Correlation functions of the integrable spin-s XXZ spin chains and some related topics [2]

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[1,2] T.D. and Chihiro Matsui, NPB **814**[FS](2009)405 ;**831**[FS](2010)359
[3] T. D. and Kohei Motegi, in preparation.

Partially in collaboration with Jun Sato

[4] Akinori Nishino and T. D., J. Stat. Phys. **133** (2008) 587
(For SCP model and the sl_2 loop algebra symmetry)

[2] RAQIS10, LAPTH, Annecy, France. The talk is given on June 15, 2010.

Contents

- **0: Integrable spin-s XXZ spin chains through fusion method**

- **Part I**:
- **(1): Quantum Inverse Scattering Problem for the spin-s XXZ spin chain** (Several tricks such as Hermitian elementary matrices)

- **(2): Multiple-integral representation of arbitrary correlation functions of the spin-s XXZ spin chain**

- **Part II**:
  Quantum Inverse Scattering Problem (QISP) for the super-integrable chiral Potts model (SCP model)

Key idea:

We construct the spin-s representation $V^{(2s)}$ of $U_q(sl_2)$ in the $2s$th tensor product of spin-1/2 representations $V^{(1)}$:

$$\begin{aligned} V^{(2s)} & \subset V_1^{(1)} \otimes \cdots \otimes V_{2s}^{(1)} \\ |2s, n\rangle & = \left(\Delta^{(2s-1)}(X^-)\right)^n \; |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s} \end{aligned}$$

We define dual vectors by anti-involution $*$ of $U_q(sl_2)$:

$$\langle 2s, n| = \left(|2s, n\rangle\right)^*$$

If q is complex, they are different from Hermitian conjugate vectors:

$$\widetilde{\langle 2s, n|} = \left(|2s, n\rangle\right)^\dagger$$

We thus introduce as Hermitian operators $\widetilde{E}^{m,n\,(2s)} = |2s, m\rangle \widetilde{\langle 2s, n|}$

PART 0: Review: Exact methods for deriving the multiple-integral representation of XXZ CFs

- ● The q-vertex operator approach (infinite system, no external field, zdero temperature)
  M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki,
  Phys. Lett. **A 168** (1992) 256–263.

- ● An algebraic approach solving the q KZ equation
  M. Jimbo and T. Miwa,
  J. Phys. A: Math. Gen. **29** (1996) 2923-2958.

- ● Algebraic Bethe-ansatz approach (finite system, external fields)
  N. Kitanine, J.M. Maillet and V. Terras,
  Nucl. Phys. B **567** [FS] (2000) 554–582.

Introduction: Monodromy matrix and the transfer matrix

We define the R-matrix and the monodromy matrix $T_{0,12\cdots L}(\lambda;\{w_j\})$ by

$$R_{12}(\lambda_1,\lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]}$$

$$T_{0,12\cdots L}(\lambda;\{w_j\}) = R_{0L}(\lambda,w_L)R_{0L-1}(\lambda,w_{L-1})\cdots R_{02}(\lambda,w_2)R_{01}(\lambda,w_1)\,.$$

The operator-valued matrix element of the monodromy matrix give the creation and annihilation operators

$$T_{0,12\cdots L}(u;\{w_j\}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_{[0]} .$$

The transfer matrix, $t(u)$, is given by the trace of the monodromy matrix with respect to the 0th space:

$$\begin{aligned} t(u;w_1,\ldots,w_L) &= \mathrm{tr}_0\left(T_{0,12\cdots L}(u;\{w_j\})\right) \\ &= A(u;\{w_j\}) + D(u;\{w_j\})\,. \end{aligned} \qquad (1)$$

The Yang-Baxter equations: Essence of the integrability

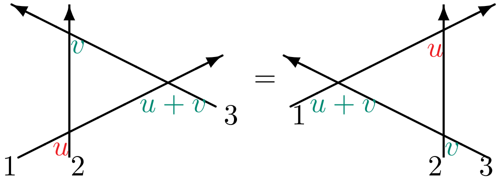

Figure: The Yang-Baxter equations:
$R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u) = R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v)$

Spectral parameter u is expressed by the angle between lines 1 and 2, where the interstion corresponds to $R_{1,2}(u)$.

Review: algebraic BA derivation of the multiple-integral representation of the spin-1/2 XXZ correlation functions

- ● Quantum Inverse Scattering Problem (QISP)
  Local spin-1/2 operators expressed by A, B, C, D (spin-1/2)

- ● Scalar product of the BA:

  If $\{\mu_j\}$ or $\{\lambda_j\}$ are Bethe roots, we have

  $\langle 0 | C(\mu_1) \cdots C(\mu_M) \, B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle = \det \Psi^{'}$ (Slavnov's formula)

  $$\frac{\langle 0 | C(\mu_1) \cdots C(\mu_M) \, B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle}{\langle 0 | C(\lambda_1) \cdots C(\lambda_M) \, B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle} = \det(\Psi^{'} / \Phi^{'})$$

  $\Phi^{'}$: the Gaudin matrix

- ● Integral equations for the matrix elements of $\Psi^{'} / \Phi^{'}$
  To evaluate the matrix elements of $\left(\Psi^{'} / \Phi^{'} \right)$ by solving the integral equations (cf. Izergin)

Hamiltonians of the spin-1/2 and spin-s XXZ spin chains

The spin-1/2 XXZ chain the Hamiltonian under the P. B. C.

$$\mathcal{H}_{\rm XXZ} = \frac{1}{2} \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) . \qquad (2)$$

Here σ_j^a $(a = X, Y, Z)$ are Pauli matrices on the jth site. We define q by

$$\Delta = (q + q^{-1})/2 \qquad (q = \exp \eta)$$

For $-1 < \Delta \leq 1$, $\mathcal{H}_{\rm XXZ}$ is gapless. ($\Delta = \cos \zeta$ by $\boldsymbol{q = e^{i\zeta}}$, $0 \leq \zeta < \pi$.)
For $\Delta > 1$ or $\Delta < -1$, it is gapful. ($\Delta = \pm \cosh \zeta$ by $q = e^{-\zeta}$, $0 < \zeta$.)

Integrable spin-s XXZ Hamiltonian $\mathcal{H}_{\rm XXZ}^{(2s)}$ is given by the logarithmic derivative of the spin-s XXZ transfer matrix. (We express it by qCGC.)

$$\text{s} = 1 \text{ XXX case} \qquad \mathcal{H}_{\rm XXX}^{(2)} = J \sum_{j=1}^{N_s} \left(\vec{S}_j \cdot \vec{S}_{j+1} - (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right) . \qquad (3)$$

Hereafter we shall often set $\ell = 2s$.

References on the spin-s XXX or XXZ correlation functions

Relevant algebraic Bethe-ansatz studies on the spin-s XXX CFs

[1] N. Kitanine,
Correlation functions of the higher spin XXX chains,
J. Phys. A: Math. Gen. **34** (2001) 8151–8169.

[2] O.A. Castro-Alvaredo and J.M. Maillet,
Form factors of integrable Heisenberg (higher) spin chains,
J. Phys. A: Math. Theor. **40** (2007) 7451–7471.

Relevant studies on the higher-spin XXZ and XYZ chains

[1] M. Idzumi, Calculation of Correlation Functions of the Spin-1 XXZ Model by Vertex Operators,
Thesis, University of Tokyo, Feb. 1993.

[2] A.H. Bougourzi and R.A. Weston,
N-point correlation functions of the spin 1 XXZ model,
Nucl. Phys. B **417** (1994) 439–462.

