Exactly solvable models and ultracold atoms

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OUTLINE

1- INTRODUCTION

review some experimental results

2- INTEGRABLE GENERALISED BEC MODELS

main emphasis: present the mathematical construction

3- ULTRACOLD ATOMIC FERMI GASES

main emphasis: discuss the physical properties

4- CONCLUSIONS

outlook of the area

1-INTRODUCTION

Prediction and experiments:

Theoretical prediction:

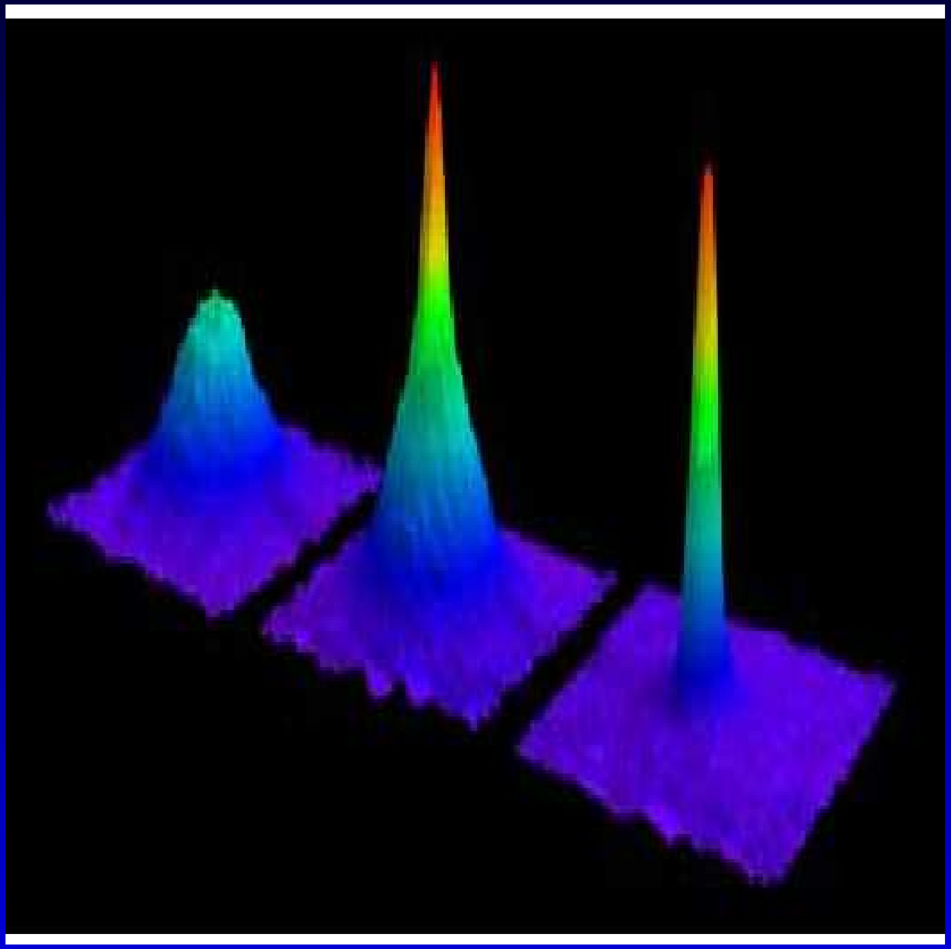
- S. N. Bose (1924)
- A. Einstein (1924-1925)

Experimental realization (70 years later)

- E. A. Cornell *et al*. (1995) $\longrightarrow {}^{87}Rb$
- W. Ketterle *et al*. (1995) $\longrightarrow {}^{23}Na$
- C. C. Bradley *et al*. (1995) $\longrightarrow {}^{7}Li$

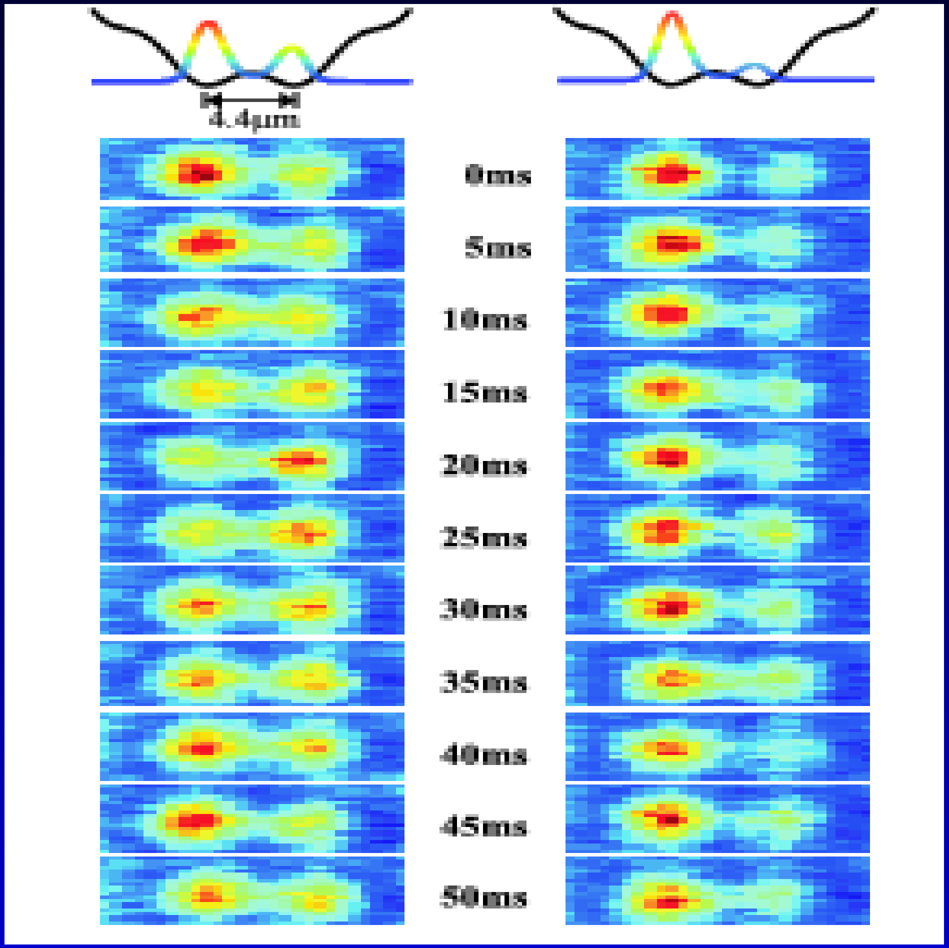
Experimental observation:

A BEC can be identified by a sharp peak in the velocity distribution of a gas of atoms below T_c



