Talk at RAQIS'10, 15-18 June 2010
LAPTH, Annecy-le-Vieux, France

Intertwining Operator Realization of Non-Relativistic Holography

V.K. Dobrev

with N. Aizawa,
Nucl. Phys. **B828** [PM] (2010) 581593

Plan

Introduction

Preliminaries

Choice of Bulk

Bulk-Boundary Correspondence

Intertwining Properties

Nonrelativistic Reduction

Introduction

The role of nonrelativistic symmetries in string theory was always important. In fact, being the theory of everything string theory encompasses together relativistic quantum field theory, classical gravity, and certainly, non-relativistic quantum mechanics, in such a way that it is not even necessary to separate these components.

Thus, it is not a surprise that the Schrödinger group - the group that is the maximal group of symmetry of the Schrödinger equation - is playing more and more a prominent role.

Originally, the Schrödinger group, actually the Schrödinger algebra, was introduced by Niederer and Hagen, as nonrelativistic limit of the vector-field realization of the conformal algebra. In the process, the space components of special conformal transformations decouple from the system. Thus, e.g., in the case of four-dimensional Minkowski space-time from the 15 generators of the conformal algebra we obtain the 12 generators of the Schrödinger algebra.

Recently, Son proposed another method of identifying the Schrödinger algebra in d+1 space-time. Namely, Son started from AdS space in d+3 dimensional space-time with metric that is invariant under the corresponding conformal algebra so(d+1,2) and then deformed the AdS metric to reduce the symmetry to the Schrödinger algebra.

In view of the relation of the conformal and Schrödinger algebra there arises the natural question. Is there a nonrelativistic analogue of the AdS/CFT correspondence, in which the conformal symmetry is replaced by Schrödinger symmetry. Indeed, this is to be expected since the Schrödinger equation should play a role both in the bulk and on the boundary.

Thus, we study the nonrelativistic analogue of the AdS/CFT correspondence in the framework of representation theory. Before explaining what we do let us remind that the AdS/CFT correspondence has 2 ingredients [Maldacena,GKP,Witten]: 1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary; 2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk. Our main focus is put on the first ingredient and we consider the simplest case of the (3+1)-dimensional bulk. It is shown that the holography principle is established using representation theory only, that is, we do not specify any action.

For the implementation of the first ingredient in the Schrödinger algebra context we use a method that is used in the mathematical literature for the construction of discrete series representations of real semisimple Lie groups, and which method was applied in the physics literature first in [DMPPT] in exactly an AdS/CFT setting, though that term was not used then.

The method utilizes the fact that in the bulk the Casimir operators are not fixed numerically. Thus, when a vector-field realization of the algebra in consideration is substituted in the Casimir it turns into a differential operator. In contrast, the boundary Casimir operators are fixed by the quantum numbers of the fields under consideration. Then the bulk/boundary correspondence forces an eigenvalue equation involving the Casimir differential operator. That eigenvalue equation is used to find the two-point Green function in the bulk which is then used to construct the boundary-to-bulk integral operator. This operator maps a boundary field to a bulk field similarly to what was done in the conformal context by Witten for the scalar and other simple cases (and in general in [D]). This is our first main result.

Our second main result is that we show that this operator is an intertwining operator, namely, it intertwines the two representations of the Schrödinger algebra acting in the bulk and on the boundary. This also helps us to establish that each bulk field has actually two bulk-to-boundary limits. The two boundary fields have conjugated conformal weights Δ, $3-\Delta$, and they are related by a boundary two-point function.

We consider also the second ingredient of the AdS/CFT correspondence in the Schrödinger context and show how our formalism involving the Casimir differential operator relates to the case of scalar field theory discussed in [Son,BMcG].