[3] H. Konno,
Free-field representation of the quantum affine algebra $U_q(\widehat{\mathrm{sl}_2})$ and form factors in the higher-spin XXZ model,
Nucl. Phys. B **432** [FS] (1994) 457–486.

[4] T. Kojima, H. Konno and R. Weston,
The vertex-face correspondence and correlation functions of the eight-vertex model I: The general formalism,
Nucl. Phys. B **720** [FS] (2005) 348–398.

Fusion construction

First trick: **Applying the R-matrix R^+ in the homogeneous grading to the fusion construction**

Through a similarity transformation we transform R to R^+

$$R^+_{12}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^-(u) & 0 \\ 0 & c^+(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{[1,2]} . \quad (4)$$

$$c^{\pm}(u) = e^{\pm} \sinh \eta / \sinh(u+\eta) \quad b(u) = \sinh u / \sinh(u+\eta)$$

The R^+ gives the intertwiner of the affine quantum group $U_q(\widehat{sl_2})$ in the homogeneous grading:

$$R^+_{12}(u) \, (\Delta(a))_{12} = (\tau \circ \Delta(a))_{12} \, R^+_{12}(u) \quad a \in U_q(sl_2)$$

where τ denotes the permutation operator: $\tau(a \otimes b) = b \otimes a$ for $a, b \in U_q(sl_2)$, and $(\Delta(a))_{12} = \pi_1 \otimes \pi_2 \, (\Delta(a))$.

Projection operators and the fusion constrution

We define permutation operator Π_{12} by

$$\Pi_{12} v_1 \otimes v_2 = v_2 \otimes v_1 \, , \tag{5}$$

and then define $\check{R}$ by

$$\check{R}^{+}_{12}(u) = \Pi_{12} R^{+}_{12} \tag{6}$$

We define spin-1 projection operator by

$$P^{(2)}_{12} = \check{R}^{+}_{12}(u = \eta) \tag{7}$$

We define $spin - \ell/2$ projection operator recursively as follows.

$$P^{(\ell)}_{12\cdots\ell} = P^{(\ell-1)}_{12\cdots\ell-1} \check{R}^{+}_{\ell-1,\ell}((\ell-1)\eta) P^{(\ell-1)}_{12\cdots\ell-1} \, , \tag{8}$$

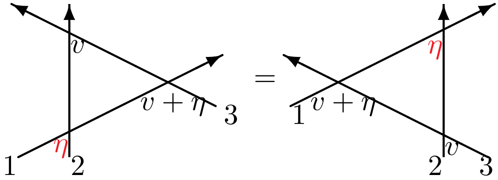

Figure: Projector $P^{(2)}$ is given by the special value of R-matrix $\check{R}(\eta)$, and hence it commutes with other R-matrices. (Cf. Kulish, Sklyanin,and Reshetikhin (1981)).

We define monodromy matrix $T_0^{(1,\,2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s})$ acting on the tensor product $V^{(1)}(\lambda_0) \otimes \left(V^{(2s)}(\xi_1) \otimes \cdots \otimes V^{(2s)}(\xi_{N_s})\right)$ as follows.

$$T_0^{(1,\,2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s}) = P_{12\ldots L}^{(2s)} \cdot R_{0,12\cdots L}^{+}(\lambda_0; w_1^{(2s)}, \ldots, w_L^{(2s)}) \cdot P_{12\ldots L}^{(2s)} . \tag{9}$$

Here inhomogenous parameters w_j are given by complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s)} = \xi_p - (k-1)\eta \quad (p = 1, 2, \ldots, N_s; k = 1, \ldots, 2s.)$$

More precisely, we shall put them as almost complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s;\,\epsilon)} = \xi_p - (k-1)\eta + \epsilon r_k \quad (p = 1, 2, \ldots, N_s; k = 1, \ldots, 2s.)$$

We express the matrix elements of the monodromy matrix as follows.

$$T_{0,\,12\cdots N_s}^{(1,\,2s)}(\lambda; \{\xi_k\}_{N_s}) = \left(\begin{array}{cc} A^{(2s)}(\lambda; \{\xi_k\}_{N_s}) & B^{(2s)}(\lambda; \{\xi_k\}_{N_s}) \\ C^{(2s)}(\lambda; \{\xi_k\}_{N_s}) & D^{(2s)}(\lambda; \{\xi_k\}_{N_s}) \end{array} \right) . \tag{10}$$

$$A^{(2s)}(\lambda; \{\xi_k\}_{N_s}) = P_{12\cdots L}^{(2s)} \cdot A^{(1)}(\lambda; \{w_j^{(2s)}\}_L) \cdot P_{12\cdots L}^{(2s)}$$

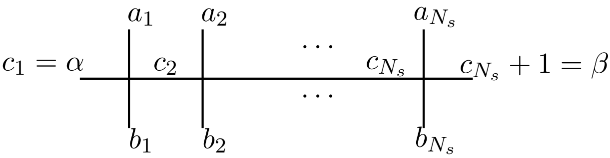

Figure: Matrix element of the monodromy matrix $(T^{(\ell,\,2s)}_{\alpha,\beta})^{a_1,\ldots,a_{N_s}}_{b_1,\ldots,b_{N_s}}$.

Here quantum spaces $V_j^{(2s)}(\xi_j)$ are $(2s+1)$-dimensional (vertical lines), Variables a_j and b_j take values $0, 1, \ldots, 2s$

while the auxiliary space $V_0^{(\ell)}(\lambda_0)$ is $(\ell+1)$-dimensional (horizontal line) . variables c_j take values $0, 1, \ldots, \ell$.

$$L = 2sN_s$$

Spin-$1/2$ chain of L-sites with inhomogeneous parameters $w_1, \ldots, w_L$, while the spin-s chain of N_s sites with $\xi_1, \ldots, \xi_{N_s}$.

We now define $T_0^{(\ell, 2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s})$ acting on the tensor product $V_0^{(\ell)}(\lambda_0) \otimes \left(V^{(2s)}(\xi_1) \otimes \cdots \otimes V^{(2s)}(\xi_{N_s})\right)$ as follows.

$$
\begin{aligned}
T_{0,\, 12\cdots N_s}^{(\ell,\, 2s)} &= P_{a_1 a_2 \cdots a_\ell}^{(\ell)} \, T_{a_1,\, 12\cdots N_s}^{(1,\, 2s)}(\lambda_{a_1}) T_{a_2,\, 12\cdots N_s}^{(1,\, 2s)}(\lambda_{a_1} - \eta) \cdots \\
&\qquad T_{a_\ell,\, 12\cdots N_s}^{(1,\, 2s)}(\lambda_{a_1} - (\ell - 1)\eta) \, P_{a_1 a_2 \cdots a_\ell}^{(\ell)} \, .
\end{aligned}
$$

AB Derivation of the multiple-integral representation for the integrable spin-s XXZ correlation functions

- ● Quantum Inverse Scattering Problem (QISP)
  Spin-s local operators expressed in terms of A, B, C, D (spin-1/2)

- ● Scalar product:
  If $\{\lambda_j\}$ are Bethe roots, we have the determinant expression:

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \langle 0 | C^{(2s)}(\mu_1) \cdots C^{(2s)}(\mu_M)\, B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle \\
& = \det \Psi'^{(2s)} \\
& \frac{\langle 0 | C^{(2s)}(\mu_1) \cdots C^{(2s)}(\mu_M)\, B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle}{\langle 0 | C^{(2s)}(\lambda_1) \cdots C^{(2s)}(\lambda_M)\, B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle} \\
& = \det (\Psi'^{(2s)} / \Phi'^{(2s)})
\end{aligned}
$$

- ● Integral equations for the spin-s
  To evaluate the matrix elements of $\left(\Psi'^{(2s)} / \Phi'^{(2s)} \right)$ by solving the integral equations

Scheme of exact derivation of spin-s XXZ correlation functions

- (1) Algebraic part:

  Quantum Inverse Scattering Problem (QISP) Formula, Slavnov's scalor product formula, Algebraic Bethe ansatz,

  Fusion method, projectors, Hermitian elementary matrices

- (2) Physical part:

  Fundamental conjecture: In the region $\mathbf{0 \le \zeta < \pi/2s}$,
  the spin-s XXZ ground state is given by a Bethe-ansatz eigenvector of $2s$-strings with small deviations

  (Cf. A. Klümper, M. Batchelor and P.A. Pearce, J. Phys. A: Math. Gen. **24** (1991) 3111–3133.)