D.S. Durfee and W. Ketterle, Optics Express **2** *(1998) 299*

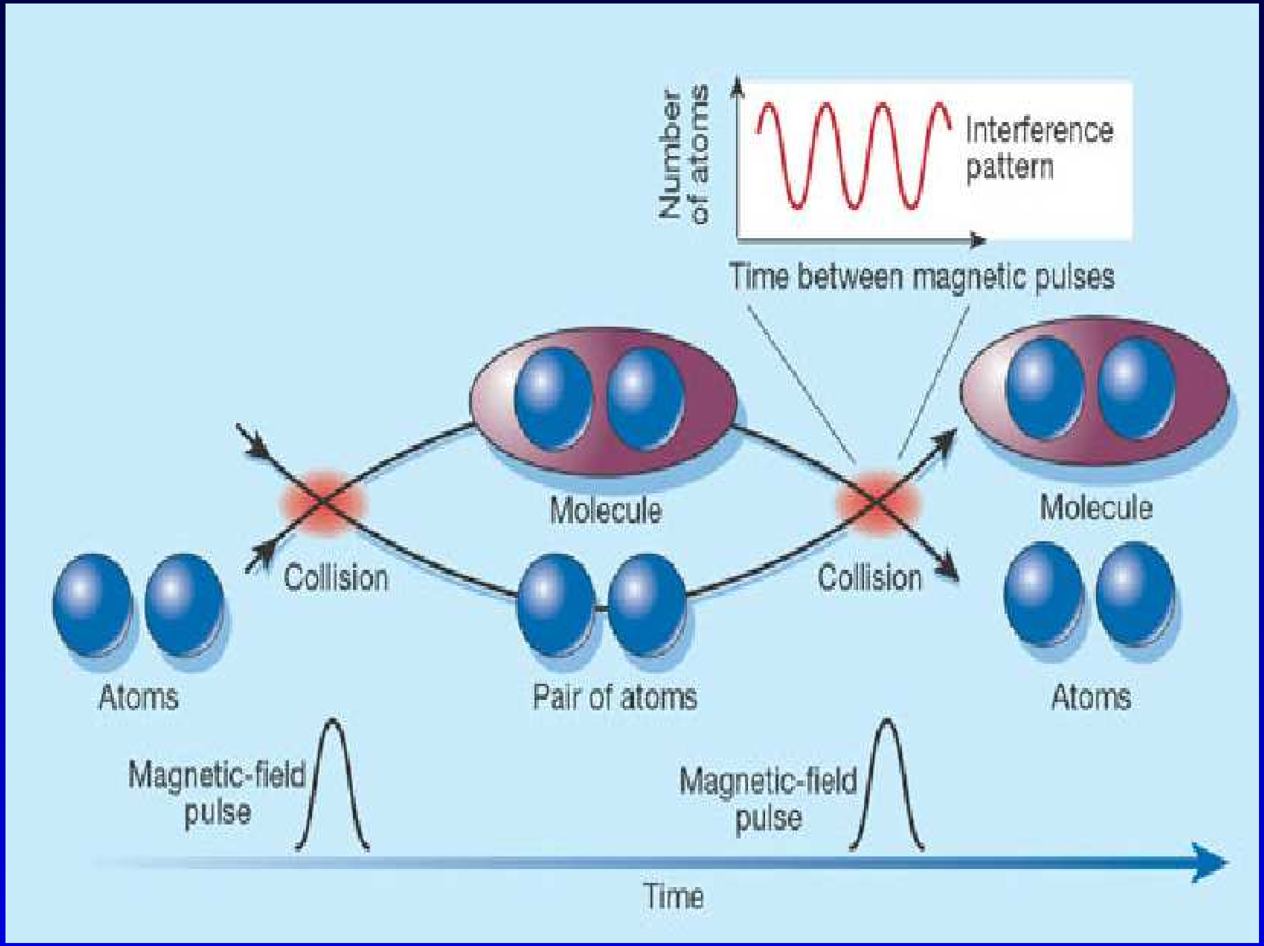
Direct observation of tunneling and self-trapping:



Albiez, M. et al., Phys. Rev. Lett. **95** (2005) 010402

Atom-molecule BEC

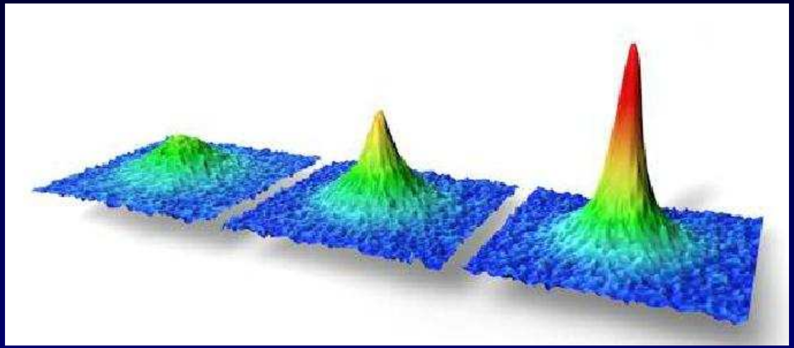
Since the experimental realization of a BEC using atoms, a significant effort was made to produce a stable BEC in a gas of molecules



Zoller, P., Nature **417** (2002) 493

Donley, E. A., Nature **417** (2002) 529

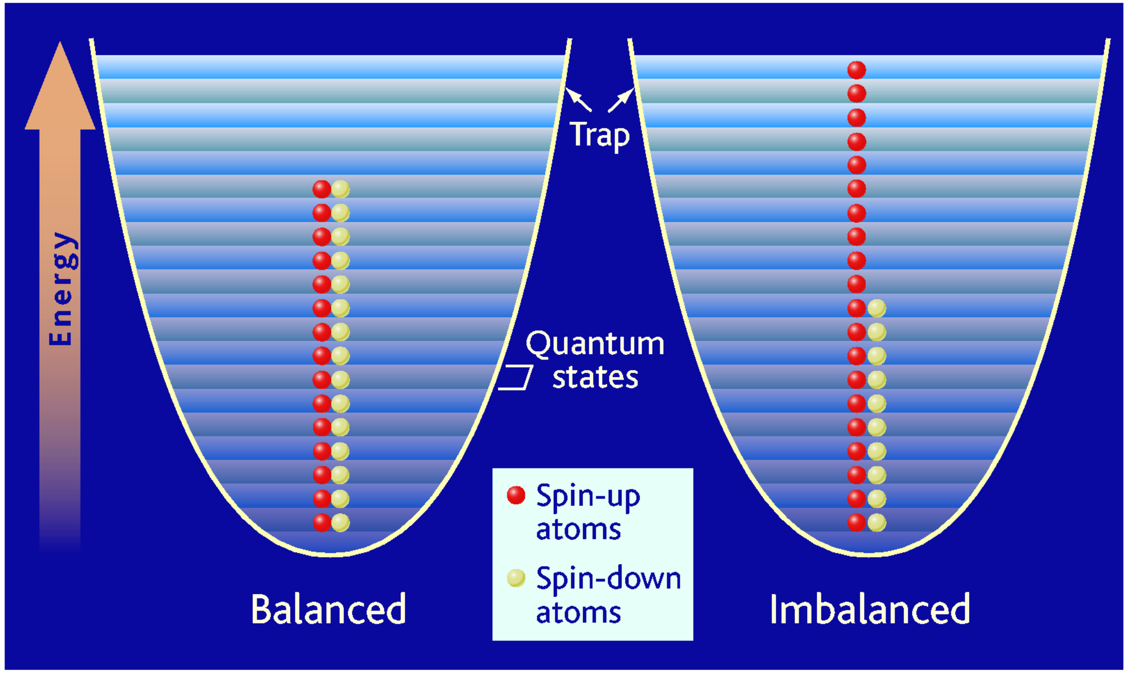
Fermionic pair condensation



D. Jin et al, Nature 424 (2003)

The first ultracold Fermi gas of ^{40}K *atoms was created in 1999 by the group of D. Jin at JILA. They also created a molecular condensate in an ultracold degenerate Fermi gas of* ^{40}K *atoms via Feshbach ressonance in 2003. At the same time, the groups of W. Ketterle and R. Grimm also obtained a molecular condensate. After that the condensation of fermionic pairs was detected.*

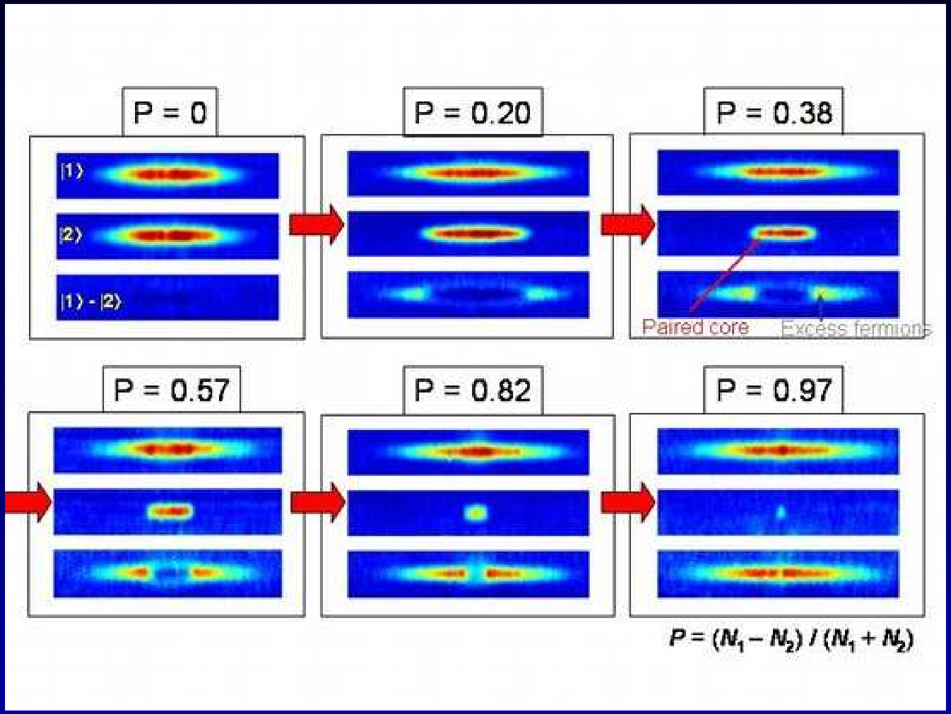
Fermions with Polarization



Find a partner. When atoms spinning one way outnumber those spinning the other, they still can pair and flow freely—perhaps like matter in a neutron star.

