Ruijsenaars duality in the framework of symplectic reduction

László Fehér, KFKI RMKI, Budapest and University of Szeged

Talk based on joint work with Ctirad Klimčík, IML, Marseille

References: arXiv:0809.1509, 0901.1983, 0906.4198, 1005.4531 [math-ph]

- Two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.

- First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).

- Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero-Sutherland type systems and their 'relativistic' deformations.

The simplest example

Rational Calogero system: $H_{\rm Cal}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$

Symplectic reduction: Consider phase space $T^* \mathrm{i}u(n) \simeq \mathrm{i}u(n) \times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(x) := \mathrm{i}x \sum_{j\neq k} E_{j,k}$$

This yields the self-dual Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $q_1 > \cdots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}x \sum_{j\neq k} \frac{E_{jk}}{q_j - q_k} \equiv L_{\rm Cal}(q,p) \quad \text{Lax matrix,} \quad \mathrm{tr}\,(dP \wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\hat{p}_1 > \cdots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\rm Cal}(\hat{p},\hat{q}) \quad \text{dual Lax matrix,} \quad \mathrm{tr}\,(dP \wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the 'action-angle map' for the two Calogero systems: alias the duality map. Ruijsenaars hinted at analogous picture in general.

A ‘dual pair’ of integrable many-body systems

Hyperbolic Sutherland system (1971):

$$H_{\mathrm{hyp-Suth}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q_j - q_k)}$$

Basic Poisson brackets: $\{q_i, p_j\} = \delta_{i,j}$, x: non-zero, real constant.

Rational Ruijsenaars-Schneider system (1986):

$$H_{\mathrm{rat-RS}}(\hat{p},\hat{q}) = \sum_{k=1}^{n} \cosh(\hat{q}_k) \prod_{j\neq k}\left[1 + \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

Poisson brackets: $\{\hat{p}_i, \hat{q}_j\} = \delta_{i,j}$ ($\hat{p}_i$ are RS ‘particle positions’).

Systems describe n ‘particles’ moving on the line, and are integrable.

Ruijsenaars (1988) constructed ‘duality symplectomorphism’ (action-angle map) between the underlying phase spaces.

Local description of two other dual pairs

Standard trigonometric Ruijsenaars-Schneider [86] system:

$$H_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

It is a relativistic generalization (here with $c = 1$) of

$$H_{\text{trigo-Suth}} = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + \frac{x^2}{2} \sum_{j \neq k} \frac{1}{\sin^2(q_k - q_j)}$$

The dual systems (Ruijsenaars [88,95]):

$$\widehat{H}_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j \neq k} \left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)} \right]^{\frac{1}{2}}$$

$$\widetilde{H}_{\text{rat-RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j \neq k} \left[1 - \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2} \right]^{\frac{1}{2}}$$

$H_{\text{trigo-RS}}$, $\widehat{H}_{\text{trigo-RS}}$: different real forms of complex trigo RS.

Further self-dual systems

Compactified trigonometric RS ($\mathrm{III_b}$) system, locally given by

$$H_{\mathrm{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

Hyperbolic Ruijsenaars-Schneider system:

$$H_{\mathrm{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

• Our purpose is to derive all of Ruijsenaars' dualities by reductions of suitable finite-dimensional, real phase spaces. Then study new cases: systems with two types of particles, $BC(n)$ systems etc.

• Today, I first overview the non-self-dual cases and then mention open problems.

Duality from symplectic reduction: the basic idea

Start with 'big phase space', of group theoretic origin, equipped with *two* commuting families of 'canonical free Hamiltonians'.

Apply suitable *single* symplectic reduction to the big phase space and construct *two* 'natural' models of the reduced phase space.

The two families of 'free' Hamiltonians turn into interesting **many-body Hamiltonians** and **particle-position variables** in terms of *both* models. Their rôle is *interchanged* in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the 'duality symplectomorphism'.

Motivated by KKS [78], the above 'scenario' was described by Gorsky and Nekrasov in the nineties (see e.g. Fock-Gorsky-Nekrasov-Roubtsov [2000]). They focused on local questions working mostly with infinite-dimensional phase spaces and in a complex holomorphic setting. **Global** structure of **real** phase spaces is relevant.