  The low-lying excitation spectrum of the spin-s XXZ chain should be characterized by the level k SU(2) WZWN model with $k = 2s$.

  Spin-s Gaudin's matrix, spin-s integral equations

The quantum algebra $U_q(sl_2)$ is an associative algebra over $\mathbf{C}$ generated by $X^{\pm}, K^{\pm}$ with the following relations:

$$\begin{aligned} KK^{-1} &= KK^{-1} = 1\,, \quad KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm}\,, \quad , \\ [X^{+}, X^{-}] &= \frac{K - K^{-1}}{q - q^{-1}}\,. \end{aligned} \tag{11}$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over $\mathbf{C}$ with comultiplication

$$\begin{aligned} \Delta(X^{+}) &= X^{+} \otimes 1 + K \otimes X^{+}\,, \quad \Delta(X^{-}) = X^{-} \otimes K^{-1} + 1 \otimes X^{-}\,, \\ \Delta(K) &= K \otimes K\,, \end{aligned} \tag{12}$$

and antipode: $S(K) = K^{-1}\,, S(X^{+}) = -K^{-1}X^{+}\,, S(X^{-}) = -X^{-}K$,
and coproduct: $\epsilon(X^{\pm}) = 0$ and $\epsilon(K) = 1$.

$[n]_q = (q^n - q^{-n})/(q - q^{-1})$: the q-integer of an integer n.
$[n]_q!$: the q-factorial for an integer n.

$$[n]_q! = [n]_q \, [n-1]_q \, \cdots \, [1]_q \, . \tag{13}$$

For integers $m \ge n \ge 0$, the q-binomial coefficient is defined by

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m-n]_q! \, [n]_q!} \, . \tag{14}$$

We define $||\ell, 0\rangle$ for $n = 0, 1, \ldots, \ell$ by

$$||\ell, 0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell \, . \tag{15}$$

Here $|\alpha\rangle_j$ for $\alpha = 0, 1$ denote the basis vectors of the spin-1/2 rep. We define $||\ell, n\rangle$ for $n \ge 1$ and evaluate them as follows .

$$\begin{aligned} ||\ell, n\rangle &= \left(\Delta^{(\ell-1)}(X^-) \right)^n ||\ell, 0\rangle \, \frac{1}{[n]_q!} \\ &= \sum_{1 \le i_1 < \cdots < i_n \le \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- |0\rangle \, q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2} \, . \end{aligned} \tag{16}$$

We have conjugate vectors $\langle \ell, n||$ explicitly as folllows.

$$\langle \ell, n|| = \left[\begin{array}{c} \ell \\ n \end{array} \right]_q^{-1} q^{n(\ell-n)} \sum_{1 \le i_1 < \cdots < i_n \le \ell} \langle 0|\sigma_{i_1}^{+} \cdots \sigma_{i_n}^{+} \, q^{i_1+\cdots+i_n - n\ell + n(n-1)/2} \, . \tag{17}$$

Here the normalization conditions: $\langle \ell, n|| \, ||\ell, n\rangle = 1$.

Conjugate vectors are given by $*$ anti-involution $\langle \ell, n|| = (||\ell, n\rangle)^*$ where

$$(X^-)^* = q^{-1} X^+ K^{-1}, \, (X^+)^* = q^{-1} X^- K, \, K^* = K$$

. with $(\Delta(a))^* = \Delta(a^*)$, $\quad (ab)^* = b^* a^*$, $\quad (a)^{**} = a$, for $a \in U_q(sl(2))$.

In the massive regime where $q = \exp \eta$ with real η, the conjugate vectors $\langle \ell, n||$ are Hermitian conjugate to vectors $||\ell, n\rangle$.

However, in the massless regime $|q| = 1$ and $q \neq \pm 1$, they are not.

For an integer $\ell \geq 0$ we define $\widetilde{\langle \ell, n||}$ for $n = 0, 1, \ldots, n$, by

$$\widetilde{\langle \ell, n||} = \left(\begin{array}{c} \ell \\ n \end{array} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0 | \sigma_{i_1}^{+} \cdots \sigma_{i_n}^{+} \, q^{-(i_1 + \cdots + i_n) + n\ell - n(n-1)/2} \, . \tag{18}$$

They are conjugate to $||\ell, n\rangle$: $\widetilde{\langle \ell, m||} \, ||\ell, n\rangle = \delta_{m,n}$. Here we have denoted the binomial coefficients as follows.

$$\left(\begin{array}{c} \ell \\ n \end{array} \right) = \frac{\ell!}{(\ell - n)! n!} \, . \tag{19}$$

Setting $\langle \ell, n|| \, \widetilde{||\ell, n\rangle} = 1$, vectors $\widetilde{||\ell, n\rangle}$ are given by

$$\widetilde{||\ell, n\rangle} = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^{-} \cdots \sigma_{i_n}^{-} |0\rangle q^{-(i_1 + \cdots + i_n) + n\ell - n(n-1)/2}$$
$$\times \left[\begin{array}{c} \ell \\ n \end{array} \right]_q q^{-n(\ell - n)} \left(\begin{array}{c} \ell \\ n \end{array} \right)^{-1} . \tag{20}$$

Hermitian elementary matrices

In the massless regime we define new elementary matrices $\widetilde{E}^{m,\,n\,(2s\,+)}$ by

$$\widetilde{E}^{m,\,n\,(2s\,+)} = ||\widetilde{2s, m}\rangle\, \langle 2s, n|| \quad \text{for } m, n = 0, 1, \ldots, 2s. \tag{21}$$

We define projection operators by

$$P^{(\ell)}_{12\cdots\ell} = \sum_{n=0}^{\ell} ||\ell, n\rangle\, \langle \ell, n||\,. \tag{22}$$

Let us now introduce another set of projection operators $\widetilde{P}^{(\ell)}_{1\cdots\ell}$ as follows.

$$\widetilde{P}^{(\ell)}_{1\cdots\ell} = \sum_{n=0}^{\ell} \widetilde{||\ell,\, n\rangle}\langle \ell,\, n||\,. \tag{23}$$

Projector $\widetilde{P}^{(\ell)}_{1\cdots\ell}$ is idempotent: $(\widetilde{P}^{(\ell)}_{1\cdots\ell})^2 = \widetilde{P}^{(\ell)}_{1\cdots\ell}$. In the massless regime where $|q| = 1$, it is Hermitian: $\left(\widetilde{P}^{(\ell)}_{1\cdots\ell}\right)^{\dagger} = \widetilde{P}^{(\ell)}_{1\cdots\ell}$. From the definition, we show the following properties:

$$
\begin{aligned}
P^{(\ell)}_{12\cdots\ell}\widetilde{P}^{(\ell)}_{1\cdots\ell} &= P^{(\ell)}_{12\cdots\ell}\,, && (24)\\
\widetilde{P}^{(\ell)}_{1\cdots\ell}P^{(\ell)}_{12\cdots\ell} &= \widetilde{P}^{(\ell)}_{1\cdots\ell}\,. && (25)
\end{aligned}
$$

In the tensor product of quantum spaces, $V_1^{(2s)} \otimes \cdots \otimes V_{N_s}^{(2s)}$, we define $\widetilde{P}^{(2s)}_{12\cdots L}$ by

$$
\widetilde{P}^{(2s)}_{12\cdots L} = \prod_{i=1}^{N_s} \widetilde{P}^{(2s)}_{2s(i-1)+1}\,. \tag{26}
$$

Here we recall $L = 2sN_s$.