In conventional superconductors the number of spin-up and spin-down particles is the same. Superfluidity may still occur in a mismatched system and several exotic phases have been proposed: Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states, etc. Experimental confirmation of these new phases is very difficult for superconductors. But it is relatively straightforward to create and manipulate a Fermi gas with unequal spin population

Pairing and Phase Separation in a Polarized Fermi gas



$$P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow} \quad 0 \leq P \leq 1$$

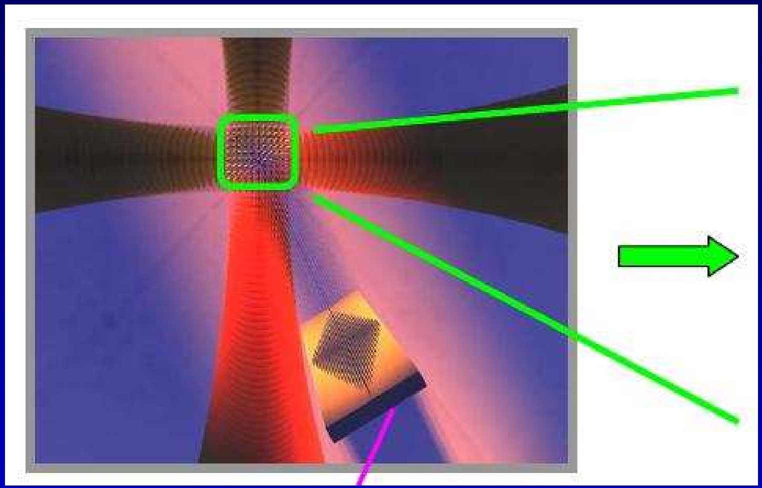
W. Li, G. Partridge, Y. Liao, and R. Hulet., Nuclear Physics A 790 (2007) 88c-95c

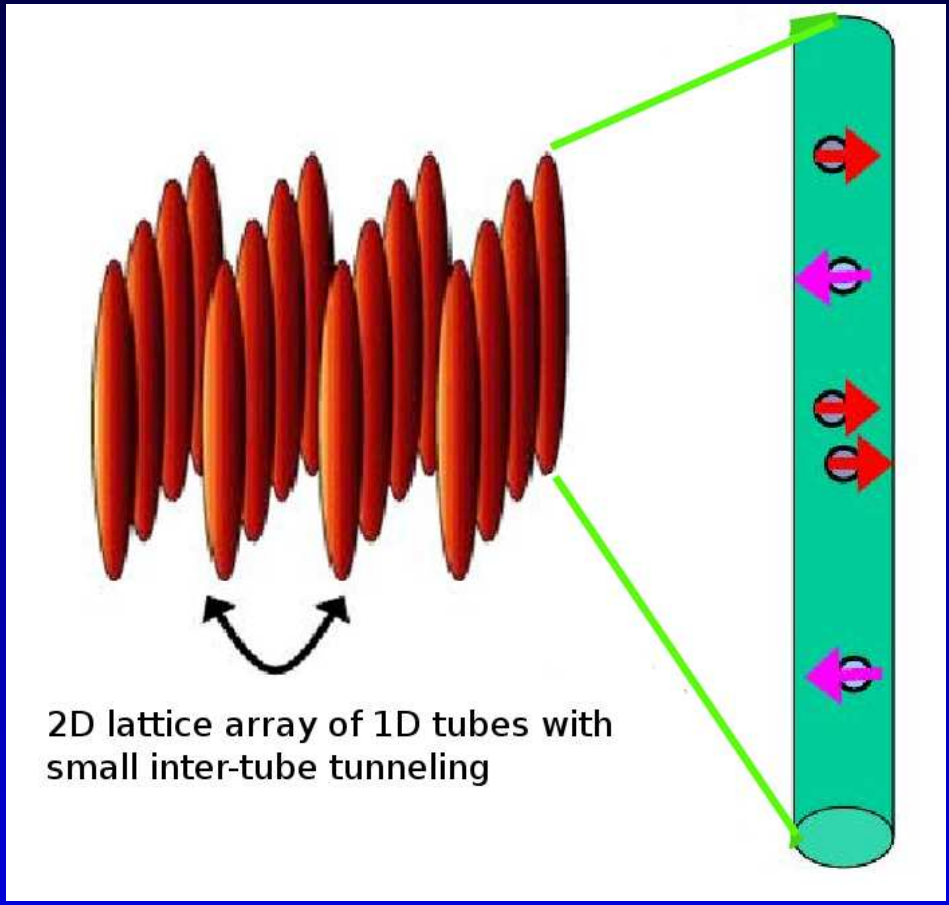
Ultracold Fermi gas of ^{6}Li with two hyperfine states $|1\rangle = |F = \frac{1}{2}, m_f = \frac{1}{2}\rangle$ and $|2\rangle = |F = \frac{1}{2}, m_f = -\frac{1}{2}\rangle$. Basic result: phase separation between a fully paired superfluid core surrounded by a shell of excess, unpaired atoms - 3D

1D Experiments [R. Hulet et al]:

Crossed beam optical trap

1D gas with spin imbalance





Spin-imbalance in a one-dimensional Fermi gas, R. Hulet et al, arXiv:0912.0092

Experimental observations are in quantitative agreement with TBA calculations

2- INTEGRABLE MODELS OF BEC

Two-site Bose Hubbard Hamiltonian:

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

- $N_i = a_i^\dagger a_i$: number of atoms in the well $(i = 1, 2)$
- K: atom-atom interaction term
- $\Delta\mu$: external potential
- $\mathcal{E}_J$: tunneling strength

G. Milburn et al, Phys. Rev. A **55** *(1997) 4318; A. Leggett, Rev. Mod. Phys.* **73** *(2001) 307 A. Foerster, J. Links and H.Q. Zhou, Class. and Quant. Nonlinear Integ. Systems (2003), edited by A. Kundu*

Integrability and exact solution:

- R-matrix:

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$b(u) = \frac{u}{u+\eta} \qquad c(u) = \frac{\eta}{u+\eta}$$

- Yang-Baxter algebra:

$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

- Monodromy-matrix:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

- Yang-Baxter algebra:

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

- Realization of the monodromy matrix:

$$L(u) = \pi(T(u)) = L_1^a(u+w)L_2^a(u-w)$$

$$L_i^a(u) = \begin{pmatrix} u+\eta N_i & a_i \\ a_i^\dagger & \eta^{-1} \end{pmatrix} \qquad i = 1, 2$$

- Transfer matrix:

$$\tau(u) = \pi(Tr(T(u))) = \pi(A(u) + D(u))$$

- Integrability:

$$[\tau(u), \tau(v)] = 0 \longrightarrow [H, \tau(v)] = 0$$

- Hamiltonian and transfer matrix:

$$H = \kappa \left(\tau(u) - \frac{1}{4}(\tau'(0))^2 - u\tau'(0) - \eta^{-2} + w^2 - u^2 \right)$$

with the identification:

$$\frac{K}{4} = \frac{\kappa\eta^2}{2}, \qquad \frac{\Delta\mu}{2} = -\kappa\eta w, \qquad \frac{\mathcal{E}_{\mathcal{J}}}{2} = \kappa$$

$$H = \frac{K}{8}(N_1 - N_2)^2 - \frac{\Delta\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}_J}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

Applying the algebraic Bethe ansatz method:

- Energy:

$$E = -\kappa(\eta^{-2} \prod_{i=1}^{N} \left(1 + \frac{\eta}{v_i - w}\right) - \frac{\eta^2 N^2}{4} - w\eta N - \eta^{-2})$$

- Bethe Ansatz Equations:

$$\eta^2 (v_i^2 - w^2) = \prod_{j \neq i}^{N} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}$$

INTEGRABLE GENERALISED MODELS:

Basic idea:

We can construct integrable generalised models in the BEC context exploring different representations of some algebra, such as the $gl(N)$ algebra and $gl(M/N)$ superalgebra.

