLIE and the eigenvalues of the observables on states appearing in models relevant for $\mathcal{N}=4$ SYM. It would be of interest to apply these techniques, for instance, to the widely studied [15, 7] $sl(2)$ sector of the theory.

5 Summary

We have reported on our project - still in progress - of writing NLIEs for the Bethe Ansatz type equations relevant for the determination of anomalous dimensions of operators in $\mathcal{N}=4$ SYM (and, correspondingly energies of strings). The NLIE describing the highest anomalous dimension operator of the BDS model and its excitations - treated in Section 2 - does not differ in form from the NLIEs studied up to now. However, the Bethe equations arising in the context of $\mathcal{N}=4$ SYM have in general intrinsic complications - namely, magnon scattering matrix which does not depend only on the difference of the rapidities and states described by roots which can condense on lines in the complex plane - which at first sight seem to prevent even the possibility of writing a NLIE. We have shown how to circumvent this problem in some cases. First we studied in Section 3 the $SU(2)$ sector with dressing factor and showed how to write a new type of NLIE which depends explicitly on the eigenvalues of the conserved charges. Then, in Section 4, we sketched a general formalism which allows to treat magnon scattering matrices with general dependence on the rapidities and states with roots on intervals on the real line (or even complex lines). We plan to give in forthcoming publications explicit applications of this new formalism, for instance in the $sl(2)$ sector of $\mathcal{N}=4$ SYM.

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Some recent results on classical super-Integrable Systems

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Abstract

In the present contribution we briefly describe a novel Hamiltonian system in n dimensions endowed with co-algebra symmetry and admitting the maximal number $2n-1$ of functionally independent (quadratic) first integrals. For the system under scrutiny we explicitly derive different complete sets of integrals of motion in involution and sketch the solution procedure for the equations of motion. The relation with Bertrand spaces is also mentioned.

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1 Introduction

At *RAQIS'05*, one of us (O.R) presented some *classical* integrable many-body systems on curved spaces, obtained through symplectic realizations of (nonstandard) $q-$deformations of $U(\mathfrak{sl}_2)$, stressing the connection among algebraic and geometric notions (see [1, 2, 3]). Here we will be even more "*classical*", showing an example of an integrable model (actually, a maximally superintegrable one) on a space of nonconstant curvature arising just from $U(\mathfrak{sl}_2)$ [4]. So, while then the key-point was the interplay between Lie-algebra deformations and curved manifolds, now the emphasis is on the construction of generalized harmonic oscillators in any dimension on a suitable Riemannian space with variable curvature. Specifically, we consider the Hamiltonian $\mathcal{H}$:

$$\mathcal{H}(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2 + \omega^2\,\mathbf{q}^2 + \sum_j b_j q_j^{-2}}{\kappa + \mathbf{q}^2} \tag{1.1}$$

where

$$\kappa > 0, \qquad \omega^2, b_j \geq 0$$

In (1.1) $\mathbf{p},\mathbf{q} \in \mathbb{R}^n$, and they are canonical for the standard symplectic form $\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p}$. Of course $\mathbf{u}^2 := \mathbf{u}\cdot\mathbf{u} = \sum_i u_i^2, \quad \forall u \in \mathbb{R}^n$
$\mathcal{H}$ yields the motion on the conformally flat Riemannian manifold $\mathcal{M}^n := (\mathbb{R}^n, \mathrm{d}s^2)$, equipped with the metric

$$\mathrm{d}s^2 = (\kappa + \mathbf{q}^2)\,\mathrm{d}\mathbf{q}^2, \tag{1.2}$$

under the potential

$$V(\mathbf{q}) = \frac{\omega^2\,\mathbf{q}^2 + \sum_j b_j q_j^{-2}}{\kappa + \mathbf{q}^2} \tag{1.3}$$

We stress that (1.1) is an example of a Hamiltonian system on a Riemannian space of nonconstant curvature which possesses the maximal number $2n-1$ of integrals of motion in $\forall n$, hence providing an intrinsic SW (Smorodinsky-Winternitz; see [5]) system on this space [6]. The centrifugal terms $b_j q_j^{-2}$ in (1.3) do not spoil maximal superintegrability, the parameters b_j being just the values of the Casimir function of the corresponding representation of $\mathfrak{sl}(2)$. When all b_j are set to 0 the Hamiltonian (1.1) can be defined as the Generalized Harmonic Oscillator (GHO) hamiltonian in the Riemannian space $\mathcal{M}^n := (\mathbb{R}^n, \mathrm{d}s^2)$. In the special case $n = 2$ this model is listed in Kalnins et al.'s [7, 8, 9, 10, 11, 12] classification of Maximally Superintegrable (MS) systems in the Darboux space of type III. We have singled out it among the others in this list because it is readily amenable to the Poisson coalgebra treatment.
We remark here that, when considered in 3 dimensions ($n = 3$), the system provides an example of a classical Bertrand system [13, 14, 15, 16, 17] namely, of a hamiltonian system in $\mathcal{T}^*\mathbb{R}^3$ where all bounded trajectories are closed. As this property is actually true for any dimension n due to the coalgebra symmetry, we can in fact regard it as an n-dimensional Bertrand system.

2 Super-Integrability and Separability

In the following, we find it more convenient to work with the slightly modified Hamiltonian $H := \frac{\mathcal{H}-\omega^2}{2}$, namely:

$$H(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2 - c + \sum_j b_j q_j^{-2}}{2(\kappa + \mathbf{q}^2)}\,, \qquad c := \kappa\omega^2. \tag{2.1}$$