Preliminaries

The Schrödinger algebra $\mathfrak{s}(d)$ in $(d+1)$-dimensional space-time is generated by:

time translation P_t

space translation P_k

Galilei boosts G_k, $k = 1, \cdots, d$

rotations $J_{k\ell} = -J_{\ell k}$ (which generate the subalgebra $so(d)$), $k, \ell = 1, \cdots, d$

dilatation D

conformal transformation K

The non-trivial commutation relations are [BR]

$$
\begin{aligned}
&[P_t, D] = 2P_t,\ \ [P_t, G_k] = P_k,\\
&[P_t, K] = D,\ \ [P_k, D] = P_k,\\
&[P_k, K] = G_k,\ \ [D, G_k] = G_k,\\
&[D, K] = 2K,\\
&[P_i, J_{k\ell}] = \delta_{i\ell}P_k - \delta_{ik}P_\ell,\\
&[G_i, J_{k\ell}] = \delta_{i\ell}G_k - \delta_{ik}G_\ell,\\
&[J_{ij}, J_{k\ell}] = \delta_{ik}J_{j\ell} + \delta_{j\ell}J_{ik} -\\
&-\delta_{i\ell}J_{jk} - \delta_{jk}J_{i\ell},
\end{aligned}
$$

Central extension of the Schrödinger algebra: $\hat{\mathfrak{s}}(d)$, obtained by adding the central element M to $\mathfrak{s}(d)$ which enters the additional commutation relations: $[P_k, G_\ell] = \delta_{k\ell}M$.

Subalgebras:

The generators J_{ij}, P_i form the $((d+1)d/2)$–dimensional Euclidean subalgebra $\mathcal{E}(d)$.

The generators J_{ij}, P_i, D form the $((d+1)d/2+1)$–dimensional Euclidean Weyl subalgebra $\mathcal{W}(d)$.

The subalgebras $\tilde{\mathcal{E}}(d)$ and $\tilde{\mathcal{W}}(d)$ generated by J_{ij}, G_i and by J_{ij}, G_i, D, resp., are isomorphic to $\mathcal{E}(d)$, $\mathcal{W}(d)$, resp.

The generators J_{ij}, P_i, G_i, P_t form the $((d+1)(d+2)/2)$–dimensional Galilei subalgebra $\mathcal{G}(d)$.
The generators J_{ij}, P_i, G_i, K form another $((d+1)(d+2)/2)$–dimensional subalgebra $\tilde{\mathcal{G}}(d)$ which is isomorphic to the Galilei subalgebra.

The isomorphic pairs mentioned above are conjugated to each other.

For the structure of $\hat{\mathfrak{s}}(d)$ it is also important to note that the generators D, K, P_t form an $sl(2, I\!\!R)$ subalgebra.

Obviously $\widehat{\mathfrak{s}}(d)$ is not semisimple and has the following Levi–Malcev decomposition ($d \neq 2$):

$$\begin{aligned} \widehat{\mathfrak{s}}(d) &= \mathcal{N}(d) \ni\!\!\!+ \mathcal{M}(d) \\ \mathcal{N}(d) &= t(d) \oplus g(d) \\ \mathcal{M}(d) &= sl(2, I\!\!R) \oplus so(d) \end{aligned} \qquad (1)$$

with $\mathcal{M}(d)$ acting on $\mathcal{N}(d)$, where the maximal solvable ideal $\mathcal{N}(d)$ is abelian, while the semisimple subalgebra (the Levi factor) is $\mathcal{M}(d)$.

[For $d = 2$ the maximal solvable ideal $t(d) \oplus g(d) \oplus so(2)$ is not abelian, while the Levi factor $sl(2, I\!\!R)$ is simple.]

For our purposes we now restrict to the 1+1 dimensional case, $d = 1$. In this case the centrally extended Schrödinger algebra has six generators:

time translation: H
space translation: P
Galilei boost: G
dilatation: D
conformal transf.: K
mass: M
with the following non-vanishing commutation relations:

$$
\begin{array}{lll}
[H, D] = 2H, & [D, K] = 2K, & [H, K] = D, \\
[P, G] = M, & [P, K] = G, & [H, G] = P, \\
[P, D] = P, & [D, G] = G. &
\end{array}
$$

For our approach we need the Casimir operator. It turns out that the lowest order nontrivial Casimir operator is the 4-th order one [Perroud]:

$$\tilde{C}_4 = (2MD - \{P,G\})^2 - \\ 2\{2MK - G^2, 2MH - P^2\}$$

In fact, there are many cancellations, and the central generator M is a common linear multiple.