A. Foerster, J. Links and H.Q. Zhou, Class. and Quant. Nonlinear Integ. Systems (2003);

A. Foerster and E. Ragoucy, Nuclear Phys. **B777** (2007) 373;

A. Tonel, G. Santos, A. Foerster, I. Roditi, Z. Santos, Physical Review **A79** (2009) 013624;

Model for atom-molecule BEC

$$\begin{aligned} H &= U_a N_a^2 + U_b N_b^2 + U_{ab} N_a N_b + \mu_a N_a + \mu_b N_b \\ &+ \Omega(a^\dagger a^\dagger b + b^\dagger a a) \end{aligned}$$

- $N_a = a^\dagger a$: number of atoms
- $N_b = b^\dagger b$: number of molecules
- U_i's: interaction strengths
- μ_i's: external potentials
- Ω: amplitude for interconversion of atoms and molecules

Triatomic-molecular BEC models:
see POSTER: ITZHAK RODITI

- Monodromy matrix:

$$\pi(T(u)) = \eta^{-1} g L^b(u - \delta - \eta^{-1}) L^K(u)$$

$$g = diag(-, +)$$

$$L^b(u) = \begin{pmatrix} u + \eta N_b & b \\ b^\dagger & \eta^{-1} \end{pmatrix}$$

$$L^K(u) = \begin{pmatrix} u + \eta K^z & \eta K^- \\ -\eta K^+ & u - \eta K^z \end{pmatrix}$$

$$K^+ = \frac{(a^\dagger)^2}{2}, \qquad K^- = \frac{a^2}{2}, \qquad K^z = \frac{2N_a + 1}{4}$$

Three-coupled BEC model

$$\mathcal{H} = \Omega_2 \left(a_2^\dagger a_1 + a_1^\dagger a_2 + a_2^\dagger a_3 + a_3^\dagger a_2 \right)$$
$$+ \Omega \left(a_1^\dagger a_3 + a_3^\dagger a_1 \right) + \mu\, n_1 + \mu\, n_3 + \mu_2\, n_2,$$

- (1): left well
- (2): middle well
- (3): right well
- Ω: tunneling between the left and the right wells
- Ω_2: left-middle and middle-right tunneling
- $\mu_2,\ \mu$: external potentials.

A. Foerster and E. Ragoucy, Nuclear Phys. **B777** (2007) 373;

3 - ULTRACOLD ATOMIC FERMI GASES

1D 2-component attractive Fermi gas with polarization

- Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{i=1}^{N_\uparrow}\sum_{j=1}^{N_\downarrow}\delta(x_i - x_j) - \frac{H}{2}(N_\uparrow - N_\downarrow)$$

- N spin $1/2$ fermions of mass m
- constrained by PBC to a line of length L
- H: external field
- $g_{1D} = \frac{\hbar^2 c}{m}$: 1D interaction strength:
  attractive for $g_{1D} < 0$ and repulsive for $g_{1D} > 0$
  HERE: ATTRACTIVE REGIME
- $\gamma \equiv \frac{c}{n}\,(n = \frac{N}{L})$: dimensionless interaction;

Bethe ansatz method

C.N. Yang, PRL **19***(1967)1312; M. Gaudin, Phys. Lett.* **24** *(1967) 55*

- Energy:

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2,$$

- BAE:

$$\exp(\mathrm{i}k_j L) = \prod_{\ell=1}^{M} \frac{k_j - \Lambda_\ell + \mathrm{i}\,c/2}{k_j - \Lambda_\ell - \mathrm{i}\,c/2}$$

$$\prod_{\ell=1}^{N} \frac{\Lambda_\alpha - k_\ell + \mathrm{i}\,c/2}{\Lambda_\alpha - k_\ell - \mathrm{i}\,c/2} = -\prod_{\beta=1}^{M} \frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\,c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\,c}$$

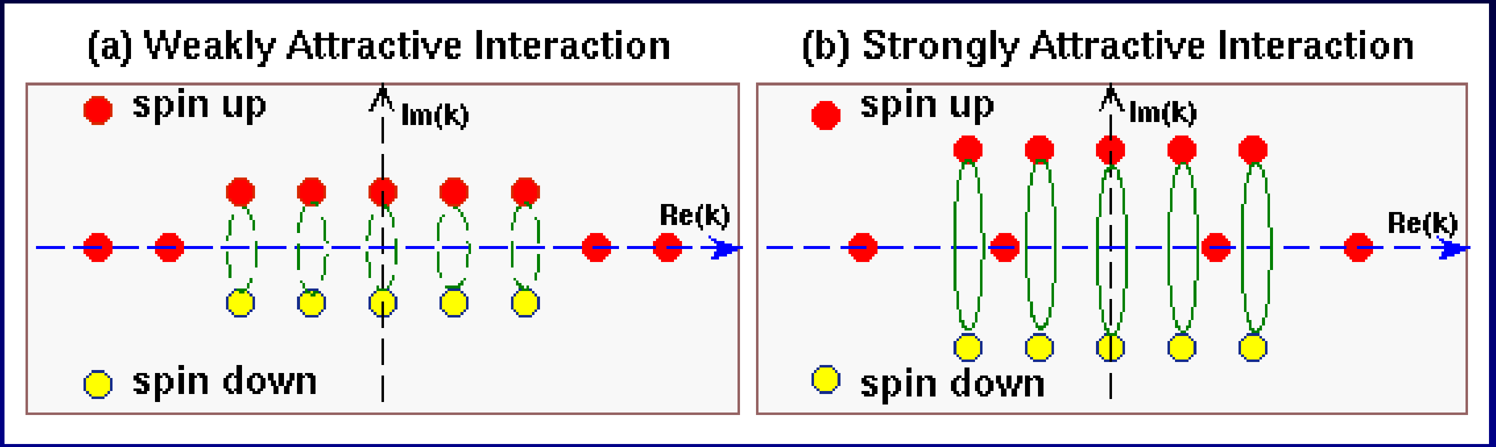
$\{k_j,\, j = 1, \ldots, N\}$ are the quasimomenta for the fermions;

$\{\Lambda_\alpha,\, \alpha = 1, \ldots, M\}$ are the rapidities for the internal spin degrees of freedom

The solutions to the BAE give the GS properties and provide a clear pairing signature

BA-root configuration for the GS

BA configuration of quasimomenta k in the complex plane for the GS: for a given polarization, the system is described by bound states ("Cooper pairs") and unpaired fermions



Guan, X. W.; Batchelor, M. T.; Lee, C.; Bortz, M., Phys. Rev. B 76, 085120 (2007)

Weak Regime

$$\frac{E}{L} \approx \frac{\hbar^2 n^3}{2m}\left(-\frac{|\gamma|}{2}(1-P^2)+\frac{\pi^2}{12}+\frac{\pi^2}{4}P^2\right)$$

Strong Regime

$$\frac{E}{L} \approx \frac{\hbar^2 n^3}{2m}\left\{-\frac{\gamma^2(1-P)}{4}+\frac{P^3\pi^2}{3}\left(1+\frac{4(1-P)}{|\gamma|}\right)+\frac{\pi^2(1-P)^3}{48}\left(1+\frac{1-P}{|\gamma|}+\frac{4P}{|\gamma|}\right)\right\}$$

Thermodynamical Bethe Ansatz - TBA

- elegant method to study thermodynamical properties
- convenient formalism to analyse QPT at $T = 0$
- 1969 C. N. Yang and C. P. Yang, "Yang-Yang approach"
- 1972 M. Takahashi, string hypothesis
- thermodynamic limit: $L \to \infty$, $N \to \infty$ with N/L finite:
  - consider a distribution function for the BA-roots;
  - the equilibrium state is determined by the condition of minimizing the Gibbs free energy: $G = E - HM^z - \mu N - TS$

TBA - equations:

set of coupled nonlinear integral equation:

$$\begin{aligned}
\epsilon^{\mathrm{b}}(k) &= 2(k^2-\mu-\frac{1}{4}c^2)+Ta_2*\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{b}}(k)/T}) \\
&\quad +Ta_1*\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{u}}(k)/T}) \\
\epsilon^{\mathrm{u}}(k) &= k^2-\mu-\frac{1}{2}H+Ta_1*\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{b}}(k)/T}) \\
&\quad -T\sum_{n=1}^{\infty}a_n*\ln(1+\eta_n^{-1}(k)) \\
\ln\eta_n(\lambda) &= \frac{nH}{T}+a_n*\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{u}}(\lambda)/T}) \\
&\quad +\sum_{n=1}^{\infty}T_{nm}*\ln(1+\eta_m^{-1}(\lambda))
\end{aligned}$$

The dressed energies: $\epsilon^{\mathrm{b}}(k) := T\ln(\sigma^h(k)/\sigma(k))$ *and* $\epsilon^{\mathrm{u}}(k) := T\ln(\rho^h(k)/\rho(k))$ *for paired and unpaired fermions; the function* $\eta_n(\lambda) := \xi^h(\lambda)/\xi(\lambda)$ *is the ratio of string densities. The Gibbs free energy per unit length (Takahashi's book):*

$$G=-\frac{T}{\pi}\int_{-\infty}^{\infty}dk\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{b}}(k)/T})-\frac{T}{2\pi}\int_{-\infty}^{\infty}dk\ln(1+\mathrm{e}^{-\epsilon^{\mathrm{u}}(k)/T})$$