Spin-s XXZ Hamiltonian expressed by the q-Clebsch-Gordan coefficiants

$$\mathcal{H}_{\rm XXZ}^{(2s)} = \frac{d}{d\lambda} \log \widetilde{t}^{(2s,\, 2s\, +)}_{12\cdots N_s}(\lambda) \Big|_{\lambda=0,\, \xi_j=0} = \sum_{i=1}^{N_s} \frac{d}{du} \widetilde{\check{R}}^{(2s,2s)}_{i,i+1}(u) \Big|_{u=0}$$

where $\widetilde{t(u)} = \widetilde{P^{(2s)}_{12\cdots L}} t(u)$. Here, the elements of the R-matrix for $V(l_1) \otimes V(l_2)$ are given by (cf. [T.D. and K. Motegi])

$$\check{R} |l_1, a_1\rangle \otimes |l_2, a_2\rangle = \sum_{b_1, b_2} \check{R}^{b_1, b_2}_{a_1, a_2} |l_1, b_1\rangle \otimes |l_2, b_2\rangle,$$

$$\check{R}^{b_1,b_2}_{a_1,a_2} = \delta_{a_1+a_2, b_1+b_2} N(l_1, a_1) N(l_2, a_2) \sum_{j=0}^{\min(l_1, l_2)} N(l_1 + l_2 - 2j, a_1 + a_2)^{-1} \times \rho_{l_1+l_2-2j} \begin{bmatrix} l_2 & l_1 & l_1+l_2-2j \\ b_1 & b_2 & a_1+a_2 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_1+l_2-2j \\ a_1 & a_2 & a_1+a_2 \end{bmatrix}$$

New spin-s QISP formula (the most important result)

For $m \geq n$ we have

$$\widetilde{E}_i^{m,\, n\, (\ell +)} = \left(\begin{array}{c} \ell \\ n \end{array} \right) \left[\begin{array}{c} \ell \\ m \end{array} \right]_q \left[\begin{array}{c} \ell \\ n \end{array} \right]_q^{-1} \widetilde{P}_{1 \cdots L}^{(\ell)} \prod_{\alpha=1}^{(i-1)\ell} (A+D)(w_\alpha)$$
$$\times \prod_{k=1}^{n} D(w_{(i-1)\ell+k}) \prod_{k=n+1}^{m} B(w_{(i-1)2s+k}) \prod_{k=m+1}^{\ell} A(w_{(i-1)\ell+k})$$
$$\times \prod_{\alpha=i\ell+1}^{\ell N_s} (A+D)(w_\alpha) \ \widetilde{P}_{1 \cdots L}^{(\ell)} \, . \qquad (27)$$

For $m \leq n$ we have a similar formula.

We express a factor of $(2s+1) \times (2s+1)$ elementary matrices in terms of a $2s$th product of 2×2 elementary matrices with entries $\{\epsilon_j, \epsilon_j^{'}\}$ as follows.

$$\widetilde{E}^{i_b, j_b \, (2s+)} = C(\{i_k, j_k\}) \, \widetilde{P}^{(2s)}_{12\ldots L} \cdot \prod_{k=1}^{2s} e_k^{\epsilon_k^{'}, \epsilon_k} \cdot \widetilde{P}^{(2s)}_{12\ldots L}. \tag{28}$$

Here, $C(\{i_k, j_k\})$ is given by

$$C(\{i_b, j_b\}) = \left(\begin{array}{c} 2s \\ j_b \end{array} \right) \left[\begin{array}{c} 2s \\ i_b \end{array} \right]_q \left[\begin{array}{c} 2s \\ j_b \end{array} \right]_q^{-1} . \tag{29}$$

Here ϵ_β and $\epsilon_\beta^{'}$ $(\beta = 1, \ldots, 2s)$ are given by

$$\epsilon_{2s(b-1)+\beta} = \left\{ \begin{array}{ll} 1 & (1 \le \beta \le j_b) \\ 0 & (j_b < \beta \le 2s) \end{array} \right. \; ; \; \epsilon_{2s(b-1)+\beta}^{'} = \left\{ \begin{array}{ll} 1 & (1 \le \beta \le i_b) \\ 0 & (i_b < \beta \le 2s) \end{array} \right. . \tag{30}$$

The fundamental conjecture of the spin-s ground state

The spin-s ground state $|\psi_g^{(2s)}\rangle$ is given by $N_s/2$ sets of $2s$-strings for the region: $0 \le \zeta < \pi/2s$

$$\lambda_a^{(\alpha)} = \mu_a - (\alpha - 1/2)\eta + \epsilon_a^{(\alpha)}\,, \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s.$$

Deviaions are given by $\epsilon_a^{(\alpha)} = \sqrt{-1}\delta_a^{(\alpha)}$ where $\delta_a^{(\alpha)}$ are real and decreasing w.r.t. α, and $|\delta_a^{(\alpha)}| > |\delta_a^{(\alpha+1)}|$ for $\alpha < s$, $|\delta_a^{(\alpha)}| < |\delta_a^{(\alpha+1)}|$ for $\alpha > s$.

It is shown analytically through the finite-size corrections of the spin-1 XXZ chain (A. Klümper, M. Batchelor and P.A. Pearce, J. Phys. A: **24** (1991)).

The density of string centers, $\rho_{\rm tot}(\mu)$, is given by

$$\rho_{\rm tot}(\mu) = \frac{1}{N}_s \sum_{p=1}^{N_s} \frac{1}{2\zeta \cosh(\pi(\mu - \xi_p)/\zeta)} \qquad (31)$$

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we denote the density of string centers by $\rho(\lambda)$.

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi\lambda/\zeta)} \, . \qquad (32)$$

Multiple integral representations of the correlation function for an arbitrary product of Hermitian elementary matrices

We define a spin-s correlation function by

$$F^{(2s+)}(\{\epsilon_j, \epsilon_j^{'}\}) = \langle \psi_g^{(2s+)} | \prod_{i=1}^{m} \widetilde{E}_i^{m_i \, , \, n_i \, (2s+)} | \psi_g^{(2s+)} \rangle / \langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle \quad (33)$$

Applying the formulas, we reduce it to the following:

$$F^{(2s+)}(\{\epsilon_j, \epsilon_j^{'}\}) = \langle \psi_g^{(2s+)} | \prod_{j=1}^{2sm} e_j^{\epsilon_j^{'}, \, \epsilon_j} | \psi_g^{(2s+)} \rangle / \langle \psi_g^{(2s+)} | \psi_g^{(2s+)} \rangle \quad (34)$$

Here $\epsilon_j, \epsilon_j^{'} = 0, 1$. We send inhomogeneous parameters w_j to a set of complete strings.

$$w_{2s(b-1)+\beta} = w_{2s(b-1)+\beta}^{(2s;\epsilon)} \to w_{2s(b-1)+\beta}^{(2s)} = \xi_b - (\beta - 1)\eta \, . \quad (35)$$