The hamiltonian H has a Poisson coalgebra symmetry yielding "for free" 2 sets of $n-1$ independent integrals of motion, the members of each set being mutually in involution. The $\mathfrak{sl}(2)$ Poisson coalgebra is defined by the basis $\{J_\epsilon : \epsilon = \pm, 0\}$ together with the usual Lie–Poisson brackets, Casimir C and primitive coproduct Δ. Accordingly, its symplectic one-dimensional realization reads:

$$J_- = q_1^2\,, \qquad J_0 = q_1 p_1\,, \qquad J_+ = p_1^2 + b_1 q_1^{-2}\,,$$

where, as anticipated in the introduction, b_1 is a real parameter labeling the representation, yielding the value of the (one-dimensional) Casimir. The symplectic realization in $\mathbb{R}^{2n}$ follows from the (primitive) n-th coproduct and reads:

$$J_- = \mathbf{q}^2\,, \qquad J_0 = \mathbf{p}\cdot\mathbf{q}\,, \qquad J_+ = \mathbf{p}^2 + \sum_j b_j q_j^{-2}\,, \tag{2.2}$$

with $b_j \in \mathbb{R}$; the Casimir $C^{(n)}$ is simply

$$C^{(n)} = \mathbf{L}^2 + \sum_j \frac{b_j \mathbf{q}^2}{q_j^2} \tag{2.3}$$

where $\mathbf{L}^2$ denotes the ($n-$body) angular momentum.
This $\mathfrak{sl}(2) \otimes \cdots \otimes \mathfrak{sl}(2)$ symmetry endows any $n-$dimensional Hamiltonian system in the enveloping algebra with $2n-3$ integrals other than the Hamiltonian given by the left and right partial Casimirs.
Indeed the following theorem holds true:

Theorem 2.1 *Any Hamiltonian $H_0(\mathbf{p},\mathbf{q}) = h(J_+, J_0, J_-)$ possesses $2n-3$ first integrals given by the left and right partial Casimirs*

$$C^{(m)} = \sum_{1\le i<j\le m} \left[(q_i p_j - q_j p_i)^2 + \frac{b_i q_j^2}{q_i^2} + \frac{b_j q_i^2}{q_j^2}\right] + \sum_{i=1}^{m} b_i\,,$$

$$C_{(m)} = \sum_{n-m<i<j\le n} \left[(q_i p_j - q_j p_i)^2 + \frac{b_i q_j^2}{q_i^2} + \frac{b_j q_i^2}{q_j^2}\right] + \sum_{i=n-m+1}^{n} b_i\,,$$

where $1 < m \le n$. Here $C^{(n)} = C_{(n)} = C$ is the Casimir defined above and the functions

$$\left\{H_0, C^{(l)}, C_{(m)} : 1 < l < n, 1 < m \le n\right\}$$

are functionally independent. Moreover, the subsets $\{H_0, C^{(m)} : 1 < m \leq n\}$ *and* $\{H_0, C_{(m)} : 1 < m \leq n\}$ *are in involution.*

This result can be rephrased in terms of the nD spherical coordinates $(r, _1, \ldots, _{n-1})$

$$\begin{aligned} q_j &= r \cos_j \prod_{k=1}^{j-1} \sin_k \qquad 1 \leq j < N \\ q_n &= r \prod_{k=1}^{n-1} \sin_k \end{aligned} \tag{2.4}$$

If we denote by p_r, p_j the conjugate momentum to r and $_j$, respectively, we find that

$$\mathbf{p}^2 = p_r^2 + r^{-2}\mathbf{L}^2 \tag{2.5}$$

the angular momentum $\mathbf{L}^2$ being given by

$$\mathbf{L}^2 = \sum_{j=1}^{n-1} p_j^2 \prod_{k=1}^{j-1} \sin_k^{-2} \ . \tag{2.6}$$

Note that adding the centrifugal terms $\sum_j b_j q_j^{-2}$ to the square of the kinetic momentum $\mathbf{p}^2$ the formula (2.5) becomes (see (2.3))

$$\mathbf{p}^2 + \sum_j b_j q_j^{-2} = p_r^2 + r^{-2} C^{(n)} \tag{2.7}$$

which can readily be substituted in the numerator of (2.1).
The complete integrability determined by the set of n constants $\{H_0, C_{(m)}\}$ $(m = 2, \ldots, n)$ leads, in spherical coordinates, to a separable system, that is, to a set of n equations each of them only depends on a pair of canonical quantities. Explicitly, the constants $C_{(m)}$ in terms of $_j$, p_j turn out to be

$$C_{(m)} = \sum_{j=n-m+1}^{n-1} p_j^2 \prod_{k=1}^{j-1} \sin_k^{-2} \tag{2.8}$$

we obtain a *common* set of $(n-1)$ 'angular' equations for any Hamiltonian H_0.

$$\begin{aligned} &C_{(2)}(_{n-1}, p_{n-1}) = p_{n-1}^2 \\ &C_{(l)}(_{n-l+1}, p_{n-l+1}) = p_{n-l+1}^2 + C_{(l-1)} \sin_{n-l+1}^{-2} \qquad l = 3, \ldots, n-1 \\ &C_{(n)}(_1, p_1) = p_1^2 + C_{(n-1)} \sin_1^{-2} \end{aligned} \tag{2.9}$$

together with a single 'radial' equation of the form:

$$H_0(r, p_r) = \frac{p_r^2}{g(r)^2} + r^{-2} C_{(n)} + V(r) \tag{2.10}$$

where we denote by p_r the conjugate momentum to r.
Now $H := \frac{\mathcal{H}-\omega^2}{2}$ is the following rational function of the $\mathfrak{sl}(2)$ generators:

$$H = \frac{J_+ - c}{2(\kappa + J_-)} .$$

The maximal superintegrability is achieved thanks to (one of) the following first integrals:

$$I_k(\mathbf{p},\mathbf{q}) = p_k^2 - 2H(\mathbf{p},\mathbf{q})\, q_k^2 + b_k q_k^{-2} = J_+^{k)} - 2H J_-^{(k)}, \tag{2.11}$$

By direct computation one can prove that the I_k (2.11) Poisson commute with (2.1) and among themselves, and that $n-1$ among them are functionally independent.
Moreover the Hamilton-Jacobi equations are (super)-separable.
Indeed, writing: $S(t,\mathbf{q}) := W(\mathbf{q}) - \frac{1}{2}Et$, one has the following Hamilton-Jacobi equations:

$$\left(\frac{\partial W}{\partial \mathbf{q}}\right)^2 - E\mathbf{q}^2 + \sum_j b_j q_j^{-2} = c + \kappa E \tag{2.12}$$

hence recovering the HJ equation for the SW system with a different set of constants, namely

$$H_{\mathrm{SW}}(\mathbf{p},\mathbf{q}) = \mathbf{p}^2 - E\mathbf{q}^2 + \sum_j b_j q_j^{-2} .$$

Consequently our hamiltonian (2.1) separates in the same coordinate systems as the SW one. In particular (2.12) shows that, in addition to the separability in spherical coordinates (holding for any $\mathfrak{sl}(2)$ coalgebra system) one has separability right in the original $(\mathbf{p},\mathbf{q})$ coordinates.