Duality between hyperbolic Sutherland and rational RS

Take real Lie algebra $gl(n,\mathbb{C})$ with bilinear form $\langle X,Y\rangle := \Re \mathrm{tr}\,(XY)$, and minimal coadjoint orbit of $U(n)$: $\mathcal{O}_x := \{\xi = \mathrm{i}x(\mathbf{1}_n - vv^\dagger)\,|\,v\in\mathbb{C}^n,\ |v|^2 = n\,\}$. Start with the 'big phase space' (M,Ω_M):

$$M := T^*GL(n,\mathbb{C})\times\mathcal{O}_x \simeq (GL(n,\mathbb{C})\times gl(n,\mathbb{C}))\times\mathcal{O}_x = \{(g,J^R,\xi)\}.$$

Introduce matrix functions $\mathcal{L}$ and $\hat{\mathcal{L}}$ on M by

$$\mathcal{L}(g,J^R,\xi) := J^R \quad \text{and} \quad \hat{\mathcal{L}}(g,J^R,\xi) := gg^\dagger.$$

These 'unreduced Lax matrices' generate 'canonical free Hamiltonians'

$$H_k := \frac{1}{k}\Re \mathrm{tr}\,(\mathcal{L}^k), \qquad \hat{H}_{\pm k} := \pm\frac{1}{2k}\mathrm{tr}\,(\hat{\mathcal{L}}^k), \qquad k = 1,\ldots,n$$

We shall reduce by symmetry group

$$K := U(n)\times U(n),$$

where $(\eta_L,\eta_R)\in K$ acts on M by symplectomorphism

$$\Psi_{\eta_L,\eta_R} : (g,J^R,\xi)\mapsto(\eta_L g\eta_R^{-1},\eta_R J^R\eta_R^{-1},\eta_L\xi\eta_L^{-1})$$

generated by moment map

$$\Phi : M\to u(n)\oplus u(n), \qquad \Phi(g,J^R) = ((gJ^Rg^{-1})_{u(n)}+\xi, -J^R_{u(n)})$$

Use **two** models of **the** reduced phase space: $M_{\mathrm{red}} := M//_0K \equiv \Phi^{-1}(0)/K$.

Consider the Weyl chamber: $\mathcal{C} := \{\, q \in \mathbb{R}^n \,|\, q_1 > q_2 > \cdots > q_n \,\}$. $T^*\mathcal{C} \simeq \mathcal{C} \times \mathbb{R}^n = \{(q,p)\}$ has Darboux form $\Omega_{T^*\mathcal{C}} = \sum_k dp_k \wedge dq_k$. Define Hermitian matrix function L on $T^*\mathcal{C}$ by

$$L(q,p)_{jk} := p_j \delta_{jk} - \mathrm{i}(1-\delta_{jk}) \frac{x}{\sinh(q_j - q_k)}$$

L is the standard Lax matrix of the hyperbolic Sutherland model.

Next, denote the elements of $T^*\mathcal{C} = \mathcal{C} \times \mathbb{R}^n$ as pairs $(\hat{p}, \hat{q})$. Define (Hermitian, positive definite) matrix-function $\hat{L}$ on $T^*\mathcal{C}$ by

$$\hat{L}(\hat{p},\hat{q})_{jk} = u_j(\hat{p},\hat{q}) \left[\frac{\mathrm{i}x}{\mathrm{i}x + (\hat{p}_j - \hat{p}_k)} \right] u_k(\hat{p},\hat{q}),$$

$$u_j(\hat{p},\hat{q}) := e^{-\hat{q}_j/2} \prod_{m \neq j} \left[1 + \frac{x^2}{(\hat{p}_j - \hat{p}_m)^2} \right]^{\frac{1}{4}}, \quad j = 1, \ldots, n.$$

Then define $\mathbb{R}^n$-valued function

$$v(\hat{p},\hat{q}) := \hat{L}(\hat{p},\hat{q})^{-\frac{1}{2}} u(\hat{p},\hat{q}) \quad \text{with} \quad u = (u_1, \ldots, u_n)^T.$$

Finally, introduce the $\mathcal{O}_x$-valued function

$$\xi(\hat{p},\hat{q}) := \xi(v(\hat{p},\hat{q})) = \mathrm{i}x(\mathbf{1}_n - v(\hat{p},\hat{q}) v(\hat{p},\hat{q})^\dagger)$$

$\hat{L}$ is the standard Lax matrix of the rational Ruijsenaars-Schneider system.