Choice of bulk and boundary

We would like to select as bulk space the four-dimensional space $(x, x_{\pm}, z)$ introduced by Son:*

$$ds^2 = -\frac{2(dx_+)^2}{z^4} + + \frac{-2dx_+dx_- + (dx)^2 + dz^2}{z^2} \qquad (2)$$

We require that the Schrödinger algebra is an isometry of the above metric. Here the variable z is the main variable distinguishing the bulk, namely,

*In the general setting of [Son] the space is $(d+3)$-dimensional.

the boundary is obtained when $z = 0$. We also need to replace the central element M by the derivative of the variable x_- which is chosen so that $\frac{\partial}{\partial x_-}$ continues to be central. Thus, a vector-field realization of the Schrödinger algebra is given by:

$$
\begin{aligned}
H &= \frac{\partial}{\partial x_+}, \qquad P = \frac{\partial}{\partial x}, \qquad M = \frac{\partial}{\partial x_-}, \\
G &= x_+\frac{\partial}{\partial x} + x\frac{\partial}{\partial x_-}, \qquad (3) \\
D &= x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + 2x_+\frac{\partial}{\partial x_+}, \\
K &= x_+\left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + x_+\frac{\partial}{\partial x_+}\right) + \\
&\quad + \frac{1}{2}(x^2 + z^2)\frac{\partial}{\partial x_-}
\end{aligned}
$$

and it generates an isometry of (3). This vector-field realization of the Schrödinger algebra acts on the bulk fields $\phi(x_\pm, x, z)$.

In this realization the Casimir becomes:

$$\begin{aligned}
\tilde{C}_4 &= M^2 C_4, \\
C_4 &= \hat{Z}^2 - 4\hat{Z} - 4z^2\hat{S} \\
&= 4z^2\partial_z^2 - 8z\partial_z + 5 - 4z^2\hat{S} \ , \qquad (4) \\
& \hat{S} \equiv 2\partial_-\partial_+ - \partial_x^2 \ , \qquad (5) \\
& \hat{Z} \equiv 2z\partial_z - 1
\end{aligned}$$

Note that (5) is the pro-Schrödinger operator.

Next we consider a realization of the Schrödinger algebra on the boundary. Actually, we use a well known such vector-field realization [BR] in which we only modify the expression for M:

$$
\begin{aligned}
H &= \frac{\partial}{\partial x_+}, \qquad P = \frac{\partial}{\partial x}, \qquad M = \frac{\partial}{\partial x_-}, \\
G &= x_+ \frac{\partial}{\partial x} + xM, \qquad\qquad (6) \\
D &= x \frac{\partial}{\partial x} + \Delta + 2x_+ \frac{\partial}{\partial x_+}, \\
K &= x_+ \left(x \frac{\partial}{\partial x} + \Delta + x_+ \frac{\partial}{\partial x_+} \right) + \frac{1}{2} x^2 M
\end{aligned}
$$

where Δ is the conformal weight. This vector-field realization of the Schroedinger algebra acts on the boundary field $\phi(x_{\pm}, x)$ with fixed conformal weight Δ.

In this realization the Casimir becomes:

$$\begin{aligned} \tilde{C}_4^0 &= M^2 C_4^0, \\ C_4^0 &= (2\Delta - 1)(2\Delta - 5) \end{aligned} \qquad (7)$$

As expected C_4^0 is a constant which has the same value if we replace Δ by $3 - \Delta$:

$$C_4^0(\Delta) = C_4^0(3 - \Delta) \qquad (8)$$

This already means that the two boundary fields with conformal weights Δ and $3 - \Delta$ are related, or in mathematical language, that the corresponding representations are (partially) equivalent. This will be very important also below.

Boundary-to-bulk correspondence

As we explained in the Introduction we first concentrate on one aspect of AdS/CFT [GKP,Witten], namely, the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field φ to a bulk field ϕ, cf. [Witten], also [D]. Mathematically, this means the following. We treat both the boundary fields and the bulk fields as representation spaces of the Schrödinger algebra. The action of the Schrödinger algebra in the boundary, resp. bulk, representation spaces is given by formulae (6), resp. by formulae (3). The boundary-to-bulk operator maps the boundary representation space to the bulk representation space. This will be done within the framework of representation theory without specifying any action.