Limit $T \to 0$: dressed energy equations

$$\epsilon^{\mathrm{b}}(\Lambda) = 2(\Lambda^2 - \mu - \frac{c^2}{4}) - \int_{-B}^{B} a_2(\Lambda - \Lambda')\epsilon^{\mathrm{b}}(\Lambda')d\Lambda' - \int_{-Q}^{Q} a_1(\Lambda - k)\epsilon^{\mathrm{u}}(k)dk$$

$$\epsilon^{\mathrm{u}}(k) = (k^2 - \mu - \frac{H}{2}) - \int_{-B}^{B} a_1(k - \Lambda)\epsilon^{\mathrm{b}}(\Lambda)d\Lambda$$

$$a_m(x) = \frac{1}{2\pi}\frac{m|c|}{(m\,c/2)^2 + x^2},\ \ \epsilon^{\mathrm{b}}(\pm B) = \epsilon^{\mathrm{u}}(\pm Q) = 0$$

The Gibbs free energy per unit length at zero temperature is given by

$$G(\mu, H) = \frac{1}{\pi}\int_{-B}^{B}\epsilon^{\mathrm{b}}(\Lambda)d\Lambda + \frac{\mathbf{1}}{\mathbf{2}\pi}\int_{-\mathbf{Q}}^{\mathbf{Q}}\epsilon^{\mathrm{u}}(\mathbf{k})\mathbf{dk}$$

From the Gibbs free energy per unit length we have the relations

$$-\partial G(\mu, H)/\partial\mu = n, \quad -\partial G(\mu, H)/\partial H = m_z = nP/2$$

Strong attraction

POLARIZATION:

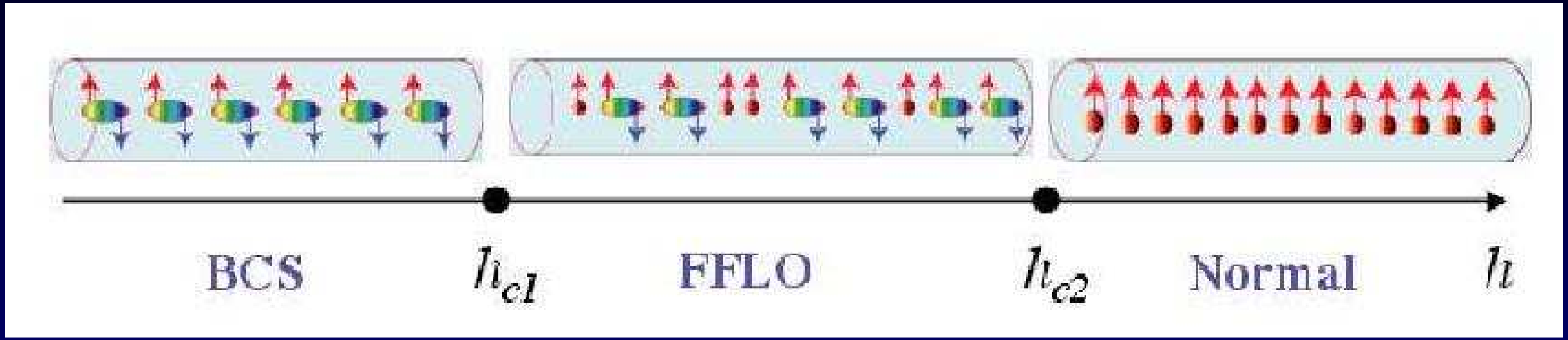
$$\frac{H}{2} \approx \frac{\hbar^2}{2m}\frac{c^2}{4} + \mu^u - \mu^b$$

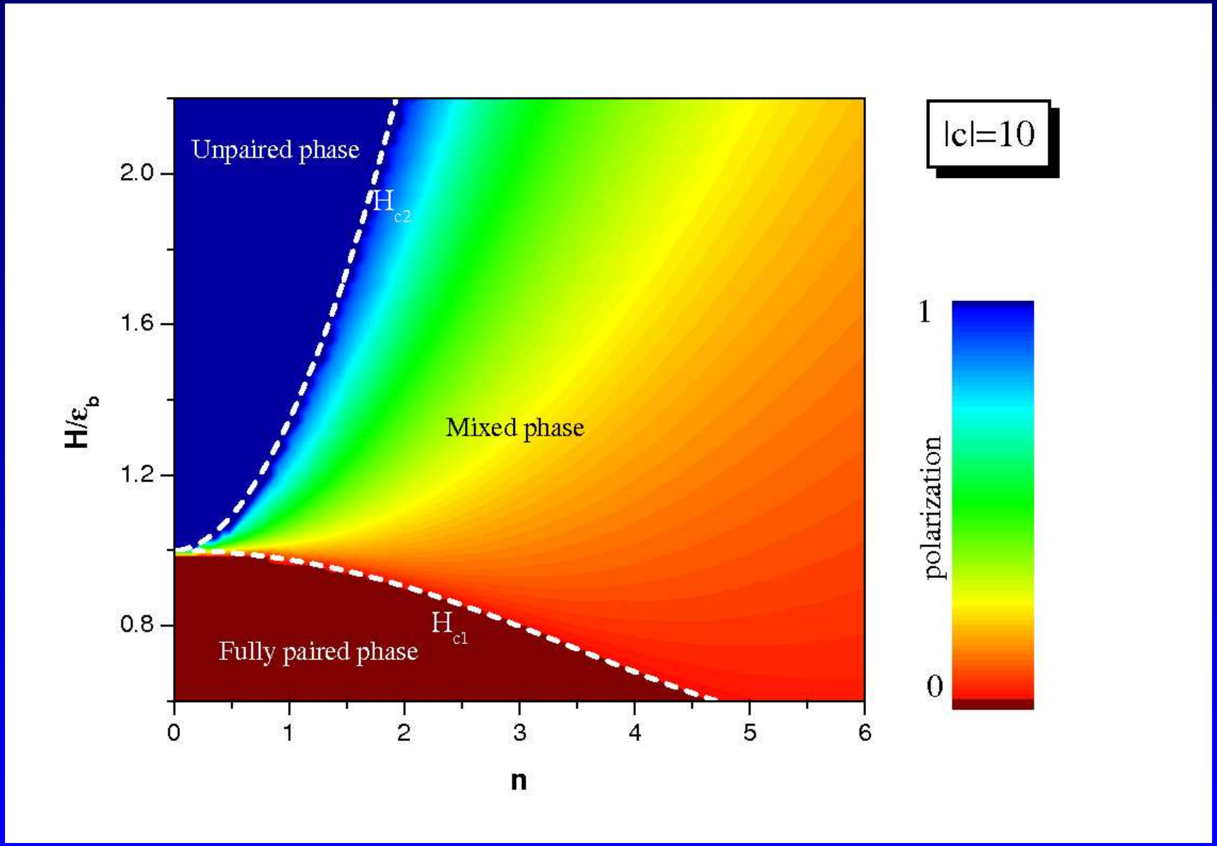
$$\begin{aligned}
\mu^u \quad &\approx \quad \frac{\hbar^2 n^2 \pi^2}{2m}\Big\{P^2 + \frac{(1-P)(49P^2-2P+1)}{12|\gamma|} \\
&+ \frac{(1-P)^2(93P^2+2P+1)}{8\gamma^2} - \frac{(1-P)}{240|\gamma|^3}\big[1441\pi^2P^4 - 7950P^4 \\
&-324\pi^2P^3 + 15720P^3 - 7620P^2 + 166\pi^2P^2 - 120P - 4\pi^2P - 30 + \pi^2\big]\Big\} \\
\mu^b \quad &\approx \quad \frac{\hbar^2 n^2 \pi^2}{2m}\Big\{\frac{(1-P)^2}{16} + \frac{(3P+1)(6P^2-3P+1)}{12|\gamma|} \\
&+ \frac{(1-P)(5+17P-P^2+491P^3)}{64\gamma^2} + \frac{1}{240|\gamma|^3}\big[15(1+2P^2) + 7470P^3 + 10\pi^2P^2 \\
&-180\pi^2P^3 + 335\pi^2P^4 - 420\pi^2P^5 - 15405P^4 - \pi^2 + 75P + 7815P^5\big]\Big\}
\end{aligned}$$