3 The equations of motion and their solution

This section contains the derivation of the trajectories of our Hamiltonian system, which are given in closed form. Skipping the details, we summarize the crucial points.
The strategy (originally introduced in [18]) is the following:

1. One first derives the time evolution of the global $\mathfrak{sl}(2)$ generators (which play the role of *collective variables*) by using the coalgebra symmetry of the problem.

2. Then, by using the extra-integrals (2.11), one solves the equations of motion for the single-particle generators (in our case, the coordinates $J_-^{(k)} = q_k^2$).

The mean-field nature of the systems endowed with co-algebra symmetry is then apparent. In fact, all individual components undergo the same time evolution, governed by parameters written in terms of the global generators.
The evolution of the collective variables is indeed very simple. With the notations ($c \neq 0$):

$$x = \kappa + J_- = \kappa + \mathbf{q}^2 .$$

$$E := 2\,H = \frac{J_+ - c}{x}$$

the evolution of the $\mathfrak{sl}(2)$ generators is given by the following set of equations:

$$\dot{x} = \{J_-, H\} = \frac{2J_0}{x}\,, \tag{3.1a}$$

$$\dot{J}_0 = \{J_0, H\} = \frac{J_+ + E(x-\kappa)}{x}\,, \tag{3.1b}$$

$$\dot{J}_+ = \{J_+, H\} = \frac{2EJ_0}{x}\,. \tag{3.1c}$$

And the Casimir reads:

$$C = -\tfrac{1}{4}x^2\dot{x}^2 + Ex^2 + (c - E\kappa)x - c\kappa \geq 0\,.$$

Let us assume $E > 0$ and set

$$\alpha := \frac{1}{2}\Big(\kappa - \frac{c}{E}\Big)\,,$$
$$\gamma^2 := \frac{1}{4}\Big(\kappa + \frac{c}{E}\Big)^2 + \frac{C}{E}\,.$$

In this case

$$x^2\dot{x}^2 = 4E\big[(x-\alpha)^2 - \gamma^2\big] \tag{3.2}$$

whence

$$\pm 2\sqrt{E}(t-\tau) = \sqrt{(x-\alpha)^2 - \gamma^2} + \alpha\,\cosh^{-1}\Big(\frac{x-\alpha}{\gamma}\Big). \tag{3.3}$$

In (3.3) τ is an arbitrary constant.
Note that the equation (3.2) for the variable x (the squared radius) coincides with the radial equation in the Kepler problem. The above expression yields t as a monotonic function of x, so that it is globally invertible (in each half-orbit).
The second (and final) step to be accomplished is that of computing the evolution of the single-particle generators.
Parametrizing the trajectory by the radial variable x, the first integral I_k reads

$$I_k = 4E\big[(x-\alpha)^2 - \gamma^2\big]\Big(\frac{\mathrm{d}q_k}{\mathrm{d}x}\Big)^2 - Eq_k^2 + b_k q_k^{-2}. \tag{3.4}$$

Setting

$$Q_k = q_k^2, \qquad \alpha_k = -\frac{I_k}{2E}, \qquad \gamma_k^2 = \alpha_k^2 + E^{-1}b_k, \tag{3.5}$$

we can write:

$$Eq_k^4 + I_k q_k^2 - b_k = E\big[(Q_k - \alpha_i k^2 - \gamma_k^2\big]\,.$$

Introducing the additional integration constants φ_k and defining $X := \cosh^{-1}(\gamma^{-1}(x-\alpha))$ the above equation (3.4) (the "separation equation") can be integrated to:

$$
\begin{aligned}
Q_k &= \alpha_k + \gamma_k \cosh(X + \varphi_k) \\
&= \alpha_k + \gamma^{-1}\gamma_k \cosh\varphi_k, (x-\alpha) + \gamma_k \sinh\varphi_k \left| 1 - \left(\frac{x-\alpha}{\gamma}\right)^2 \right|^{1/2}
\end{aligned} \tag{3.6}
$$

It is worth remarking that all the orbits with positive energy are recovered through an appropriate choice of the $2n$ parameters

$$
\left\{ \tau, \alpha_i, \varphi_j : 1 \leq i \leq n-1,\ 1 \leq j \leq n \right\},
$$

where τ was defined in (3.3).
Indeed, summing (3.6) and recalling that $x = \kappa + \sum_k Q_k$ we get:

$$
x - \kappa = \sum_k a l_k + \gamma^{-1}(x-\alpha) \sum_i \gamma_k \cosh\varphi_k + \left| 1 - \left(\frac{x-\alpha}{\gamma}\right)^2 \right|^{1/2} \sum_k \gamma_k \sinh\varphi_k \,.
$$

yielding the following compatibility conditions:

$$
\sum_i \alpha_i + \kappa = \alpha \tag{3.7a}
$$

$$
\sum_i \gamma_i \cosh\varphi_i = \gamma \qquad (ii)\,, \tag{3.7b}
$$

$$
\sum_i \gamma_i \sinh\varphi_i = 0 \qquad (iii) \tag{3.7c}
$$

Equation (3.7a) gives the value of α in terms of $\{\alpha_j k\}$, (3.7b) consistently provides the value of the Casimir, (3.7c) imposes the following constraint

$$
\sum_i \sinh\varphi_i \left(\alpha_i^2 + E^{-1} b_i\right)^{1/2} = 0
$$

on the set $\{\alpha_k, \varphi_l\}$, reducing the number of those independent parameters to $2n-1$. Note that the above equations can be combined yielding E as a function of $\{\alpha_k\}$:

$$
E = -\frac{c}{\kappa + 2\sum_i \alpha_i} \,.
$$

We omit the discussion of the case $E \leq 0$, which goes along the same lines.