The fields on the boundary are fixed by the value of the conformal weight Δ, correspondingly, as we saw, the Casimir has the eigenvalue determined by Δ:

$$\begin{aligned} C_4^0 \varphi(x_\pm, x) &= \lambda \varphi(x_\pm, x) \ , \\ \lambda &= (2\Delta - 1)(2\Delta - 5) \end{aligned} \tag{9}$$

Thus, the first requirement for the corresponding field on the bulk $\phi(x_\pm, x, z)$ is to satisfy the same eigenvalue equation, namely, we require:

$$\begin{aligned} C_4 \phi(x_\pm, x, z) &= \lambda \phi(x_\pm, x, z) \ , \qquad (10) \\ \lambda &= (2\Delta - 1)(2\Delta - 5) \end{aligned}$$

where C_4 is the differential operator given in (4). Thus, in the bulk the eigenvalue condition is a differential equation.

The other condition is the behaviour of the bulk field when we approach the boundary:

$$\phi(x_\pm, x, z) \ \rightarrow \ z^\alpha \varphi(x_\pm, x) \ , \quad (11)$$
$$\alpha = \Delta, 3 - \Delta$$

To find the boundary-to-bulk operator we follow the method of [DMPPT], namely, we find the two-point Green function in the bulk solving the differential equation:

$$(C_4 - \lambda)\, G(\chi, z\, ;\, \chi', z') = z'^4\, \delta^3(\chi - \chi')\, \delta(z - z') \qquad (12)$$

where $\chi = (x_+, x_-, x)$.

It is important to use an invariant variable which in our case is:

$$u =$$

$$= \frac{4zz'}{(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-) + (z+z')^2}$$

The normalization is chosen so that for coinciding points we have $u = 1$.

In terms of u the Casimir becomes:

$$C_4 = 4u^2(1-u)\frac{d^2}{du^2} - 8u\frac{d}{du} + 5 \quad (13)$$

The eigenvalue equation can be reduced to the hypergeometric equation by the substitution:

$$\begin{aligned} G(\chi, z; \chi', z') &= G(u) \\ &= u^{\alpha} \widehat{G}(u) \end{aligned}$$

and the two solutions are:

$$\widehat{G}(u) = F(\alpha, \alpha - 1; 2(\alpha - 1); u)$$

where $F = {}_2F_1$ is the standard hypergeometric function, $\alpha = \Delta, 3 - \Delta$.

As expected at $u = 1$ both solutions are singular: by [BaEr], they can be recast into:

$$G(u) = \frac{u^{\Delta}}{1-u} F(\Delta - 2, \Delta - 1; 2(\Delta - 1); u),$$

$$G(u) = \frac{u^{3-\Delta}}{1-u} F(1-\Delta, 2-\Delta; 2(2-\Delta); u).$$

Following the general method the boundary-to-bulk operator is obtained from the two-point bulk Green function by bringing one of the points to the boundary, however, one has to take into account all info from the field on the boundary.

More precisely, in mathematical terms we express the function in the bulk with boundary behaviour (11) through the function on the boundary by the formula:

$$\phi(\chi, z) = \int d^3\chi' \, S_\alpha(\chi - \chi', z) \, \varphi(\chi'), \tag{14}$$

where $d^3\chi' = dx'_+ dx'_- dx'$ and $S_\alpha(\chi - \chi', z)$ is defined by

$$S_\alpha(\chi - \chi', z) = \lim_{z' \to 0} z'^{-\alpha} \, G(u) =$$

$$= \left[\frac{4z}{(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-) + z^2} \right]^\alpha$$

Intertwining properties

Let us denote by L_α the bulk-to-boundary operator :

$$(L_\alpha \ \phi)(\chi) \doteq \lim_{z\to 0} z^{-\alpha}\phi(\chi, z), \qquad (15)$$

where $\alpha = \Delta, 3-\Delta$ consistently with (11). The intertwining property is:

$$L_\alpha \circ \hat{X} = \tilde{X}_\alpha \circ L_\alpha, \qquad X \in \hat{\mathfrak{s}}(1), \quad (16)$$

where $\tilde{X}_\alpha$ denotes the action of the generator X on the boundary (6) (with Δ replaced by α from (11)), $\hat{X}$ denotes the action of the generator X in the bulk (3).