CRITICAL FIELDS:

$$\begin{aligned}
H_{c1} \quad &\approx \quad \frac{\hbar^2 n^2}{2m}\left(\frac{\gamma^2}{2} - \frac{\pi^2}{8}\left(1 - \frac{3}{4\gamma^2} - \frac{1}{|\gamma|^3}\right)\right) \\
H_{c2} \quad &\approx \quad \frac{\hbar^2 n^2}{2m}\left(\frac{\gamma^2}{2} + 2\pi^2\left(1 - \frac{4}{3|\gamma|} + \frac{16\pi^2}{15|\gamma|^3}\right)\right)
\end{aligned}$$

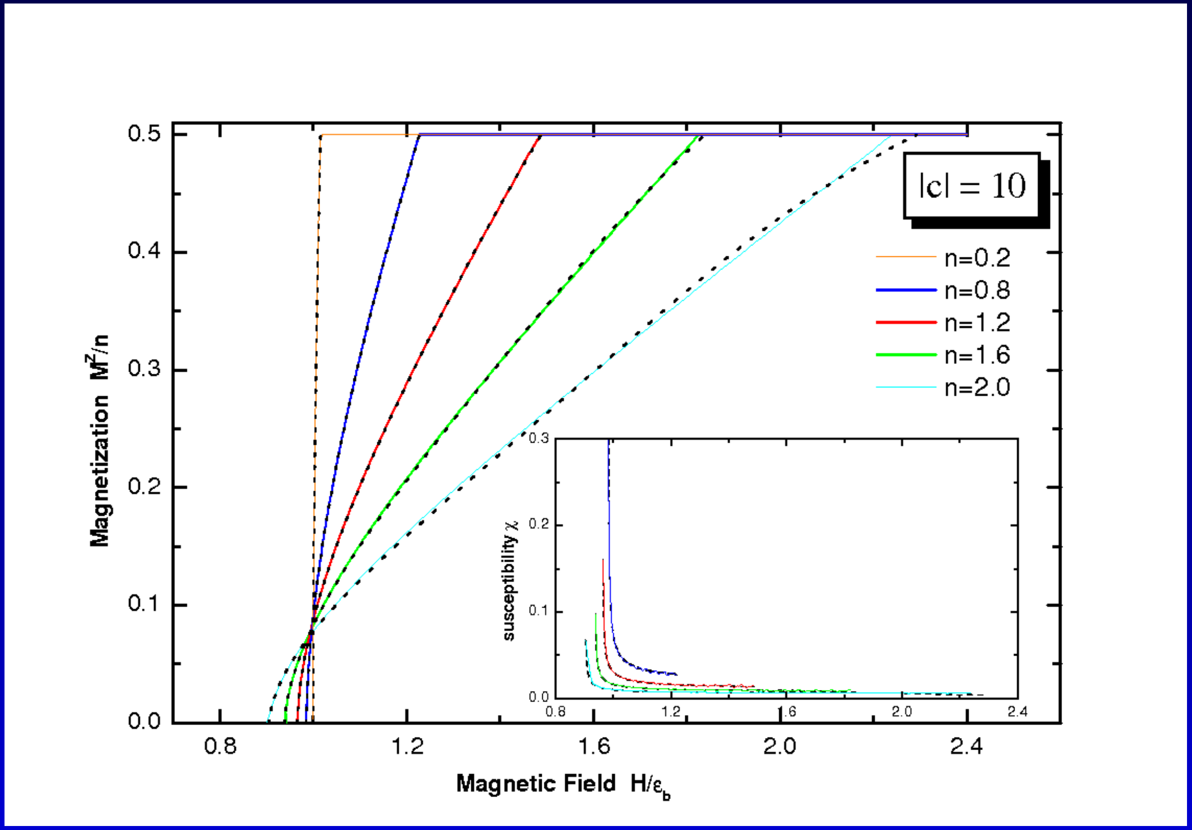
Phase diagram and schematic representation:





Magnetization

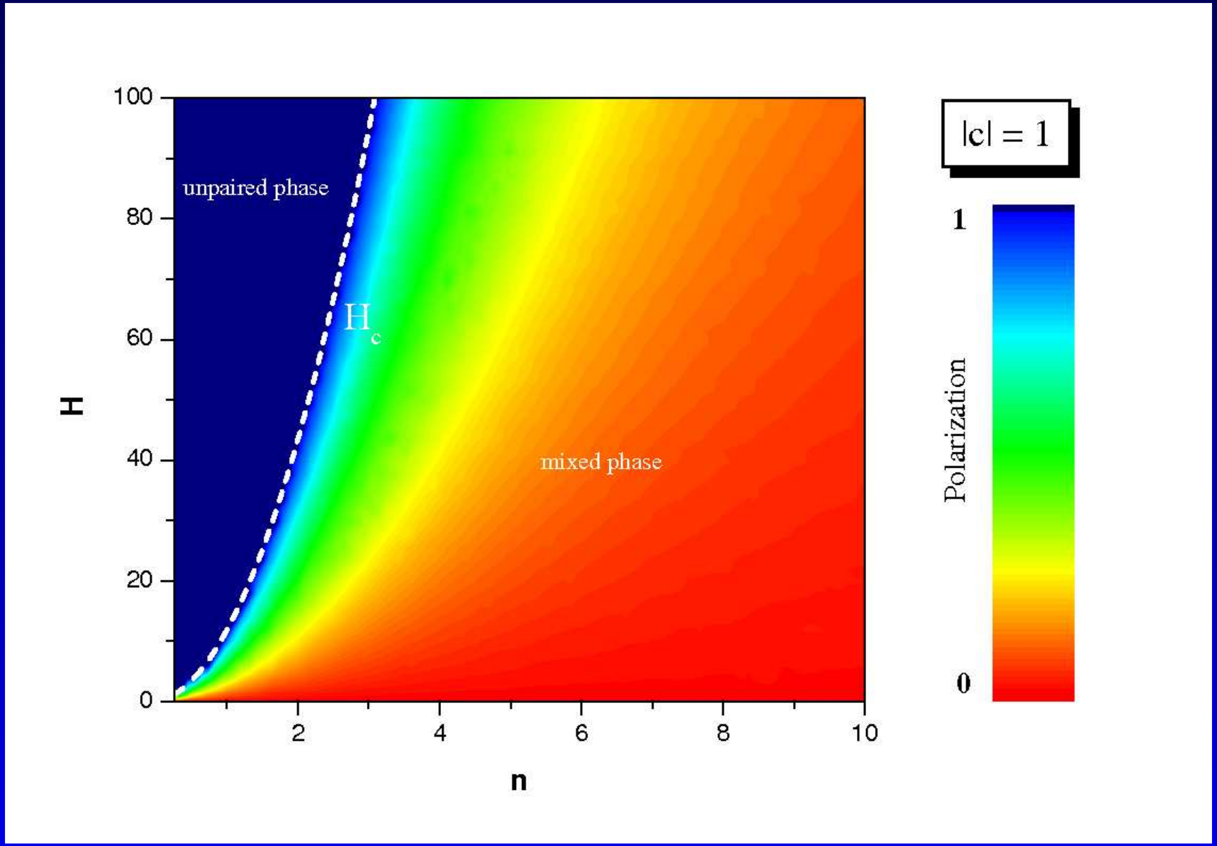
$$M^z \approx \frac{2(H-H_{c1})}{n\pi^2}\left(1+\frac{2}{|\gamma|}+\frac{11}{2\gamma^2}\right) \qquad M^z \approx \frac{n}{2}\left(1-\frac{H_{c2}-H}{4n^2\pi^2}(1+\frac{4}{|\gamma|}+\frac{12}{\gamma^2})\right)$$



The analytical results (dashed lines) coincide well with the numerical solution (solid lines)