Let us define $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ by

$$\boldsymbol{\alpha}^- = \{j; \epsilon_j = 0\} \, , \quad \boldsymbol{\alpha}^+ = \{j; \epsilon_j^{'} = 1\} \, . \quad (36)$$

For sets $\boldsymbol{\alpha}^-$ and $\boldsymbol{\alpha}^+$ we define $\tilde{\lambda}_j$ for $j \in \boldsymbol{\alpha}^-$ and $\tilde{\lambda}_j'$ for $j \in \boldsymbol{\alpha}^+$, respectively, by the following relation:

$$(\tilde{\lambda}'_{j'_{max}}, \ldots, \tilde{\lambda}'_{j'_{min}}, \tilde{\lambda}_{j_{min}}, \ldots, \tilde{\lambda}_{j_{max}}) = (\lambda_1, \ldots, \lambda_{2sm}) \, . \qquad (37)$$

We have

$$\begin{aligned} & F^{(2s+)}(\{\epsilon_j, \epsilon_j^{'}\}) = \\ = \quad & \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_1 \\ & \cdots \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\lambda_{s'} \\ & \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_{s'+1} \\ & \cdots \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\lambda_m \\ & \quad \times Q(\{\epsilon_j, \epsilon_j^{'}\}; \lambda_1, \ldots, \lambda_{2sm}) \, \det S(\lambda_1, \ldots, \lambda_{2sm}) \qquad (38) \end{aligned}$$

Here factor Q is given by

$$
\begin{aligned}
& Q(\{\epsilon_j, \epsilon_j^{'}\}) \\
= \quad & (-1)^{\alpha_+} \frac{\prod_{j \in \boldsymbol{\alpha}^-} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j - w_k^{(2s)} + \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) \right)}{\prod_{1 \le k < \ell \le 2sm} \varphi(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})} \\
& \times \frac{\prod_{j \in \boldsymbol{\alpha}^+} \left(\prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j^{'} - w_k^{(2s)} - \eta) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j^{'} - w_k^{(2s)}) \right)}{\prod_{1 \le k < \ell \le 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})} \qquad (39)
\end{aligned}
$$

The matrix elements of S are given by

$$
S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2) \, \delta(\alpha(\lambda_j), \beta(k)) \, , \quad \text{for} \quad j, k = 1, 2, \ldots, 2sm \, . \qquad (40)
$$

Here $\delta(\alpha, \beta)$ denotes the Kronecker delta and $\alpha(\lambda_j)$ are given by a if $\lambda_j = \mu_j - (a - 1/2)\eta$ $(1 \le a \le 2s)$, where μ_j correspond to centers of complete $2s$-strings.

In the denominator, we have set $\epsilon_{k,l}$ associated with λ_k and λ_l as follows.

$$
\epsilon_{k,l} = \begin{cases} i\epsilon & \text{for} \quad \mathcal{I}m(\lambda_k - \lambda_l) > 0 \\ -i\epsilon & \text{for} \quad \mathcal{I}m(\lambda_k - \lambda_l) < 0. \end{cases} \qquad (41)
$$

For $s=1$ and $m=1$ ($w_1^{(2)}=\xi_1$, $w_2^{(2)}=\xi_1-\eta$), we have

$$\begin{aligned}\langle \widetilde{E}_1^{11\,(2+)} \rangle &= \langle \psi_g^{(2+)} | \widetilde{E}_1^{11\,(2+)} | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle \\ &= 2 \langle \psi_g^{(2+)} | A(w_1^{(2)}) D(w_2^{(2)}) | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle \\ &= 2 \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \int_{-\infty-i\zeta+i\epsilon}^{\infty-i\zeta+i\epsilon} \right) d\lambda_1 \left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \int_{-\infty-i\zeta-i\epsilon}^{\infty-i\zeta-i\epsilon} \right) d\lambda_2 \\ &\quad \times Q(\lambda_1, \lambda_2) \, \det S(\lambda_1, \lambda_2) \end{aligned} \tag{42}$$

where $Q(\lambda_1, \lambda_2)$ is given by

$$Q(\lambda_1, \lambda_2) = (-1) \frac{\varphi(\lambda_2 - w_2^{(2)}) \varphi(\lambda_1 - w_1^{(2)} - \eta)}{\varphi(\lambda_2 - \lambda_1 + \eta + \epsilon_{2,1}) \varphi(\eta)} \tag{43}$$

and matrix $S(\lambda_1, \lambda_2)$ is given by

$$\left(\begin{array}{cc} \rho(\lambda_1 - w_1^{(2)} + \eta/2) \delta(\alpha(\lambda_1), 1) & \rho(\lambda_1 - w_2^{(2)} + \eta/2) \delta(\alpha(\lambda_1), 2) \\ \rho(\lambda_2 - w_1^{(2)} + \eta/2) \delta(\alpha(\lambda_2), 1) & \rho(\lambda_2 - w_2^{(2)} + \eta/2) \delta(\alpha(\lambda_2), 2) \end{array} \right) . \tag{44}$$

Here we have applied the following formula

$$\widetilde{E}_1^{1,\,1\,(2+)} \;=\; 2\,\widetilde{P}^{(2)}_{1\cdots L}\,D^{(1+)}(w_1)A^{(1+)}(w_2)\prod_{\alpha=3}^{2N_s}(A^{(1+)}+D^{(1+)})(w_\alpha)\;\widetilde{P}^{(2)}_{1\cdots L}\,.$$

The correlation function is expressed in terms of a single product of the multiple-integral representation.

By evaluating the double integral, the integral over λ_1 is decomposed as follows.

$$\begin{aligned}
& \left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon}+\int_{-\infty-i\zeta+i\epsilon}^{\infty-i\zeta+i\epsilon}\right) d\lambda_1\, Q(\lambda_1,\lambda_2)\,\det S(\lambda_1,\lambda_2) \\
= \; & \left(\int_{-\infty-i\zeta/2}^{\infty-i\zeta/2}+\int_{-\infty-i3\zeta/2}^{\infty-i3\zeta/2}\right) d\lambda_1\, Q(\lambda_1,\lambda_2)\,\det S(\lambda_1,\lambda_2) \\
& + \oint_{\Gamma_1} d\lambda_1\, Q(\lambda_1,\lambda_2)\,\det S(\lambda_1,\lambda_2) + \oint_{\Gamma_2} d\lambda_1\, Q(\lambda_1,\lambda_2)\,\det S(\lambda_1,\lambda_2)\,.
\end{aligned}$$

Thus, the integral is calculated as

$$
\begin{aligned}
& \langle \widetilde{\psi}_g^{(2+)} | \widetilde{E}_1^{11\,(2+)} | \widetilde{\psi}_g^{(2+)} \rangle / \left(2 \langle \widetilde{\psi}_g^{(2+)} | \widetilde{\psi}_g^{(2+)} \rangle \right) \\
= & -2\pi i \int_{-\infty}^{\infty} \frac{\sinh(x-\eta/2)\sinh(x-3\eta/2)}{\sinh\eta} \rho^2(x)\,dx \\
& +2\cosh\eta \int_{-\infty}^{\infty} \frac{\sinh(x-\eta/2)}{\sinh(x+\eta/2)} \rho(x)\,dx \\
& -\int_{-\infty}^{\infty} \rho(x)dx + (-1)2\cosh\eta \int_{-\infty}^{\infty} \frac{\sinh(x-\eta/2)}{\sinh(x+\eta/2)} \rho(x)\,dx \\
= & \frac{\cos\zeta(\sin\zeta - \zeta\cos\zeta)}{2\zeta\sin^2\zeta} \, . \qquad (45)
\end{aligned}
$$

We thus obtain

$$
\langle \widetilde{E}_1^{1,1} \rangle = \frac{\cos\zeta\,(\sin\zeta - \zeta\cos\zeta)}{\zeta\sin^2\zeta} \, . \qquad (46)
$$

Evaluating the integral we obtain the spin-1 EFP with $m = 1$ as follows.

$$\tau^{(2)}(1) = \langle \widetilde{E}_1^{2,\,2} \rangle = \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta}\,. \tag{47}$$

In the XXX limit, we have

$$\lim_{\zeta\to 0} \frac{\zeta - \sin\zeta\cos\zeta}{2\zeta\sin^2\zeta} = \frac{1}{3}\,. \tag{48}$$

The limiting value $1/3$ coincides with the spin-1 XXX result [Kitanine (2001)]. As pointed out in Kitanine (2001), $\langle E_1^{22} \rangle = \langle E_1^{11} \rangle = \langle E_1^{00} \rangle = 1/3$ for the XXX case since it has the rotational symmetry.