The Sutherland gauge slice S: *The manifold S defined by*

$$S := \{\, (e^q, L(q,p), -\mu(x)) \,|\, (q,p) \in \mathcal{C} \times \mathbb{R}^n \,\}$$

is a **global cross section** *of the gauge orbits in* $\Phi^{-1}(0)$. *In the coordinates* q, p *on* S *the reduced symplectic form reads* $\sum_k dp_k \wedge dq_k$. *Thus, the symplectic manifold* $(S, \sum_k dp_k \wedge dq_k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$ *is a model of the reduced phase space.*

• Goes back to Olshanetsky-Perelomov [76], Kazhdan-Kostant-Sternberg [78].

The Ruijsenaars gauge slice $\hat{S}$: *The manifold $\hat{S}$ defined by*

$$\hat{S} := \{\, (\hat{L}(\hat{p},\hat{q})^{\frac{1}{2}}, 2\hat{p}, \xi(\hat{p},\hat{q})) \,|\, (\hat{p},\hat{q}) \in \mathcal{C} \times \mathbb{R}^n \,\}$$

is a **global cross section** *of the gauge orbits in* $\Phi^{-1}(0)$. *In terms of the coordinates* $\hat{p}$, $\hat{q}$ *on* $\hat{S}$ *the reduced symplectic form is* $\sum_k d\hat{q}_k \wedge d\hat{p}_k$. *Hence, the symplectic manifold* $(\hat{S}, \sum_k d\hat{q}_k \wedge d\hat{p}_k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$ *is a model of the reduced phase space.*

• We proved in arXiv:0901.1983: **Gauge transformation between the two gauge slices is Ruijsenaars' duality symplectomorphism.**

How to obtain the trigonometric Sutherland systems and their Ruijsenaars duals?

Consider cotangent bundle $T^*U(n)$ of $U(n)$ (in right-trivialization):

$$T^*U(n) = \{(g, J_L) \,|\, g \in U(n),\ J_L \in u(n)^* \simeq u(n)\}$$

It carries the natural symplectic form

$$\Omega(g, J_L) = d\,\mathrm{tr}\,(J_L dg g^{-1})$$

and 'canonical free Hamiltonians' $\{H_k\}$ and $\{\widehat{H}_{\pm k}\}$ defined by

$$H_k(g, J_L) := \frac{1}{k}\mathrm{tr}\,(-\mathrm{i}J_L)^k, \quad \widehat{H}_k(g, J_L) := \frac{1}{k}\mathrm{tr}\,(g^k + g^{-k}), \quad \widehat{H}_{-k}(g, J_L) := \frac{1}{\mathrm{i}k}\mathrm{tr}\,(g^k - g^{-k})$$

- One can write down their Hamiltonian flows explicitly.

- They are invariant under the adjoint action of $U(n)$ on $T^*U(n)$:

$$\eta \triangleright (g, J_L) = (\eta g \eta^{-1}, \eta J_L \eta^{-1}) \qquad \forall \eta \in U(n),$$

generated by the moment map $J : T^*U(n) \to u(n)^*$ given by

$$J(g, J_L) = J_L + J_R \quad \text{with} \quad J_R(g, J_L) := -g^{-1} J_L g.$$

J is sum of moment maps generating left/right multiplication.

KKS [78] found that the moment map constraint $J = \mu(x)$ produces the trigonometric Sutherland system from the Hamiltonian system describing the free particle on $U(n)$: $(T^*U(n), \Omega, H_2)$. The Hamiltonians $\{H_k\}$ give action variables of Sutherland system (and $\{\widehat{H}_{\pm k}\}$ become in effect Sutherland particle positions).