4 Concluding remarks

First of all, looking at the expression of the extra-integrals of motions (2.11), we notice that if, on one hand, they play the role of separation equations, on the other hand they are written in terms of "single-particle" $\mathfrak{sl}(2)$ generators. However the latter property is merely due to the fact that the coproduct of any generator in $\mathfrak{sl}(2)$ coalgebra is primitive

(being in fact just the direct sum of the single particle objects). Indeed, should we had worked in the more general framework of q-deformed coalgebras, it would have become clear that the proper variables to use to enforce separation (and then integration of the equations of motions) are in fact the cluster-type variables given by the *partial coproducts* [18].
Furthermore, as our system enjoys $\mathfrak{sl}(2)$ coalgebra symmetry, it turns out that the 2-particle coproducts of the Casimirs $\mathcal{C}^{j,k}$ are in fact the (squared) generators of $SO(n)$. Hence, an alternative set of $(n-1)$ integrals of motion in involution are provided by the "Uhlenbeck" integrals:

$$\mathcal{H}_j = \sum_{k=1,\ldots,n}{}' \frac{\mathcal{C}^{(j,k)}}{\lambda_j - \lambda_k}, \qquad (j = 1, \ldots, n)$$

where λ_j are arbitrary parameters.
Finally, we stress that the above construction can be (and in fact, it has partially already been) extended in several directions, i.e.:

1. The addition of suitable external potentials preserving MS or QMS, like for instance a suitable generalization of the Kepler potential (in this direction, see [19]). A systematic investigation of coalgebra n-dimensional generalization of Darboux spaces is actually being performed.

2. The extension of the above results to the q-deformed $\mathfrak{sl}(2)_q$ case. The construction of QMS integrable systems equipped with $\mathfrak{sl}(2)_q$ coalgebra symmetry has been considered in a few recent papers [1], [2]), [3]. In this case, the partial coproducts of the Casimirs are essentially nonlocal, in contrast with the undeformed case where they are just sums of bi-local terms. Moreover, we don't know of any connection among the generators of $\mathfrak{sl}(2)_q$ and those of $\mathfrak{so}(n)_q$. As a consequence, the existence of a q-deformed version of the "Uhlenbeck" integrals is still unclear.

3. The investigation of higher rank Lie algebras, starting with $\mathfrak{sl}(n)$. Obviously, for higher rank Lie algebras, the Casimir functions are generically not enough to ensure complete integrability at the "single algebra" level, and so they cannot "propagate" it to any n-th tensor product. Further invariants have to be used.

4. The quantum case. Though we have so far only preliminary results, we are confident that at least in $\mathfrak{sl}(2)$ (undeformed case) the construction of $n-$dimensional QMS and MS quantum-mechanical systems with coalgebra symmetry will go through smoothly.

We would like to end our paper by a special acknowledgement to Stefan Rauch for his pioneering contributions, obtained in the early 80s, in investigating the role of $\mathfrak{sl}(2)$ symmetry for constructing integrable many-body systems [20].

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Defects in affine Toda field theories

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Abstract

In this talk some classical and quantum aspects concerning a special kind of integrable defect - called a jump-defect - will be reviewed. In particular, recent results obtained in an attempt to incorporate this defect in the affine Toda field theories, in addition to the sine-Gordon model, will be presented.

1 Jump-defects

The jump-defect is a purely transmitting defect, which can be incorporated in certain integrable field theories in such a way as to allow the integrability of the system to be preserved. Its existence has been proved originally for the sine-Gordon model, both in the classical [1] and in the quantum context [2], and subsequently extended to other integrable systems (see for instance [3]). From the very start, the jump-defect has displayed interesting features, which appear to be quite different from the ones enjoyed by a typical δ-type impurity.

Consider two free massive scalar fields $\phi(x,t)$, $x<0$ and $\psi(x,t)$, $x>0$, with Lagrangian density given by

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)\mathcal{D}(\phi,\psi). \tag{1.1}$$

The terms $\mathcal{L}_\phi$ and $\mathcal{L}_\psi$ represent the bulk Lagrangian densities ($-\infty < x < \infty$) for the fields ϕ and ψ respectively, while $\mathcal{D}$ defines the condition for the defect, which is located at $x=0$. The function $\mathcal{D}$ can be chosen in many ways. For instance, for a δ-type impurity

$$\mathcal{D} = -\frac{1}{2}\left[\sigma\,\phi\psi - (\phi_x+\psi_x)(\phi-\psi)\right], \tag{1.2}$$

which leads to the following set of equations of motion and defect conditions

$$\begin{aligned} &\partial^2\phi = -m^2 \quad x<0, && \phi=\psi \quad x=0,\\ &\partial^2\psi = -m^2 \quad x>0, && \psi_x-\phi_x=\sigma\phi \quad x=0. \end{aligned} \tag{1.3}$$

Because of the presence of a defect, which breaks the space translation invariance, it can be verified that momentum is not conserved. Moreover, the system described by the Lagrangian density (1.1) with defect term (1.2) allows both transmission and reflection and possesses a bound state, provided $\sigma<0$.

On the other hand, a defect term, which defines the condition for a jump-defect reads

$$\mathcal{D} = \frac{1}{2}\left(\phi\psi_t - \psi\phi_t\right) + \frac{m\,\sigma}{4}(\phi+\psi)^2 + \frac{m}{4\,\sigma}(\phi-\psi)^2. \tag{1.4}$$

As a consequence, the Lagrangian density (1.1) with $\mathcal{D}$ given by (1.4) leads to the following set of equations

$$\begin{aligned} &\partial^2\phi = -m^2 \quad x<0, && \partial_x\phi-\partial_t\psi = -\sigma\left(\frac{\phi+\psi}{2}\right) - \frac{1}{\sigma}\left(\frac{\phi-\psi}{2}\right) \quad x=0,\\ &\partial^2\psi = -m^2 \quad x>0, && \partial_x\psi-\partial_t\phi = \;\;\sigma\left(\frac{\phi+\psi}{2}\right) - \frac{1}{\sigma}\left(\frac{\phi-\psi}{2}\right) \quad x=0. \end{aligned} \tag{1.5}$$

This time, surprisingly, it was discovered that momentum is conserved [1], provided a suitable contribution from the defect is added. Besides, the system described by equations (1.5) is purely transmitting and does not have a bound state. Finally, it is worth pointing

out that contrary to the δ-type impurity situation (1.3), if there is a jump-defect, the two fields ϕ and ψ do not match at the defect location.

This very simple example is useful to elucidate some of the more striking features of a jump-defect, which persist when it is incorporated in some integrable field theory such as the sine-Gordon model. In addition, contrary to a δ-type impurity, the jump-defect is able to preserve integrability.