Furthermore, due to the uniaxial symmetry we have $\langle \widetilde{E}_1^{0,\,0} \rangle = \langle \widetilde{E}_1^{2,\,2} \rangle$, and hence, directly evaluating the integrals, we confirm

$$\langle \widetilde{E}_1^{0,\,0} \rangle + \langle \widetilde{E}_1^{1,\,1} \rangle + \langle \widetilde{E}_1^{2,\,2} \rangle = 1\,. \tag{49}$$

Symmetric expression of the multiple integrals of $F^{(2s+)}(\{\epsilon_j, \epsilon_j^{'}\})$

$$\frac{1}{\prod_{1\le\alpha<\beta\le 2s}\sinh^m(\beta-\alpha)\eta}\prod_{1\le k<l\le m}\frac{\sinh^{2s}(\pi(\xi_k-\xi_l)/\zeta)}{\prod_{j=1}^{2s}\prod_{r=1}^{2s}\sinh(\xi_k-\xi_l+(r-j)\eta)}$$

$$\sum_{\sigma\in\mathcal{S}_{2sm}/(\mathcal{S}_m)^{2s}}\prod_{j=1}^{\alpha_+}\left(\int_{-\infty+i\epsilon}^{\infty+i\epsilon}+\cdots+\int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon}\right)d\mu_{\sigma j}$$

$$\prod_{j=\alpha_++1}^{2sm}\left(\int_{-\infty-i\epsilon}^{\infty-i\epsilon}+\cdots+\int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon}\right)d\mu_{\sigma j}$$

$$\times(\mathrm{sgn}\,\sigma)\left(\prod_{j=1}^{2sm}\frac{\prod_{b=1}^{m}\prod_{\beta=1}^{2s-1}\sinh(\lambda_j-\xi_b+\beta\eta)}{\prod_{b=1}^{m}\cosh(\pi(\mu_j-\xi_b)/\zeta)}\right)$$

$$\times\frac{i^{2sm^2}}{(2i\zeta)^{2sm}}\prod_{\gamma=1}^{2s}\prod_{1\le b<a\le m}\sinh(\pi(\mu_{2s(a-1)+\gamma}-\mu_{2s(b-1)+\gamma})/\zeta)$$

$$Q(\{\epsilon_j,\epsilon_j^{'}\};\lambda_{\sigma 1},\ldots,\lambda_{\sigma(2sm)}))\ .$$

It is straightforward to take the homogeneous limit: $\xi_k \to 0$.

Part II: sl(2) loop algebra symmetry and the super-integrable chiral Potts model

$$\mathcal{H}_{XXZ} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right) \, .$$

When q is **a root of unity:** $q^{2N}=1$ where $\Delta = (q+q^{-1})/2$,
the $\mathcal{H}_{XXZ}$ (and $\tau_{6\mathrm{V}}(z)$) commutes with the sl_2 loop algebra, $U(L(sl_2))$
in sector A: $S^Z \equiv 0$ (mod N) (and in sector B: $S^Z \equiv N/2$ (mod N)):

$$S^{\pm(N)} = (S^{\pm})^N/[N]! \, , \qquad T^{\pm(N)} = (T^{\pm})^N/[N]!$$

Here $S^{\pm}$ and $T^{\pm}$ are generators of $U_q(\hat{sl}_2)$

$$S^{\pm} = \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2}$$

$$T^{\pm} = \sum_{j=1}^{L} q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}$$

$S^{\pm(N)}$ and $T^{\pm(N)}$ generate the sl_2 loop algebra $U(L(sl_2))$

Generators h_k and $x_k^{\pm}$ ($k \in \mathbf{Z}$) satisfy the defining relations of the sl_2 loop algebra ($j, k \in \mathbf{Z}$)

$$[h_j, x_k^{\pm}] = \pm 2 x_{j+k}^{\pm}\,, \qquad [x_j^{+}, x_k^{-}] = h_{j+k}$$

$$[h_j, h_k] = 0\,, \qquad [x_j^{\pm}, x_k^{\pm}] = 0$$

We have

$$x_0^{+} = S^{+(N)}\,, \quad x_0^{-} = S^{-(N)}\,, \tag{50}$$

$$x_{-1}^{+} = T^{+(N)}\,, \quad x_1^{-} = T^{-(N)}\,, \tag{51}$$

$$h_0 = \frac{2}{N} S^{Z}\,. \tag{52}$$

Main Reference of PART II.

[D1] T. D., Regular XXZ Bethe states at roots of unity as highest weight vectors of the sl_2 loop algebra, J. Phys. A: Math. Theor. Vol. 40 (2007) 7473–7508

[D2] T.D., J. Stat. Mech. (2007) P05007

[ND2008] **A. Nishino** and T. D., An algebraic derivation of the eigenspaces associated with an Ising-like spectrum of the superintegrable chiral Potts model, J. Stat. Phys. **133** (2008) pp. 587–615

[DFM] T. D., K. Fabricius and B. M. McCoy,
The sl_2 loop algebra symmetry of the six-vertex model at roots of unity, J. Stat. Phys. **102** (2001) 701-736.

Many interesting papers relevant to this talk

[AY] H. Au-Yang and J. Perk, J. Phys. A **41** (2008) 275201; J. Phys. A **42** (2009) 375208; J. Phys. A **43** (2010) 025203; arXiv:0907.0362

[B] R. Baxter, arXiv:0906.3551; arXiv:0912.4549; arXiv:1001.0281

[vG] N. Iorgov, V. Shadura, Yu. Tykhyy, S. Pakuliak, and G. von Gehlen, arXiv:0912.5027

[FM2010] K. Fabricius and B.M. McCoy, arXiv:1001.0614

Definition of D-highest weight vectors (Drinfeld-highest weight)

We call a representation of $U(L(sl_2))$ *D-highest weight* if it is generated by a vector Ω satisfying
(i) Ω is annihilated by generators x_k^+:

$$x_k^+ \Omega = 0 \quad (k \in \mathbf{Z})$$

(ii) Ω is a simultaneous eigenvector of generators, h_k's:

$$h_k \Omega = d_k \Omega \,, \quad \text{for} \quad k \in \boldsymbol{Z}.$$

Here d_k denotes the eigenvalues of h_k's.

We call Ω a highest weight vector.

Definition (D-highest weight polynomial)

Let λ_k be eigenvalues as follows:

$$(x_0^+)^k (x_1^-)^k / (k!)^2 \quad \Omega = \lambda_k \Omega \quad (k = 1, 2, \ldots, r)$$

For the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we define polynomial $\mathcal{P}^\lambda(u)$ by

$$\mathcal{P}^\lambda(u) = \sum_{k=0}^{r} \lambda_k \, (-u)^k \, , \qquad (53)$$

We call $\mathcal{P}^\lambda(u)$ the D-highest weight polynomial for highest weight d_k.