It can be shown that using another model of the reduced phase space $\{\widehat{H}_{\pm k}\}$ yield the commuting Hamiltonians of the Ruijsenaars dual of the Sutherland system (and $\{H_k\}$ become in effect the dual particle positions).

Recently in 1005.4531 [math-ph] (V. Ayadi and L.F.: Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction), we considered covering homomorphisms

$$G_2 := \mathbb{R} \times SU(n) \longrightarrow G_1 := U(1) \times SU(n) \longrightarrow G := U(n)$$

and 'KKS reductions' of the 3 cotangent bundles by the effective symmetry group

$$\bar{G} := G/\mathbb{Z}_G \simeq G_1/\mathbb{Z}_{G_1} \simeq G_2/\mathbb{Z}_{G_2}.$$

This 'explained' the web of dualities and coverings due to Ruijsenaars [95]:

$$\begin{array}{ccc}
T^*\mathbb{R} \times T^*SQ(n) & \xrightarrow{\mathrm{id}_2 \times \mathcal{R}_0} & T^*\mathbb{R} \times \mathbb{C}^{n-1} \\
\Big\downarrow {\scriptstyle \psi_2^{\mathrm{I}}} & & \Big\downarrow {\scriptstyle \psi_2^{\mathrm{II}}} \\
T^*U(1) \times T^*SQ(n) & \xrightarrow{\mathrm{id}_1 \times \mathcal{R}_0} & T^*U(1) \times \mathbb{C}^{n-1} \\
\Big\downarrow {\scriptstyle \psi_1^{\mathrm{I}}} & & \Big\downarrow {\scriptstyle \psi_1^{\mathrm{II}}} \\
P = T^*Q(n) & \xrightarrow{\mathcal{R}} & \widehat{P}_c = \mathbb{C}^{n-1} \times \mathbb{C}^{\times}
\end{array}$$

$Q(n) = \mathbb{T}_n^0 / S_n$ is the configuration space of n indistinguishable non-colliding point particles moving on the circle and $SQ(n)$ belongs to the relative motion of n distinguishable particles. On the right-side the corresponding completed dual phase spaces appear and the vertical maps are coverings.

As our final example we deal with the standard trigo RS system, whose phase space is $P := T^*Q(n)$. Here, $Q(n) := \mathbb{T}_n^0/S_n$ with $\mathbb{T}_n^0$ being the regular part of the maximal torus $\mathbb{T}_n < U(n)$.

The corresponding Lax matrix L and symplectic form ω are:

$$L_{jk}(q,p) = \frac{e^{p_k}\sinh(-x)}{\sinh(\mathrm{i}q_j - \mathrm{i}q_k - x)} \prod_{m\neq j}\left[1 + \frac{\sinh^2 x}{\sin^2(q_j - q_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}}$$

$$\omega = \sum_k dp_k \wedge dq_k, \qquad p_k \in \mathbb{R}, \qquad 0 \leq q_k < \pi, \quad q_1 > q_2 > \cdots > q_n$$

The dual system can be *locally* characterized by

$$\hat{L}_{jk}(e^{\mathrm{i}\hat{q}}, \hat{p}) = \frac{e^{\mathrm{i}\hat{q}_k}\sinh(-x)}{\sinh(\hat{p}_j - \hat{p}_k - x)} \prod_{m\neq j}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_j - \hat{p}_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_m)}\right]^{\frac{1}{4}}$$

$\hat{p} = \mathrm{diag}(\hat{p}_1, \ldots, \hat{p}_n) \in \mathfrak{C}_x := \{\hat{p} \,|\, \hat{p}_j - \hat{p}_{j+1} > |x|, \;\; j = 1, \ldots, (n-1)\}$
$e^{\mathrm{i}\hat{q}} \in \mathbb{T}_n$ with $\hat{q} = \mathrm{diag}(\hat{q}_1, \ldots, \hat{q}_n)$. Dual phase space $\hat{P} = \mathbb{T}_n \times \mathfrak{C}_x$ is open submanifold of cotangent bundle of $\mathbb{T}_n$, with $\hat{\omega} = d\hat{p}_k \wedge d\hat{q}_k$.