2 Jump-defects and affine Toda field theories

The aim of this talk is to summarize some recent results obtained in the context of the affine Toda field theories (ATFTs) [4] (see also the review [5] and references therein) with a jump-defect, namely the progress achieved in stretching the investigation beyond the sine-Gordon model. For this purpose, only the ATFTs associated with the root data of the Lie algebra a_r will be taken into account. Classically, integrable jump-defects incorporated into these models have been described extensively in [6], where their integrability was established via a Lax pair argument. However, only recently, it has been possible to add a quantum description of the a_2 affine Toda model with a jump-defect [7].

2.1 Classical setting

The bulk Lagrangian density for a complex ATFT with root data of the Lie algebra a_r reads

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi \cdot \partial^\mu \phi) + \frac{m^2}{\beta^2} \sum_{j=0}^{r} (e^{i\beta \alpha_j \cdot \phi} - 1), \qquad |\alpha_j|^2 = 2 \tag{2.1}$$

where m and β are constants, and r is the rank of the algebra. The vectors α_j with $j = 1, \ldots, r$ are simple roots and α_0 is the extended root, defined by

$$\alpha_0 = -\sum_{j=1}^{r} \alpha_j .$$

The field $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$ takes values in the r-dimensional Euclidean space spanned by the simple roots $\{\alpha_j\}$. The ATFTs described by the Lagrangian density (2.1) are massive integrable field theories. They possess infinitely many conserved charges, a Lax pair representation, and many other interesting properties, both in the classical and quantum domains. The simplest choice $r = 1$ coincides with the sine-Gordon model. Apart this model, all other ATFTs described by the Lagrangian density (2.1) are not unitary theories.

What is particular interesting in the context of the present investigation is the fact that

these models possess soliton solutions [8], for which the explicit expression reads

$$\phi^a = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j \ln\left(1 + E_a\, \omega^{aj}\right) \quad a = 1, \ldots, r, \qquad E_a = e^{a_a x - b_a t + \xi_a}, \qquad \omega = e^{2\pi i/h}, \tag{2.2}$$

where $(a_a, b_a) = m_a\,(\cosh\theta, \sinh\theta)$, $h = (r+1)$ is the Coxeter number of the algebra, ξ_a is a complex parameter, and θ is the soliton rapidity. These soliton solutions are complex, with the exception for the sine-Gordon soliton. Nevertheless, they possess real energy and momentum, and their masses are given by [8]

$$M_a = \frac{4\,h\,m}{\beta^2} \sin\left(\frac{\pi a}{h}\right). \tag{2.3}$$

Each solution (2.2) is characterized by a topological charge, which is defined to be

$$Q^a = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx\, \partial_x \phi^a = \frac{\beta}{2\pi} \phi^a(\infty, t), \tag{2.4}$$

which lies in the weight lattice $\Lambda_W(a_r)$ of the Lie algebra a_r. In particular, it can be noticed that for each $a = 1, \ldots, r$ there are several solitons whose topological charges lie in the set of weights of the fundamental a^{th} representation of a_r [9]. Looking at the expression (2.2), it can be noticed that the value of the topological charge depends on the imaginary part of parameter ξ_a. Shifting ξ_a by $2\pi i a/h$ changes the topological charge, since that amount sets the boundaries between different topological charge sectors.

The system with a single jump-defect located at $x = 0$, which links two a_r fields $\phi(x,t)$, $x < 0$ and $\psi(x,t)$, $x > 0$, is described by the Lagrangian density (1.1) with defect term given by

$$\mathcal{D} = \left(\frac{1}{2}\phi \cdot E \partial_t \phi + \phi \cdot D \partial_t \psi + \frac{1}{2}\psi \cdot E \partial_t \psi - \mathcal{B}(\phi, \psi)\right). \tag{2.5}$$

The requirement that integrability must be preserved, forces the matrices E to be anti-symmetric with $D = 1 - E$ and fixes the form of the defect potential $\mathcal{B}$ to be

$$\mathcal{B} = -\frac{m}{\beta^2} \sum_{j=0}^{r} \left(\sigma\, e^{i\beta\alpha_j \cdot (D^T\phi + D\psi)/2} + \frac{1}{\sigma}\, e^{i\beta\alpha_j \cdot D(\phi - \psi)/2}\right), \tag{2.6}$$

where σ represent the defect parameter. Moreover, the matrix D satisfies the following constraints

$$\alpha_k \cdot D\alpha_j = \begin{cases} 2 & k = j, \\ -2 & k = \pi(j), \\ 0 & \text{otherwise}, \end{cases} \qquad (D + D^T) = 2, \tag{2.7}$$

where $\pi(j)$ indicates a permutation of the simple roots. Note that for $r = 1$, and after setting $\alpha_1 = 1/\beta = \sqrt{2}$, the linearized version of (2.5) and (2.6) reduce to (1.4). It should be mentioned that the jump-defect setting presented in this section is not unique, as was explained in [6]. However, the alternative case will not be considered here.

Choosing a particular cyclic permutation, namely

$$\alpha_{\pi(j)} = \alpha_{j-1} \quad j = 1, \ldots, r, \qquad \alpha_{\pi(0)} = \alpha_r,$$

it is possible to write explicitly the matrix D as follows

$$D = 2 \sum_{j=1}^{r} w_j \, (w_j - w_{j+1})^T \,, \qquad w_0 \equiv w_{r+1} = 0, \tag{2.8}$$

where w_j with $j = 1, \ldots, r$ are the fundamental highest weights of the Lie algebra a_r ($\alpha_i \cdot w_j = \delta_{ij}$). The Lagrangian density (1.1) with defect term (2.5) leads to the following equations of motion

$$\partial^2 \phi = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j \, e^{i\beta\alpha_j \cdot \phi} \quad x < 0, \qquad \partial^2 \psi = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j \, e^{i\beta\alpha_j \cdot \psi} \quad x > 0, \tag{2.9}$$

and defect conditions

$$\partial_x \phi - E\partial_t \phi - D\partial_t \psi = 0 - \partial_\phi \mathcal{B} \quad x = 0, \qquad \partial_x \psi - D^T \partial_t \phi + E\partial_t \psi = 0 \partial_\psi \mathcal{B} \quad x = 0. \tag{2.10}$$