When $U(L(sl_2))\Omega$ is irreducible, we call it the Drinfeld polynomial of the representation.

Conjecture on highest weight vectors

Conjecture (Fabricius and McCoy)

Bethe eigenvectors are highest weight vectors of the sl_2 loop algebra, and they have the Drinfeld polynomials [FM].

[FM] K. Fabricius and B. M. McCoy, Progress in Math. Phys. Vol. 23 (*MathPhys Odyssey 2001*), (Birkhäuser, Boston, 2002) 119–144.

It was proved in sector A: $S^Z \equiv 0$ (mod N) when $q^{2N} = 1$ (and in sector B: $S^Z \equiv N/2$ (mod N) when $q^N = 1$ with N being odd): T.D., J. Phys. A: Math. Theor. (2007)

Taking the limit of infinite rapidities, we have

$$\hat{A}(\pm\infty) = q^{\pm S^Z}, \quad \hat{B}(\infty) = T^-, \quad \hat{B}(-\infty) = S^-,$$
$$\hat{D}(\pm\infty) = S^Z, \quad \hat{C}(\infty) = S^+, \quad \hat{C}(-\infty) = T^+, \tag{54}$$

Let N be a positive integer.

Definition (Complete N-string)

We call a set of rapidities z_j a complete N-string, if they satisfy

$$z_j = \Lambda + \eta(N + 1 - 2j) \quad (j = 1, 2, \ldots, N). \tag{55}$$

We call Λ the center of the N-string.

Sufficient conditions of a highest weight vector

Lemma

Suppose that $x_0^{\pm}$, x_{-1}^{+}, x_1^{-} and h_0 satisfy the defining relations of $U(L(sl_2))$, and $x_k^{\pm}$ and h_k ($k \in \boldsymbol{Z}$) are generated from them. If a vector $|\Phi\rangle$ satisfies the following:

$$x_0^{+}|\Phi\rangle = x_{-1}^{+}|\Phi\rangle = 0\,, \tag{56}$$

$$h_0|\Phi\rangle = r|\Phi\rangle\,, \tag{57}$$

$$(x_0^{+})^{(n)}(x_1^{-})^{(n)}|\Phi\rangle = \lambda_n\,|\Phi\rangle \quad \text{for } n = 1, 2, \ldots, r\,, \tag{58}$$

where r is a nonnegative integer and λ_n are complex numbers. Then $|\Phi\rangle$ is highest weight, i.e. we have

$$x_k^{+}|\Phi\rangle = 0 \quad (k \in \boldsymbol{Z})\,, \tag{59}$$

$$h_k|\Phi\rangle = d_k|\Phi\rangle \quad (k \in \boldsymbol{Z})\,, \tag{60}$$

where d_k are complex numbers.

Diagonal property

In addition to regular Bethe roots at generic q, $t_1, t_2, \ldots, t_R$, we introduce kN rapidities, $z_1, z_2, \ldots, z_{kN}$, forming a complete kN-string:
$z_j = \Lambda + (kN + 1 - 2j)\eta$ for $j = 1, 2, \ldots, kN$.
We calculate the action of $(S_\xi^{+(N)})^k (T_\xi^{-(N)})^k$ on the Bethe state at q_0,

$$|R\rangle = B(\tilde{t}_1) \cdots B(\tilde{t}_R)|0\rangle,$$

and we have explicitly shown

$$\left(S_\xi^{+(N)}\right)^k \left(T_\xi^{-(N)}\right)^k |R\rangle = \lim_{q \to q_0} \Bigg(\lim_{\Lambda \to \infty} \frac{1}{([N]_q!)^k} \left(\hat{C}(\infty)\right)^{kN} \times \frac{1}{([N]_q!)^k} \hat{B}(z_1, \eta) \cdots \hat{B}(z_{kN}, \eta_n) \, B(t_1, \eta) \cdots B(t_R, \eta) \, |0\rangle \Bigg)$$

$$= \lambda_k |R\rangle$$

Connection to SCP model

The 1D transverse Ising model

$$\mathcal{H} = A_0 + hA_1$$

where A_0 and A_1 are given by

$$A_0 = \sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z \ , \qquad A_1 = \sum_{j=1}^{L} \sigma_j^x$$

The A_0 and A_1 satisfy the Dolan-Grady conditions:

$$[A_i, [A_i, [A_i, A_{1-i}]]] = 16[A_i, A_{1-i}], (i = 0, 1)$$

and generate the Onsager algebra (OA). [L. Onsager (1944)]

$$\begin{aligned} [A_\ell, A_m] &= 4G_{\ell-m} \\ [G_\ell, A_m] &= 2A_{m+\ell} - 2A_{m-\ell} \\ [G_\ell, G_m] &= 0 \end{aligned}$$

L. Onsager, Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition, Phys. Rev. **65** (1944) 117 – 149.

The Z_N-symmetric Hamiltonian by von Gehlen and Rittenberg:

$$
\begin{aligned}
H_{\mathrm{Z_N}} &= A_0 + k' A_1 \\
&= \frac{4}{N} \sum_{i=1}^{L} \sum_{m=1}^{N-1} \frac{1}{1-\omega^{-m}} Z_i^{2m} + k' \frac{4}{N} \sum_{i=1}^{L} \sum_{m=1}^{N-1} \frac{1}{1-\omega^{-m}} X_i^{-m} X_{i+1}^{m}
\end{aligned}
$$

Here, $q^N = 1$ and $\omega = q^2$, and the operators $Z_i, X_i \in \mathrm{End}((\boldsymbol{C}^N)^{\otimes L})$ are defined by

$$
\begin{aligned}
Z_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) &= q^{\sigma_i} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}, \\
X_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) &= v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i+1} \otimes \cdots \otimes v_{\sigma_L},
\end{aligned}
$$

for the standard basis $\{v_\sigma | \sigma = 0, 1, \cdots, N-1\}$ of $\mathbb{C}^N$ and under the P.B.C.s: $Z_{L+1} = Z_1$ and $X_{L+1} = X_1$.

The Hamiltonian $H_{\mathrm{Z_N}}$ is derived from the expansion of the SCP transfer matrix with respect to the spectral parameter.

The superintegrable τ_2 model

The transfer matix of the superintegrable τ_2 model commutes with that of the superintegrable chiral Potts (SCP) model.

The L-opetrator of the superintegrable τ_2 model is given by that of the integrable spin-$(N-1)/2$ XXZ spin chain with twisted boundary conditions.

We define **SCP polynomial** (superintegrable chiral Potts polynomial) by

$$P_{\rm SCP}(z^N) = \omega^{-p_b} \sum_{j=0}^{N-1} \frac{(1-z^N)^L (z\omega^j)^{-p_a-p_b}}{(1-z\omega^j)^L F_{\rm CP}(z\omega^j) F_{\rm CP}(z\omega^{j+1})}, \qquad (61)$$

Here $F_{\rm CP}(z) = \prod_{i=1}^{R}(1+zu_i\omega)$ and $\{u_i\}$ satisfy the Bethe ansatz equations.

Proposition

If $q^N = 1$ and L is a multiple of N, the transfer matrix $\tau_{\tau_2}(z)$ has the $sl(2)$ loop algebra symmetry in the sector with $S^Z \equiv 0$ (mod N).

Proposition

The superintegrable chiral Potts (SCP) polynomial $P_{\rm SCP}(\zeta)$ is equivalent to the D-highest weight polynomial $P_{\rm D}(\zeta)$ in the sector $S^Z \equiv 0$ (mod N).