The commuting flows associated with $\hat{L}$ are **not** complete on $\hat{P}$. Ruijsenaars [95] constructed completion using direct methods.

Poisson-Lie analogue of Kazhdan-Kostant-Sternberg reduction

According to Semenov-Tian-Shansky [85] and Lu-Weinstein [90]:

- P-L analogue of $T^*U(n)$ is Heisenberg double of Poisson $U(n)$.

The Heisenberg double of $U(n)$ is the *real* manifold $GL(n,\mathbb{C})$.

Every $K \in GL(n,\mathbb{C})$ admits two Iwasawa decompositions:

$$K = b_L g_R^{-1} \quad \text{and} \quad K = g_L b_R^{-1} \quad \text{with} \quad g_{L,R} \in U(n),\ b_{L,R} \in B$$

B: group of upper triangular matrices with positive diagonal entries

Define Iwasawa maps $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ and $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$

$$\Lambda_{L,R}(K) := b_{L,R} \quad \text{and} \quad \Xi_{L,R}(K) := g_{L,R}$$

$GL(n,\mathbb{C})$ has natural symplectic form (Alekseev-Malkin [94])

$$\omega_+ = \frac{1}{2}\Im \mathrm{tr}\,(d\Lambda_L \Lambda_L^{-1} \wedge d\Xi_L \Xi_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}\,(d\Lambda_R \Lambda_R^{-1} \wedge d\Xi_R \Xi_R^{-1})$$

Commuting Hamiltonians from dual P-L groups

Iwasawa maps $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$ and $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ are **Poisson maps** if $U(n)$ and B are equipped with their standard Poisson structures. In fact, the Poisson bracket $\{\ ,\ \}_+$ defined by ω_+ closes on

$$\Xi^*_{L,R} C^\infty(U(n)) \quad \text{and on} \quad \Lambda^*_{L,R} C^\infty(B)$$

Induced Poisson bracket on $U(n)$ is standard Sklyanin bracket.

$C^\infty(U(n))^{U(n)}$: the adjoint (conjugation) invariant functions

$C^\infty(B)^c \equiv C^\infty(B)^{U(n)}$: the center of the Poisson bracket on $C^\infty(B)$ provided by the dressing invariants

$$\Lambda^*_L C^\infty(B)^c = \Lambda^*_R C^\infty(B)^c \quad \text{and} \quad \Xi^*_R C^\infty(U(n))^{U(n)}$$

form **Abelian subalgebras** in $C^\infty(GL(n,\mathbb{C}))$ w.r.t. $\{\ ,\ \}_+$

These Abelian algebras give **'canonical free Hamiltonians'**. Their flows are written down explicitly in 0906.4198.

Quasi-adjoint symmetry

Following Lu [90]:
Poisson map from phase space into P-L group B is called (equivariant) *P-L moment map*. Every such map generates infinitesimal Poisson action of $U(n)$

$\Lambda_{L,R} : GL(n, \mathbb{C}) \to B$ moment maps generating left/right multiplications by $U(n)$.
The product $\Lambda := \Lambda_L \Lambda_R : GL(n, \mathbb{C}) \to B$ is also P-L moment map.
Λ generates infinitesimal 'quasi-adjoint' action of $U(n)$.

Concretely, for any $Y \in u(n)$ define vector field $\tilde{Y}$ on $GL(n, \mathbb{C})$ by

$$\mathcal{L}_{\tilde{Y}} f := \Im \mathrm{tr}\,(Y\{f, \Lambda\}_+ \Lambda^{-1}), \qquad \forall f \in C^\infty(GL(n, \mathbb{C}))$$

Integration of infinitesimal action yields $U(n)$ action on $GL(n, \mathbb{C})$:

$$\eta \triangleright K := \eta K \Xi_R(\eta \Lambda_L(K)), \qquad \eta \in U(n), \quad K \in GL(n, \mathbb{C})$$

Now can reduce $(GL(n, \mathbb{C}), \omega_+)$ by choosing $\nu \in B$ and imposing

moment map constraint: $\Lambda(K) = \nu, \quad K \in GL(n, \mathbb{C})$.