Let us denote by $\tilde{L}_\alpha$ the boundary-to-bulk operator in (14):

$$\phi(\chi, z) = (\tilde{L}_\alpha \varphi)(\chi, z) \doteq$$
$$\doteq \int d^3\chi' \, S_\alpha(\chi - \chi', z) \, \varphi(\chi')$$

The intertwining property now is:

$$\tilde{L}_\alpha \circ \tilde{X}_{3-\alpha} = \hat{X} \circ \tilde{L}_\alpha, \qquad X \in \hat{\mathfrak{s}}(1). \tag{17}$$

Next we check consistency of the bulk-to-boundary and boundary-to-bulk operators, namely, their consecutive application in both orders should be the identity map:

$$L_{3-\alpha} \circ \tilde{L}_\alpha \;=\; \mathbf{1}_{\text{boundary}}, \qquad (18)$$

$$\tilde{L}_\alpha \circ L_{3-\alpha} \;=\; \mathbf{1}_{\text{bulk}}. \qquad (19)$$

Checking (18) means:

$$(L_{3-\alpha} \circ \tilde{L}_\alpha \, \varphi)(\chi) =$$
$$= \lim_{z\to 0} z^{\alpha-3} \, (\tilde{L}_\alpha \, \varphi) \, (\chi, z)$$
$$= \lim_{z\to 0} z^{\alpha-3} \int d^3\chi' \, S_\alpha(\chi - \chi', z) \, \varphi(\chi')$$
$$= \lim_{z\to 0} z^{\alpha-3} \int d^3\chi' \, \left(\tfrac{4z}{A}\right)^\alpha \, \varphi(\chi') \;,$$

$$A = (x - x')^2 - 2(x_+ - x'_+)(x_- - x'_-) + z^2$$

For the above calculation we interchange the limit and the integration, and use the following formula:

$$\lim_{z\to 0} z^{\alpha-3} \left(\frac{4z}{A}\right)^{\alpha} = \qquad (20)$$

$$= 2^{2\alpha}\pi^{3/2}\,\frac{\Gamma(\alpha-\frac{3}{2})}{\Gamma(\alpha)}\,\delta^3(\chi-\chi')\ ,$$

$$\alpha - 3/2 \notin \mathbb{Z}_-$$

Using (20) we obtain:

$$(L_{3-\alpha} \circ \tilde{L}_\alpha \, \varphi)(\chi) = 2^{2\alpha} \pi^{3/2} \frac{\Gamma(\alpha - \frac{3}{2})}{\Gamma(\alpha)} \varphi(\chi) \tag{21}$$

Thus, in order to obtain (18) exactly, we have to normalize, e.g., $\tilde{L}_\alpha$.

We note the excluded values $\alpha - 3/2 \notin \mathbb{Z}_-$ for which the two intertwining operators are not inverse to each other. This means that at least one of the representations is reducible. This reducibility was established [DDM] for the associated Verma modules with lowest weight determined by the conformal weight Δ.[†]

Checking (19) is now straightforward, but also fails for the excluded values.

[†]For more information on the representation theory and related hierarchies of invariant differential operators and equations, cf. [ADDMS].

Note that checking (18) we used (15) for $\alpha \to 3-\alpha$, i.e., we used one possible limit of the bulk field (14). But it is important to note that this bulk field has also the boundary as given in (15). Namely, we can consider the field:

$$\varphi_0(\chi) \doteq (L_\alpha\ \phi)(\chi) = \lim_{z\to 0} z^{-\alpha}\phi(\chi, z), \tag{22}$$

where $\phi(\chi, z)$ is given by (14). We obtain immediately:

$$\varphi_0(\chi) = \int d^3\chi'\, G_\alpha(\chi-\chi')\,\varphi(\chi'), \tag{23}$$

where

$$G_\alpha(\chi) = \left[\frac{4}{x^2 - 2x_+x_-}\right]^\alpha . \tag{24}$$

If we denote by G_α the operator in (23) then we have the intertwining property:

$$\tilde{X}_\alpha \circ G_\alpha = G_\alpha \circ \tilde{X}_{3-\alpha} \quad . \tag{25}$$

Thus, the two boundary fields corresponding to the two limits of the bulk field are equivalent (partially equivalent for $\alpha \in \mathbb{Z} + 3/2$). The intertwining kernel has the properties of the conformal two-point function.