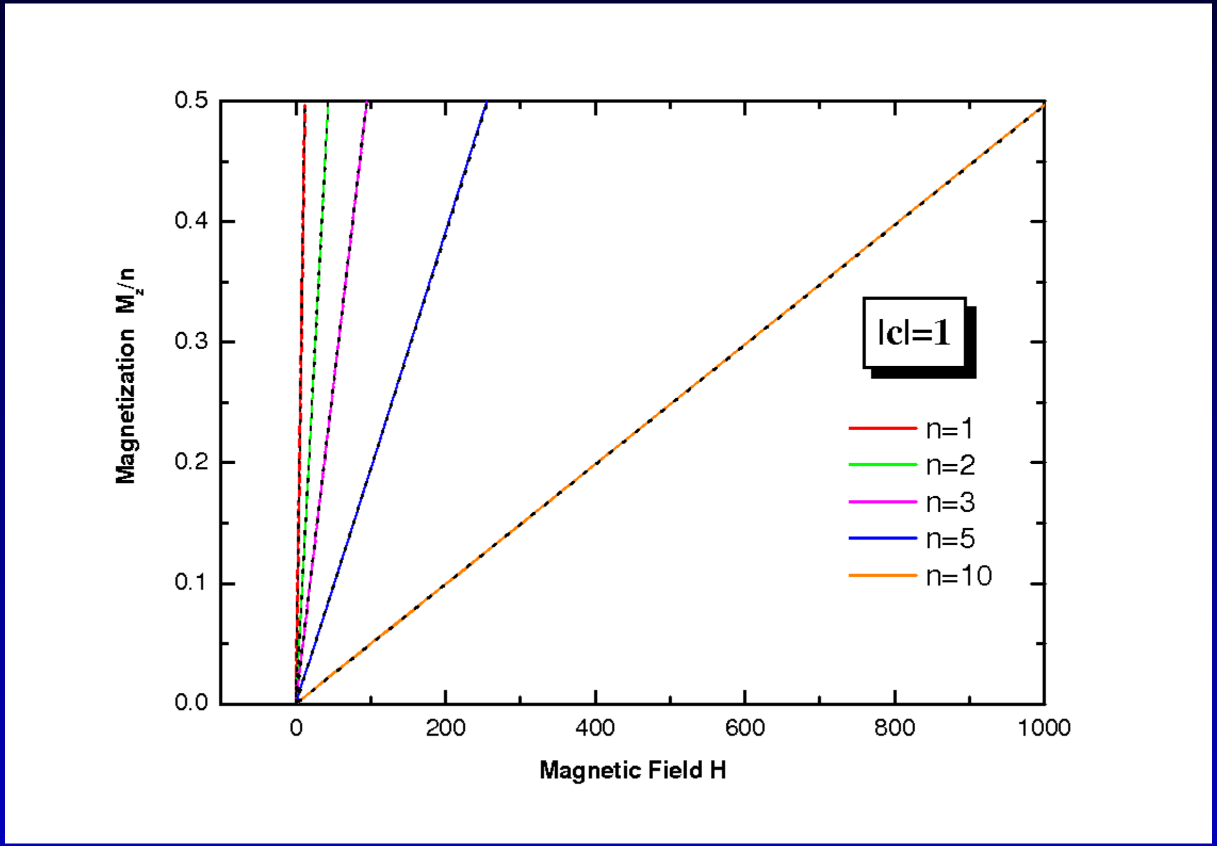
Weak attraction

$$H \approx \frac{\hbar^2 n^2}{2m}\left[2\pi^2 m^z + 4|\gamma| m^z\right], \qquad H_c = n^2[\pi^2 + 2|\gamma|]$$



J. He, A. Foerster, X-W. Guan, M. Batchelor, NJP 11(2009) 073009

Magnetization



The universality class of linear field dependent behaviour of the magnetization holds throughout the whole attractive regime

Three-component attractive Fermi gas

- Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{1\le i<j\le N}\delta(x_i - x_j)$$

- Energy and BAE

$$E = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2, \quad \exp(\mathrm{i}k_j L) = \prod_{\ell=1}^{M_1}\frac{k_j - \Lambda_\ell + \mathrm{i}\,c/2}{k_j - \Lambda_\ell - \mathrm{i}\,c/2},$$

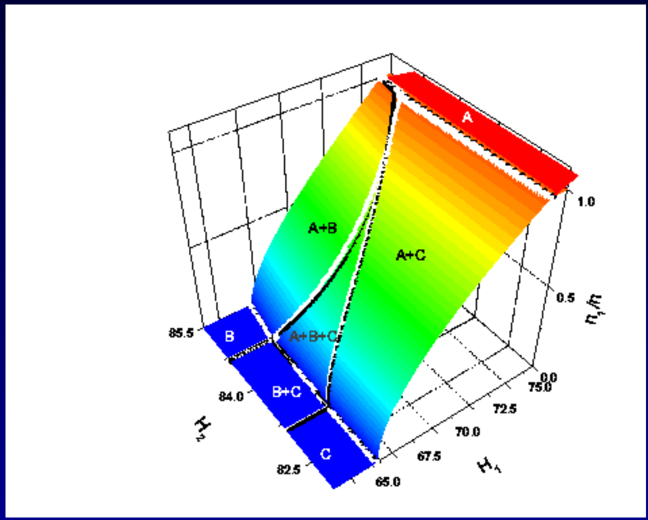
$$\prod_{\ell=1}^{N}\frac{\Lambda_\alpha - k_\ell + \mathrm{i}\,c/2}{\Lambda_\alpha - k_\ell - \mathrm{i}\,c/2} = -\prod_{\beta=1}^{M_1}\frac{\Lambda_\alpha - \Lambda_\beta + \mathrm{i}\,c}{\Lambda_\alpha - \Lambda_\beta - \mathrm{i}\,c}\prod_{\beta=1}^{M_2}\frac{\Lambda_\alpha - \lambda_\beta - \mathrm{i}\,c/2}{\Lambda_\alpha - \lambda_\beta + \mathrm{i}\,c/2},$$

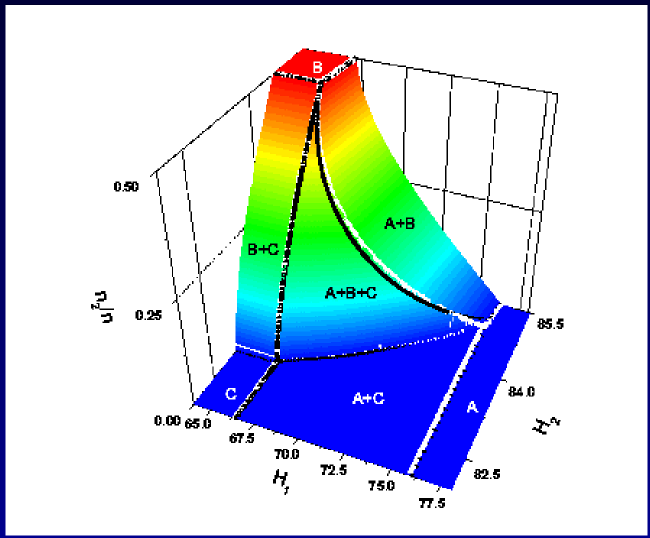
$$\prod_{\beta=1}^{M_1}\frac{\lambda_\mu - \Lambda_\beta + \mathrm{i}\,c/2}{\lambda_\mu - \Lambda_\beta - \mathrm{i}\,c/2} = -\prod_{\beta=1}^{M_2}\frac{\lambda_\mu - \lambda_\beta + \mathrm{i}\,c}{\lambda_\mu - \lambda_\beta - \mathrm{i}\,c}$$

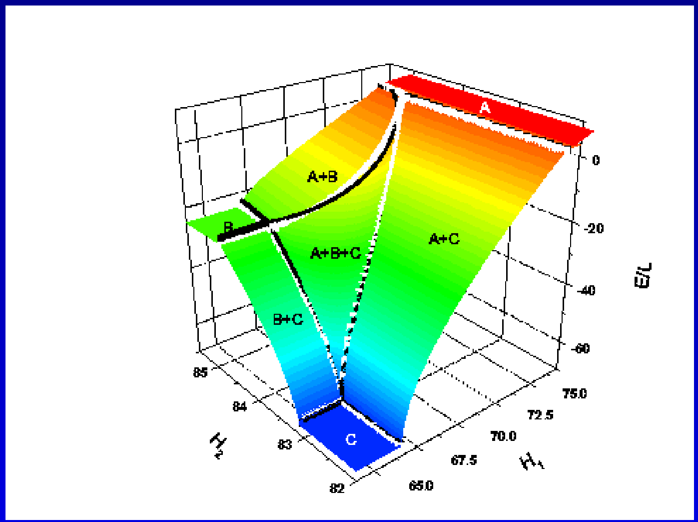
B. Sutherland PRL **20***(68)98; M. Takahashi Prog. Th*

Phase diagram in the strong regime: ($|c| = 10$)