'Canonical free Hamiltonians' are invariant under the quasi-adjoint action of $U(n)$; thus can be reduced simultaneously.

‘Unreduced Lax matrices’

generators of $C^\infty(B)^c$: $\quad f_k(b) := \frac{1}{k}\mathrm{tr}\,(bb^\dagger)^k \quad \forall k \in \mathbb{Z}^*$
/$C^\infty(B)^c = C^\infty(B)^{U(n)}$ – dressing invariants/

$$\text{generators of } C^\infty(U(n))^{U(n)}: \quad \phi_k(g) := \frac{1}{k}\mathrm{tr}\,(g^k + g^{-k})$$

$$\phi_{-k}(g) := \tfrac{1}{k\mathrm{i}}\mathrm{tr}\,(g^k - g^{-k}) \quad \forall k \in \mathbb{Z}_+$$

Canonical Hamiltonians $H_k := f_k \circ \Lambda_R$ and $\hat{H}_k := \phi_k \circ \Xi_R$ are **spectral invariants** of matrix functions $\mathcal{L}$ and $\hat{\mathcal{L}}$ defined on the double by

$$\mathcal{L} := \Lambda_R \Lambda_R^\dagger \quad \text{and} \quad \hat{\mathcal{L}} := \Xi_R$$

We call $\mathcal{L}$ and $\hat{\mathcal{L}}$ unreduced Lax matrices.

The quasi-adjoint action operates on the ‘unreduced Lax matrices’ $\mathcal{L}$ and $\hat{\mathcal{L}}$ by similarity transformations. Hence $\mathcal{L}$ and $\hat{\mathcal{L}}$ yield Lax matrices for reduced systems obtained from $\{H_k\}$ and from $\{\hat{H}_k\}$.

Definition of the reduction and the main result

- First, fix value of moment map Λ to some constant $\nu \in B$.
- Second, factor level set $\Lambda^{-1}(\nu)$ by isotropy group G_ν of ν.

The crux is the choice $\nu := \nu(x)$: with $x \neq 0$ real parameter

$$\nu(x)_{kk} = 1, \quad \forall k, \quad \nu(x)_{kl} = (1 - e^{-2x})e^{(l-k)x}, \quad \forall k < l$$

'Constraint-surface' $\Lambda^{-1}(\nu(x))$ is embedded submanifold of $GL(n,\mathbb{C})$.

Central $U(1) < U(n)$ acts trivially, effective gauge group $G_{\nu(x)}/U(1)$ acts freely.

- We constructed two explicit models of the reduced phase space $\Lambda^{-1}(\nu(x))/G_{\nu(x)}$ and identified them with trigo RS phase space (P, ω) and with natural completion of dual phase space $(\hat{P}, \hat{\omega})$.

- Unreduced Lax matrix $\mathcal{L}$ descends to trigo RS Lax matrix L in terms of the model (P, ω) and $\hat{\mathcal{L}}$ yields dual Lax matrix $\hat{L}$ on $(\hat{P}, \hat{\omega})$.

- **In this way we derived 'trigonometric Ruijsenaars duality' from Poisson-Lie duality.** For full details, see our paper in arXiv:0906.4198.

Conclusion and remarks on open problems

Presented group theoretical method that yields many-body systems together with geometric interpretation of their duality relations.

Technically simplifies parts of original work of Ruijsenaars [88,95].

Main advantage:

Completion of local phase spaces and duality symplectomorphisms result automatically, once the correct starting point is 'guessed'.

A few problems for future work:

- Study compactified, hyperbolic and elliptic RS systems.

- Explore reduced systems at arbitrary moment map value.

- Quantum Hamiltonian reduction ($\sim$ works on special functions)
Etingof-Kirillov [94], Noumi [96]: Q.G. interpretation of Macdonald polynomials

- Connections to bispectrality and to separation of variables.

- Investigate 'soliton-antisoliton' systems.

- Derive $BC(n)$ (van Diejen) systems by symplectic reduction.