As already pointed out in the case of the free massive field in section (1), a generalized momentum is conserved. Again, the system allows only transmission, and it is instructive to look at what happens when a soliton solution ϕ^a ($x < 0$) (2.2) travels across the jump-defect from the left to the right ($\theta > 0$). As expected, the emerging soliton ψ^a ($x > 0$) will experience a delay since its form will be

$$\psi^a = \frac{m^2 i}{\beta} \sum_{j=0}^{r} \alpha_j \ln(1 + z_a \, E_a \, \omega^{aj}), \tag{2.11}$$

where the explicit expression for the delay z_a is provided by the defect conditions (2.10), namely

$$z_a = \left(\frac{i \, e^{-(\theta-\eta)} + i e^{-i\gamma_a}}{e^{-(\theta-\eta)} + i e^{i\gamma_a}} \right), \qquad \gamma_a = \frac{\pi \, a}{h}, \qquad \sigma = e^{-\eta}. \tag{2.12}$$

This expression is in general complex and diverges when

$$\theta = \eta + \frac{i\pi}{2} \left(1 - \frac{2a}{h} \right). \tag{2.13}$$

However, for the self-conjugate soliton $a = h/2$ (provided r is odd), the delay becomes real and coincides with the delay for the sine-Gordon model [1]. When this happens the soliton can be absorbed by the defect since the pole (2.13) appears for a real value of the rapidity, namely $\theta = \eta$. Finally, in [6] it was also pointed out that a soliton might be turned into one and only one of the adjacent solitons by the jump-defect, provided the argument of the delay (2.12) is sufficiently large. In fact, the argument is given by

$$\tan(\arg z_a) = - \left(\frac{\sin 2\gamma_a}{e^{-2(\theta-\eta)} + \cos 2\gamma_a} \right), \tag{2.14}$$

and therefore the phase shift produced by the defect can vary between zero (as $\theta \to -\infty$) and $-2\gamma_a$ (as $\theta \to \infty$) allowing a change in the topological charge of the incoming soliton, since, as pointed out before, the topological charge sectors are separated exactly by $2\gamma_a$.

2.2 Quantum domain

In this section, recent developments concerning the quantization of the a_r ATFTs will be presented. In particular, the example elucidated in this talk concerns the a_2 affine Toda model, for which a complete analysis has been carried out in [7]. The purpose of that investigation was to find the transmission matrices, describing the interaction amongst a jump-defect and the soliton and antisoliton solutions of the model. Two different approaches were used for this purpose and they will be sketched in the next section. Both methods make use of the assumption that the topological charge of the system containing two a_2 fields ϕ $(x < 0)$, ψ $(x > 0)$ and the jump-defect located in $x = 0$ is conserved. This fact relies on the classical investigation, briefly presented in section 2.1, which suggests that both solitons and defects carry a topological charge that can be exchanged due to their mutual interaction.

Both procedures allow to determine the transmission matrices up to an overall function of the rapidity. The first method consists of a functional integral approach, which makes use of the Lagrangian density (1.1) with defect term (2.5), together with a bootstrap procedure. The second method consists in solving directly the triangular equations, which represent a set of consistency conditions among the bulk scattering S-matrices and the unknown transmission matrices. Primary ingredients needed for both investigations are the S-matrices for the a_2 ATFT. Together with the S-matrices for the other a_r models (with the exception of the sine-Gordon model), they have been conjectured by Hollowood [10], who made use of the trigonometric solutions of the Yang-Baxter equation found originally by Jimbo [11] (see also references therein).

The Lie algebra a_2 has two fundamental representations, and the weights belonging to the first representation can be written in terms of simple roots as follows

$$l_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \qquad l_2 = -\frac{1}{3}(\alpha_1 - \alpha_2), \qquad l_3 = -\frac{1}{3}(\alpha_1 + 2\alpha_2). \tag{2.15}$$

As a consequence this representation contains three solitons, while the corresponding antisolitons have weights which are the negative of these and lie in the second representation. The knowledge of the soliton-soliton S-matrix suffices since it can be used to derive the other S-matrices, which describe the interactions soliton-antisoliton and antisoliton-antisoliton, by means of a bootstrap procedure. Having said that, the soliton-soliton S-matrix for the a_2 model can be written in the following explicit form

$$S_{kl}^{mn}(\theta_{12}) = R_{kl}^{mn}(x_{12})\,\rho(\theta_{12}), \qquad \theta_{12} = (\theta_1 - \theta_2), \qquad x_{12} = \frac{x_1}{x_2}, \tag{2.16}$$

where k, l label the incoming particles and m, n label the outgoing particles in a two-body scattering process, with the particle k, n having rapidity θ_1, and the particle l, m having

rapidity θ_2. The explicit form for the R-matrix is

$$R(x_{12}) = \begin{pmatrix} a(x_{12}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{12}^{1/3}\,c & 0 & b(x_{12}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{12}^{-1/3}\,c & 0 & 0 & 0 & b(x_{12}) & 0 & 0 \\ 0 & b(x_{12}) & 0 & x_{12}^{-1/3}\,c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a(x_{12}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{12}^{1/3}\,c & 0 & b(x_{12}) & 0 \\ 0 & 0 & b(x_{12}) & 0 & 0 & 0 & x_{12}^{1/3}\,c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(x_{12}) & 0 & x_{12}^{-1/3}\,c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(x_{12}) \end{pmatrix}$$

with

$$a(x_{12}) = (q\,x_{12} - q^{-1}\,x_{12}^{-1}), \qquad b(x_{12}) = (x_{12} - x_{12}^{-1}), \qquad c = (q - q^{-1}), \tag{2.17}$$

and

$$x_k = e^{h\gamma\theta_k/2} \quad k = 1, 2, \qquad q = -e^{-i\pi\gamma}, \qquad \gamma = \frac{4\pi}{\beta^2} - 1.$$

Finally, ρ is a scalar function constrained by consistency relations such as bootstrap constraints, and requirements such as crossing, which a scattering matrix must satisfy. Its expression can be found in [10].