Observations: (1) SCP and the D-highest weight polynomials have the same 'Bethe roots'. (2) The dimensions are the same for the OA representation and the highest weight representation of the loop algebra of the τ_2 model.

Conjecture

The OA representation space V_{OA} should correspond to the highest weight representation of the sl(2) loop algebra generated by the regular Bethe states $|R, \{z_i\}\rangle$ of τ_2 model.

Corollary

The eigenvectors of the Z_N symmetric Hamiltonian, $|V\rangle$ are given by

$$|V\rangle = (\text{some element of } L(sl_2)) \times B(z_1) \cdots B(z_R)|0\rangle$$

Here $B(z_1) \cdots B(z_R)|0\rangle$ denotes a regular Bethe state of the τ_2 model and generates the highest weight rep. of the $sl(2)$ loop algebra.

$N = 2$ case

The Hamiltonians of the SCP model and the τ_2-model are given in the forms

$$H_{\mathsf{SCP}} = \sum_{i=1}^{L} \sigma_i^z + \lambda \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x, \qquad H_{\tau_2} = \sum_{i=1}^{L} (\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x),$$

where σ^x, σ^y and σ^z are Pauli's matrices. In terms of fermion operators:

$$c_i = \sigma_i^+ \prod_{j=1}^{i-1} \sigma_j^z, \qquad \tilde{c}_k = \frac{1}{L} \sum_{i=1}^{L} e^{-\imath(ki + \frac{\pi}{4})} c_i,$$

the Hamiltonian H_{τ_2} in the sector with $S^z \equiv \frac{1}{2} \sum_{i=1}^{L} \sigma_i^z \equiv 0 \mod 2$ is written as

$$H_{\tau_2} = \sum_{k \in K} \sin(k) \tilde{c}_k^\dagger \tilde{c}_k,$$

where $K = \{\frac{\pi}{L}, \frac{3\pi}{L}, \dots, \frac{(L-1)\pi}{L}\}$.

For even L, the generators of the $\mathfrak{sl}_2$ loop algebra symmetry of the τ_2-model is given by

$$h_n = \sum_{k \in K} \cot^{2n}\left(\frac{k}{2}\right)(H)_k, \quad x_n^{+} = \sum_{k \in K} \cot^{2n+1}\left(\frac{k}{2}\right)(E)_k,$$
$$x_n^{-} = \sum_{k \in K} \cot^{2n-1}\left(\frac{k}{2}\right)(F)_k, \qquad (62)$$

where $(H)_k, (E)_k, (F)_k$ are given by

$$(H)_k = 1 - \tilde{c}_k^{\dagger}\tilde{c}_k - \tilde{c}_{-k}^{\dagger}\tilde{c}_{-k}, \quad (E)_k = \tilde{c}_{-k}\tilde{c}_k, \quad (F)_k = \tilde{c}_k^{\dagger}\tilde{c}_{-k}^{\dagger}.$$

The reference state $|0\rangle$, generates a $2^{L/2}$-dimensional irreducible representation, a degenerate eigenspace of the Hamiltonian H_{τ_2}.

$$H_{\mathrm{SCP}} = 2\sum_{k \in K}(H)_k + 2\lambda \sum_{k \in K}\Big(\cos(k)(H)_k + \sin(k)\big((E)_k + (F)_k\big)\Big).$$

The $2^{L/2}$ eigenvalues of H_{SCP} and the eigenstates are given by

$$
\begin{aligned}
E(K_+;K_-) &= 2\sum_{k\in K_+}\sqrt{1-2\lambda\cos(k)+\lambda^2} \\
&\quad -2\sum_{k\in K_-}\sqrt{1-2\lambda\cos(k)+\lambda^2}, \\
|K_+;K_-\rangle &= \prod_{k\in K_+}(\cos\theta_k+\sin\theta_k(F)_k)\prod_{k\in K_-}(\sin\theta_k-\cos\theta_k(F)_k)|0\rangle,
\end{aligned}
$$

where K_+ and K_- are such disjoint subsets of K that $K = K_+ \cup K_-$ and $\tan(2\theta_k) = \frac{2\lambda\sin(k)}{2(\lambda\cos(k)-1)}$.

QISP for the SCP model

We consider the spin-1/2 chain of $(N-1)L$ lattice sites. We set $\ell = N-1$.
For $m = n$ $(0 \le m \le N-1)$, upto gauge transformations, we have

$$
\begin{aligned}
& \widetilde{E}_i^{n,\,n\,(N-1\,+)} \\
= & \begin{pmatrix} \ell \\ n \end{pmatrix} \widetilde{P}^{(\ell)}_{1\cdots(N-1)L} \prod_{\alpha=1}^{(i-1)\ell} (A^{(\ell\,+)} + e^{\varphi} D^{(\ell\,+)})(w_{\alpha}^{(\ell\,+)}) \\
& \prod_{k=1}^{n} D^{(\ell\,+)}(w^{(\ell\,+)}_{(i-1)\ell+k}) \times \prod_{k=n+1}^{\ell} A^{(\ell\,+)}(w_{(i-1)\ell+k}) \\
& \prod_{\alpha=i\ell+1}^{\ell L} (A^{(\ell\,+)} + e^{\varphi} D^{(\ell\,+)})(w_{\alpha}^{(\ell\,+)}) \widetilde{P}^{(\ell)}_{1\cdots L(N-1)} \, .
\end{aligned}
$$

Here we set $e^{\varphi} = q$.

We next consider the case of $N=3$, $q^3 = 1$ and $\omega = q^2$

$$
\begin{aligned}
Z_i &= E_i^{00} + \omega E_i^{11} + \omega^2 E_i^{22} \\
&= \widetilde{P}^{(2)}_{1\cdots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) \\
& \quad \times \prod_{k=1}^{2} A^{(2+)}(w_{2(i-1)+k}^{(2+)}) \prod_{\alpha=2i+1}^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) \ \widetilde{P}^{(2)}_{1\cdots 2L} \\
&+ \omega\, \widetilde{P}^{(2)}_{1\cdots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) D^{(2+)}(w_{2(i-1)+1}^{(2+)}) \\
& \quad \times A^{(2+)}(w_{2(i-1)+2}^{(2+)}) \prod_{\alpha=2i+1}^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) \ \widetilde{P}^{(2)}_{1\cdots 2L} \\
&+ \omega^2\, \widetilde{P}^{(2)}_{1\cdots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) \prod_{k=1}^{2} D^{(2+)}(w_{2(i-1)+k}^{(2+)}) \\
& \quad \times \prod^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w_{\alpha}^{(2+)}) \ \widetilde{P}^{(2)}_{1\cdots 2L}
\end{aligned}
$$

Conclusion

- **I-1: QISP for the spin-s XXZ spin chains** both in the massless and massive regimes
  New tricks such as Hermitian elementary matrices
- **I-2: Single-product form of the multiple-integral representation of an arbitrary spin-s XXZ correlation function**
  We have evaluated one-point functions for spin-1 and show
  $\langle \widetilde{E}^{00} \rangle + \langle \widetilde{E}^{11} \rangle + \langle \widetilde{E}^{22} \rangle = 1. \ (0 \le \zeta < \pi/2s)$
- **II: QISP for SCP model**

Possible future work:

- ● Spin-s correlation functions in the massive regime
- ● Factorization property: Are all the multiple integrals reduced into single integrals ? (A question given by B.M. McCoy)
- ● Confirmation of conjectures by Au-Yang and Perk for deriving correlation functions of Z_N-symmetric Hamiltonian (SCP model)

Thank you for your attention !

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