Thus, for generic Δ the bulk fields obtained for the two values of α are not only equivalent - they coincide, since both have the two fields φ_0 and φ as boundaries.

Remark: For the relativistic AdS/CFT correspondence the above analysis relating the two fields in (23) was given in [D]. An alternative treatment relating these two fields via the Legendre transform was given later in [KleWit].

As in the relativistic case there is a range of dimensions when both fields $\Delta, 3-\Delta$ are physical:

$$\Delta_-^0 \equiv 1/2 < \Delta < 5/2 \equiv \Delta_+^0 \ . \qquad (26)$$

At these bounds the Casimir eigenvalue $\lambda = (2\Delta - 1)(2\Delta - 5)$ becomes zero.

Nonrelativistic reduction

In order to connect our approach with that of previous works [Son,BMcG,FuMo], we consider the action for a scalar field in the background (3):

$$I(\phi) = -\int d^3\chi dz \sqrt{-g}\,(\partial^\mu \phi^* \partial_\mu \phi + m_0^2 |\phi|^2). \tag{27}$$

By integrating by parts, and taking into account a non-trivial contribution from the boundary, one can see that $I(\phi)$ has the following expression:

$$I(\phi) = \int d^3\chi dz \sqrt{-g}\,\phi^* (\Box - m_0^2)\phi - \\ - \lim_{z\to 0} \int d^3\chi \frac{1}{z^3} \phi^*\, z\partial_z \phi \tag{28}$$

The second term is evaluated using (14). For $z \to 0$, one has

$$z\partial_z \phi \ \sim$$
$$\alpha(4z)^\alpha \int d^3\chi' \frac{\varphi(\chi')}{[(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-)]^\alpha}$$
$$+O(z^{\alpha+2})$$

It follows that

$$\lim_{z\to 0} \int d^3\chi \frac{1}{z^3} \phi^* \, z\partial_z \phi$$
$$= \lim_{z\to 0} \alpha \int d^3\chi d^3\chi' z^{\alpha-3} \phi^*(\chi, z) \left(\frac{4}{A}\right)^\alpha \varphi(\chi')$$
$$= 4^\alpha \alpha \int d^3\chi d^3\chi' \frac{\varphi(\chi)^* \varphi(\chi')}{[(x-x')^2 - 2(x_+ - x'_+)(x_- - x'_-)]^\alpha}$$

The equation of motion being read off from the first term in (28) can be expressed in terms of the differential operator (4):

$$(\Box - m_0^2)\,\phi = \left(\frac{C_4 - 5}{4} + 2\partial_-^2 - m_0^2\right)\phi = 0 \tag{29}$$

Now we set an Ansatz for the fields on the boundary: $\varphi(\chi) = e^{Mx_-}\varphi(x_+, x)$ and compactify the x_- coordinate: $x_- + a \sim x_-$. This leads to a separation of variables for the fields in the bulk in the following way:

$$\phi(\chi, z) = e^{Mx_-} \int dx'_+ dx' \int_0^a d\xi \times$$

$$\left(\frac{4z}{(x-x')^2 - 2(x_+ - x'_+)\xi + z^2}\right)^{\alpha} e^{-M\xi}\varphi(x'_+, x')$$

Thus we are allowed to make the identification $\partial_- = M$ both in the bulk and on the boundary [Son,BMcG]. We remark that under this identification the operator (5) becomes the Schrödinger operator. Integration over ξ turns out to be incomplete gamma function:

$$\phi(\chi, z) = e^{Mx_-}\phi(x_+, x, z), \tag{30}$$

$$\phi(x_+, x, z) = (-2z)^\alpha M^{\alpha-1}\gamma(1-\alpha, Ma)$$
$$\times \int \frac{dx'_+ dx'}{(x_+ - x'_+)^\alpha} \exp\left(-\frac{(x-x')^2 + z^2}{2(x_+ - x'_+)} M\right)$$
$$\times\ \varphi(x'_+, x').$$

This formula was obtained first in [FuMo]. The equation of motion (29) now reads

$$\left(\frac{\lambda - 5}{4} - m^2\right)\phi(x_+, x, z) = 0, \qquad (31)$$

where $m^2 = m_0^2 - 2M^2$.