2.3 Transmission matrices: two different approaches

Consider the following static field configurations

$$(\phi, \psi) = \frac{2\pi}{\beta}(r, s), \tag{2.18}$$

where r, s are any two elements of the root lattice. It is not difficult to check, looking at (1.1) and (2.5) that, despite having a discontinuity at the location of the defect, the constant configurations (2.18) all have the same energy and momentum, namely

$$(\mathcal{E}_0,\, \mathcal{P}_0) = -\frac{2hm}{\beta^2}(\cosh\eta,\ -\sinh\eta),$$

and they are the vacuum configurations of the system. Suppose that a jump-defect is labelled by these vacuum configurations, in the sense that when the fields ϕ, ψ have the constant values (2.18), the label (r, s) is ascribed to the defect. The idea, first presented in [2], is to compare the transmission matrix elements describing the evolution of the field configurations in the presence of two different defects: one labelled (r, s) and the other $(0, 0)$. For doing so, the fields ϕ, ψ are shifted as follows

$$\phi \to \phi - \frac{2\pi r}{\beta}, \qquad \psi \to \psi - \frac{2\pi s}{\beta}.$$

Note that the bulk and the defect potential $\mathcal{B}$ (2.6) do not change under this shift, but the part linear in time derivatives appearing in the defect term (2.5) does. As a consequence, the functional integrals, which represent the transmission factors related to the two differently labelled defects, will differ by a constant amount, namely

$$T(r,s) = e^{i\tau(r,s)}\, T(0,0), \tag{2.19}$$

where

$$\tau(r,s) = \frac{\pi}{\beta}\left(-\delta\phi\cdot(Er+Ds) + (rD+sE)\cdot\delta\psi\right),$$

and $\delta\phi$, $\delta\psi$ are the changes in the field configurations from initial to final states. To obtain explicit expressions for the elements of the soliton transmission matrix for the a_2 model, consider that a soliton passing the defect will either retains its topological charge or change it to one of the other weights l_k listed in (2.15). Therefore, the effect of a soliton passing the defect must be to change the defect labels by

$$r \to r - l_i, \qquad s \to s - l_j, \tag{2.20}$$

which implies

$$\delta\phi = -\frac{2\pi l_i}{\beta}, \qquad \delta\psi = -\frac{2\pi l_j}{\beta} \qquad i,j = 1,2,3.$$

Consequently, expression (2.19) becomes

$$T(r,s,l_i,l_j) = e^{i\tau(r,s,l_i,l_j)}\, T(0,0,l_i,l_j) \tag{2.21}$$

where

$$\tau(r,s,l_i,l_j) = \frac{2\pi^2}{\beta^2}\left(l_i\cdot(Er+Ds) - (rD+sE)\cdot l_j\right).$$

In the end, the functional integral approach suggests the following form for the elements of the transmission matrix (see [7] for details)

$$T_{i\alpha}^{j\beta}(\theta) = Q^{\,\alpha\cdot[E(l_i-l_j)+l_i+l_j]/2}\, T_i^j(\theta)\, \delta_\alpha^{\beta-l_i+l_j}, \tag{2.22}$$

where

$$\alpha = s - r, \qquad Q = -e^{i\pi\gamma}.$$

Note that the matrix (2.22) is infinite dimensional with roman and greek labels denoting soliton states and defect charges, respectively. Naturally, this kind of argument does not provide any information concerning the rapidity dependent part of the transmission matrix (2.22). However, important information concerning this unknown quantity can be collected making use of a bootstrap procedure.

Consider D_α to be the defect operator. Then, it is formally possible to describe the interaction between a defect and a soliton or antisoliton as follows ($\theta > 0$),

$$A_i(\theta)D_\alpha = T_{i\alpha}^{j\beta}(\theta)D_\beta A_j(\theta), \qquad \bar{A}_i(\theta)D_\alpha = \bar{T}_{i\alpha}^{j\beta}(\theta)D_\beta \bar{A}_j(\theta) \qquad i = 1,2,3, \tag{2.23}$$

where A_i, $\bar{A}_i$ are operators representing the soliton and antisoliton states, respectively. Since the antisoliton states $\bar{A}_i$ can be built making use of the soliton states A_i, the two expressions in (2.23) can be combined together to provide a link amongst the elements of T and $\bar{T}$. The constraints obtained allow to fix, up to an overall scalar function of the rapidity, both the matrices T, $\bar{T}$ and, surprisingly, to determine the constraints (2.7) that the classical quantity $D = 1 - E$ has to satisfy. A complete discussion and explicit calculations are reported in [7].

Before revealing the explicit expressions of the two transmission matrices, a few words must be said on the alternative approach mentioned in section (2.2). It consists in solving directly the triangular equations, which relate, for instance, the elements of the soliton transmission matrix T to the scattering soliton-soliton S-matrix elements. Adopting the same conventions as before for the roman and greek labels, and considering solitons travelling along the positive x-axis ($\theta_1 > \theta_2$), the triangular equations read

$$S_{kl}^{mn}(\theta_{12})\, T_{n\alpha}^{t\beta}(\theta_1)\, T_{m\beta}^{s\gamma}(\theta_2) = T_{l\alpha}^{n\beta}(\theta_2)\, T_{k\beta}^{m\gamma}(\theta_1)\, S_{mn}^{st}(\theta_{12}). \tag{2.24}$$

These equations have been discussed first in the context of purely transmitting defects by Delfino, Mussardo and Simonetti in [12]. Making use of the S-matrix (2.16) and of the following ansatz for the transmission matrix elements

$$T_{i\alpha}^{n\beta}(\theta) = t_{i\alpha}^{n}(\theta)\, \delta_{\alpha}^{\beta - l_i + l_n} \qquad i, n = 1, 2, 3, \tag{2.25}$$

it is possible to classify the solutions of (2.24). This much has been done in [7], where it was found that one of the solutions obtained coincides exactly with the soliton transmission matrix T conjectured by the functional integral approach. Some of the other solutions may be related to an alternative setting for the jump-defect with respect to the one presented in section (2.1), while others do not seem to be relevant for the jump-defect problem. Details are available in [7]. To summarize, the transmission matrices for solitons and antisolitons related to the jump-defect presented in section (2.1) are, respectively,

$$T_{i\alpha}^{n\beta}(\theta) = g(\theta) \begin{pmatrix} Q^{\alpha\cdot l_1}\, \delta_{\alpha}^{\beta} & \hat{x}^2\, \delta_{\alpha}^{\beta - \alpha_1} & \hat{x}\, Q^{-\alpha\cdot l_2}\, \delta_{\alpha}^{\beta + \alpha_0} \\ \hat{x}\, Q^{-\alpha\cdot l_3}\, \delta_{\alpha}^{\beta + \alpha_1} & Q^{\alpha\cdot l_2}\, \delta_{\alpha}^{\beta} & \hat{x}^2\, \delta_{\alpha}^{\beta - \alpha_2} \\ \hat{x}^2\, \delta_{\alpha}^{\beta - \alpha_0} & \hat{x}\, Q^{-\alpha\cdot l_1}\, \delta_{\alpha}^{\beta + \alpha_2} & Q^{\alpha\cdot l_3}\, \delta_{\alpha}^{\beta} \end{pmatrix}, \tag{2.26}$$

and

$$\bar{T}_{i\alpha}^{n\beta}(\theta) = \bar{g}(\theta) \begin{pmatrix} Q^{-\alpha\cdot l_1}\, \delta_{\alpha}^{\beta} & \hat{x}\, \delta_{\alpha}^{\beta + \alpha_1} & 0 \\ 0 & Q^{-\alpha\cdot l_2}\, \delta_{\alpha}^{\beta} & \hat{x}\, \delta_{\alpha}^{\beta + \alpha_2} \\ \hat{x}\, \delta_{\alpha}^{\beta + \alpha_0} & 0 & Q^{-\alpha\cdot l_3}\, \delta_{\alpha}^{\beta} \end{pmatrix}, \tag{2.27}$$

where

$$\bar{g}(\theta) = g(\theta - i\pi/3)\, g(\theta + i\pi/3)\, (1 + \hat{x}^3), \qquad \hat{x} = e^{\gamma(\theta - \Delta)}.$$

Eventually, the constant Δ will be related to the Lagrangian defect parameter σ introduced in (2.6), but, first, a few comments are in order. First of all, note the striking

asymmetry of T and $\bar{T}$. Classically, there is little difference in behaviour between solitons and antisolitons, and in section (2.1) is was pointed out that in either case the jump-defect causes a phase shift. Depending on the size of this shift, the topological charge of a soliton or antisoliton passing through the defect could be converted to just one of the adjacent topological charges. Comparing expression (2.26) and (2.27) with the argument of the classical delay (2.14), it can be seen that $\bar{T}$ provides a good match to the classical situation because of the presence of zeros in expected positions, while T does not possess the expected zeros corresponding to the classical selection rule. It appears that in the quantum context a soliton passing through the defect may change into either of the solitons adjacent to it, though the classically allowed transition remains the most probable.

It should be pointed out that solutions (2.26) and (2.27) are related by a bootstrap procedure, in the sense that starting with a T matrix (2.26) for the solitons, the bootstrap leads to the $\bar{T}$ matrix (2.27) for the antisolitons. Similarly, starting with the antisoliton matrix (2.27), the bootstrap leads to the soliton matrix (2.26). A different setting for the jump-defect would present a situation in which the asymmetry of T and $\bar{T}$ is maintained but the role of solitons and antisolitons is interchanged.

2.4 The overall scalar function: additional constraints

Some additional requirements are needed to be able to fix the overall functions of the transmission matrices. They are provided by crossing

$$\bar{T}^{i\beta}_{n\alpha}(\theta) = \tilde{T}^{n\beta}_{i\alpha}(i\pi - \theta), \tag{2.28}$$

which allows to relate the transmission matrix for antisolitons $\bar{T}$ to the transmission matrix $\tilde{T}$, which represents a process in which the incoming particles meet the defect from the right. In the jump-defect problem, parity is explicitly violated and therefore the matrix $\tilde{T}$ is expected to differ from the matrix T. Nevertheless, the two matrices T and $\tilde{T}$ are expected to be related by

$$T^{b\beta}_{a\alpha}(\theta)\,\tilde{T}^{c\gamma}_{b\beta}(-\theta) = \delta^c_a \delta^\gamma_\alpha. \tag{2.29}$$

This constraint replaces the usual unitarity condition, which does not hold here due to the fact that the model investigated is not unitary. Making use of solutions (2.26) and (2.27) in (2.28) and (2.29) leads to a relationship between the functions g and $\bar{g}$, from which the following minimal solution for g is derived

$$g(\theta) = \frac{f(\theta)}{(2\pi)^{2/3}\,\hat{x}} \tag{2.30}$$

with

$$f(\theta) = \Gamma[(1+\gamma)/2 - z] \prod_{k=1}^{\infty} \frac{\Gamma[(1+\gamma)/2 + 3k\gamma - z]\,\Gamma[(1-\gamma)/2 + (3k-2)\gamma + z]}{\Gamma[(1-\gamma)/2 + 3k\gamma + z]\,\Gamma[(1+\gamma)/2 + (3k-1)\gamma - z]}, \tag{2.31}$$

where $z = i3\gamma(\theta - \Delta)/2\pi$. Note the presence of a pole in (2.30) at

$$\theta_P = \Delta - \frac{i\pi}{3} - \frac{i\pi}{3\gamma}. \tag{2.32}$$

Comparing this in the classical limit, $1/\gamma \to 0$ ($\beta \to 0$), with the pole (2.13) appearing in the classical delay allows a determination of the relationship between the parameter Δ appearing in the transmission matrix and the defect parameter σ appearing in the Lagrangian density. This relationship reads

$$\Delta = \eta + \frac{i\pi}{2}, \qquad \sigma = e^{-\eta}. \tag{2.33}$$

The identification (2.33) is also supported by the results found during the calculation of the transmission factors for the lightest breathers, as explained in [7]. Besides, the computation of the energy of the state associated with the pole (2.32) reveals that it corresponds to an unstable bound state, provided $\frac{1}{2} < \gamma < 2$. Consequently, in the classical limit, this unstable state disappears completely. This fact agrees nicely with the classical finding that a soliton with real rapidity cannot be absorbed by the defect. It is worth pointing out that the latter phenomenon differs from the sine-Gordon case in which a soliton can be absorbed by the defect and consequently a quantum unstable bound state is always present, independently of the range of the coupling constant.

3 Conclusion

Recent results in the context of the a_2 affine Toda field theory concerning the existence of a special integrable defect - called a jump-defect - have been presented. For this model, it was possible to provide a complete and consistent description both in the classical and quantum domains. In particular, the interaction between the soliton solutions of the a_2 affine Toda model and a jump-defect was found to be described, in the quantum context, by infinite dimensional matrices that are solutions of the triangular equations. Unfortunately, there was no room to discuss here further interesting issues, such as the connection with Bäcklund transformations or the scattering of defects in motion.

The jump-defect problem can be extended to all the a_r affine Toda models. On the other hand, the existence of integrable, purely transmitting defects in the other ATFTs appears to be more difficult to prove. In principle, infinite dimensional solutions of the triangular equations can be found for some other Toda models, but it remains to be seen if these solutions can be regarded as transmission matrices.

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