Requiring $\phi(x_+, x, z)$ to be a solution to the equation of motion makes the connection between the conformal weight and mass:

$$\Delta_\pm = \frac{1}{2}(3 \pm \sqrt{9 + 4m^2}). \qquad (32)$$

This result is identical to the relativistic AdS/CFT correspondence [GKP,Witten].

The action (28) evaluated for this classical solutions has the following form ($\alpha = \Delta_\pm$):

$$I(\phi) = -(-2)^\alpha M^{\alpha-1}\alpha\gamma(1-\alpha, Ma)$$
$$\times \int \frac{dx dx_+ dx' dx'_+}{(x_+ - x'_+)^\alpha} \exp\left(-\frac{(x-x')^2}{2(x_+ - x'_+)} M\right)$$
$$\times \varphi(x_+, x)^* \varphi(x'_+, x'). \qquad (33)$$

The two-point function of the operator dual to ϕ computed from (33) coincides with the result of [Son, BMcG, Henkel, StoHen].

We remark that the Ansatz for the boundary fields $\varphi(\chi) = \exp(Mx_- - \omega x_+ + ikx)$

used in [Son,BMcG] is not necessary to derive (33).

One can also recover the solutions in [Son,BMcG] rather simply in our group theoretical context. We use again the eigenvalue problem of the differential operator (4):

$$C_4\,\phi(x_+,x,z) = \lambda\,\phi(x_+,x,z). \qquad (34)$$

but make separation of variables $\phi(x_+,x,z) = \psi(x_+,x)f(z)$. Then (34) is written as follows:

$$\frac{1}{f(z)}\left(\partial_z^2 - \frac{2}{z}\partial_z + \frac{5-\lambda}{4z^2}\right)f(z) = \frac{1}{\psi(x_+,x)}\widehat{S}\psi(x_+,x) = p^2 \text{ (const)}$$

Schrödinger part is easily solved:

$\psi(x_+, x) = \exp(-\omega x_+ + ikx)$

which gives

$$p^2 = -2M\omega + k^2. \qquad (35)$$

The equation for $f(z)$ now becomes

$$\partial_z^2 f(z) - \tfrac{2}{z}\,\partial_z f(z) + \qquad (36)$$
$$+ \left(2M\omega - k^2 - \tfrac{m^2}{z^2}\right) f(z) = 0$$

This is the equation given in [Son,BMcG] for $d = 1$. Thus, solutions to equation (36) are given by modified Bessel functions: $f_\pm(z) = z^{3/2} K_{\pm\nu}(pz)$ where ν is related to the effective mass m

[Son,BMcG]. In our group theoretic approach one can see its relation to the eigenvalue of C_4 : $\nu = \sqrt{\lambda+4}/2$.

We finish by giving the expression of (33) for the alternate boundary field φ_0. To this end, we again use the Ansatz $\varphi(\chi) = e^{Mx_-}\varphi(x_+, x)$ for (23). Then performing the integration over x'_- it is immediate to see that:

$$\varphi_0(x, x_+) \sim e^{Mx_-} \int \frac{dx' dx'_+}{(x_+ - x'_+)^\alpha} \times \quad (37)$$
$$\times \exp\left(-\frac{(x-x')^2}{2(x_+ - x'_+)} M\right) \varphi(x'_+, x')$$

One can invert this relation since $G_{3-\alpha} \circ G_\alpha = 1_{\text{boundary}}$. Substitution of (37) and its inverse to (33) gives the following expression:

$$I(\phi) \sim \int \frac{dx dx_+ dx' dx'_+}{(x_+ - x'_+)^{3-\alpha}} \times \tag{38}$$

$$\exp\left(-\frac{(x-x')^2}{2(x_+ - x'_+)} M\right) \varphi_0(x_+, x)^* \varphi_0(x'_+, x')$$

Outlook

We plan to extend our considerations for the higher-dimensional cases [Aiz-Dob]. Higher dimensional Schrödinger group has the rotation group as a subgroup. Thus our formalism can be naturally extended to the cases with arbitrary spin. In the relativistic case higher spins were leading to degeneracies, maybe the same phenomenon will appear in the non-relativistic setting.