Exhibits a rich scenario with new phases: TRIONS: three-body bound states



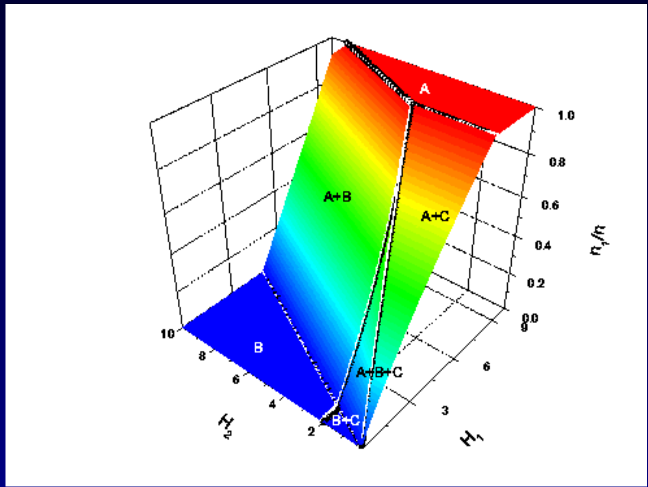


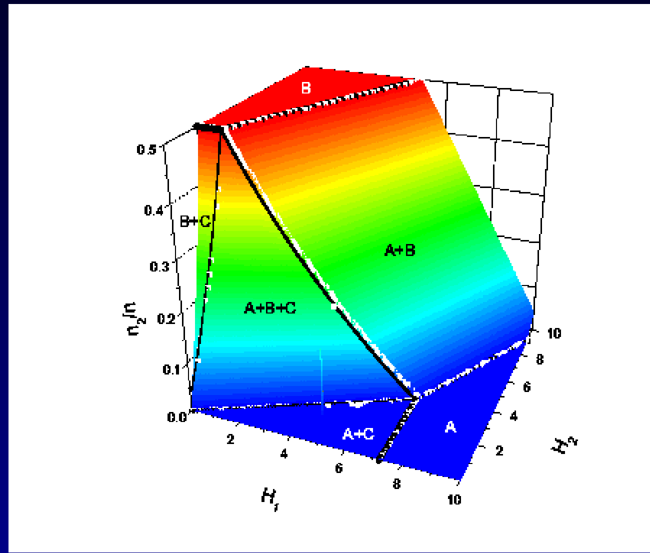


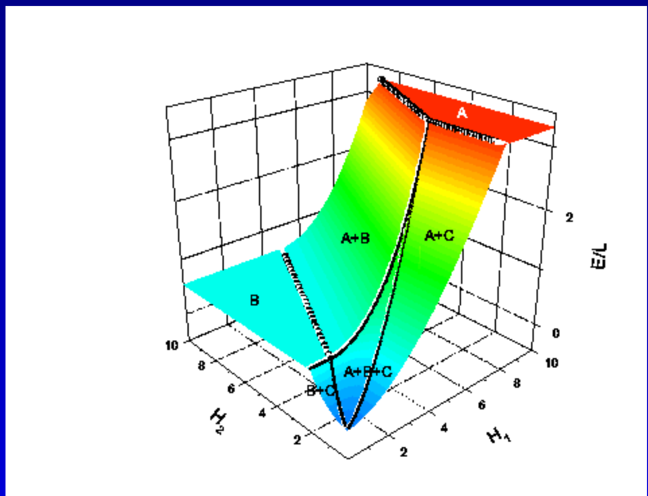
Unpaired phase A, pairing phase B, trion phase C and four different mixtures of these phases;

Above: unpaired fermion density n_1, pair density n_2 and GS energy *vs* the fields H_1,H_2

Phase diagram in the weak regime: ($|c| = 0.5$)







The trionic phase C is supressed. The pure paired phase can be sustained.

The numerical boundaries (white dots) coincide well with the analytical results (black lines)

C. Kuhn, A. Foerster, arXiv:1003.5314v1(2010)

4- Conclusions:

- " **After 75 years the Bethe ansatz is alive and well**. It has been used to solve genuine interacting quantum many-body systems for which perturbative approaches and mean-field theories often fail. **In the case of interacting bosons and fermions, the spatial confinement to one dimension leads to** enhanced dynamics and correlations, and ultimately to **new quantum phases. Because of its underlying mathematical structure and the richness of its results, the Bethe ansatz has had a remarkable impact on several fields, with many surprises along the way. Given the recent advances in the manipulation of atoms in optical lattices,no doubt many more surprises lie ahead.**"
- *M. T. Batchelor, Physics Today 60 (2007) 36*

Collaborators

- Prof. Murray Batchelor, ANU-Australia
- Prof. Xiwen Guan, ANU-Australia
- Prof. Jon Links, UQ-QLD-Australia
- Prof. Eric Ragoucy, LAPTH-France
- Prof. Itzhak Roditi, CBPF-Brazil
- Prof. Arlei Tonel, Unipampa-Brazil
- Dr. Gilberto S. Filho, UFRGS-Brazil
- Carlos Kuhn, UFRGS-Brazil
- Eduardo Mattei, UFRGS-Brazil
- Diefferson Lima, UFRGS-Brazil
- Jardel Cestari, UFRGS-Brazil

THANK YOU!

Appendix: Quantum dynamics:

- Temporal operator U:
  determines the time evolution of any physical quantity

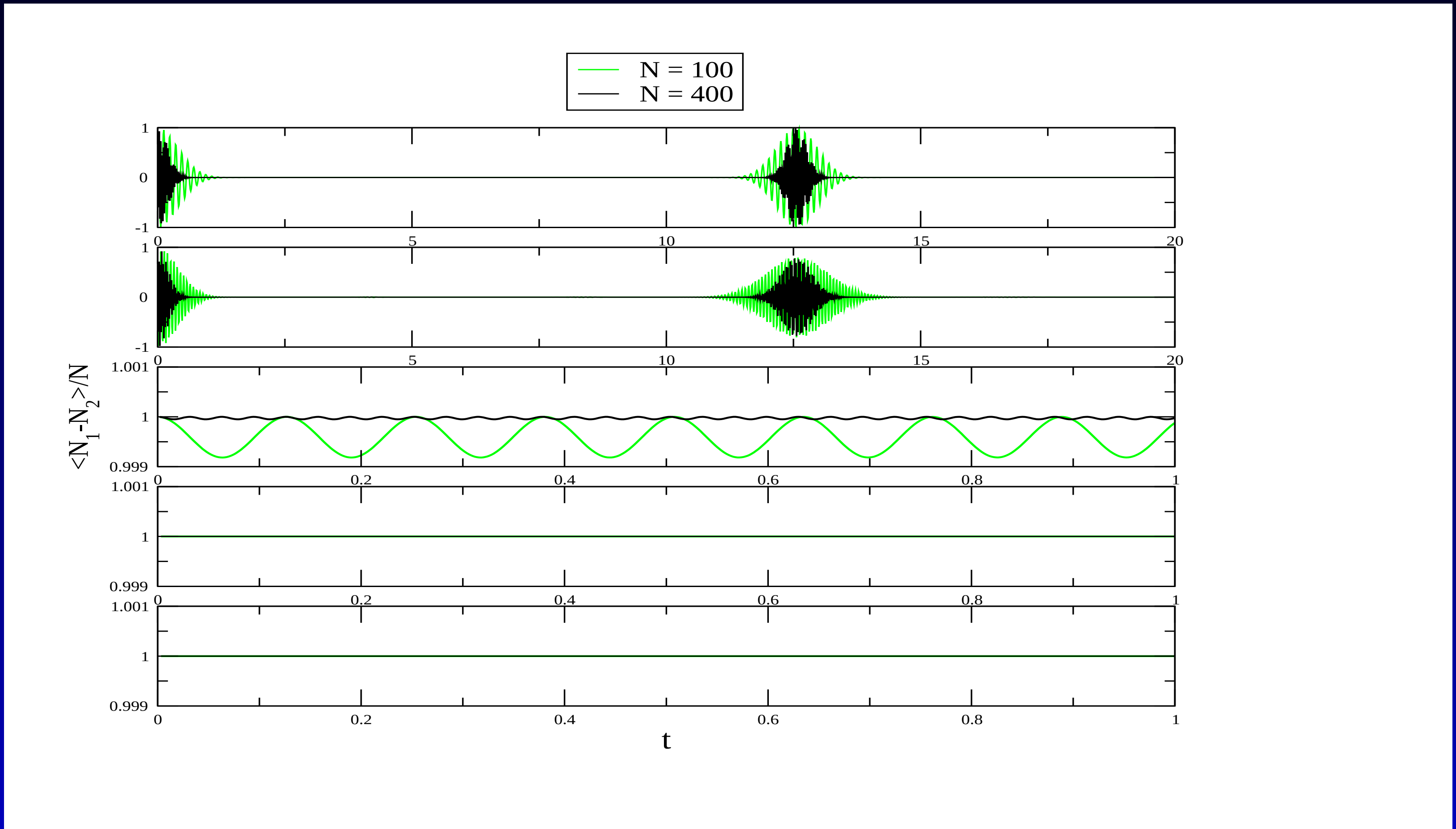
$$U = \sum_{n=0}^{N} e^{-i\lambda_n t} |\psi_n\rangle\langle\psi_n|$$

  $\{\lambda_n\}$; $\{|\psi_n\rangle\}$: eigenvalues and eigenvectors of H

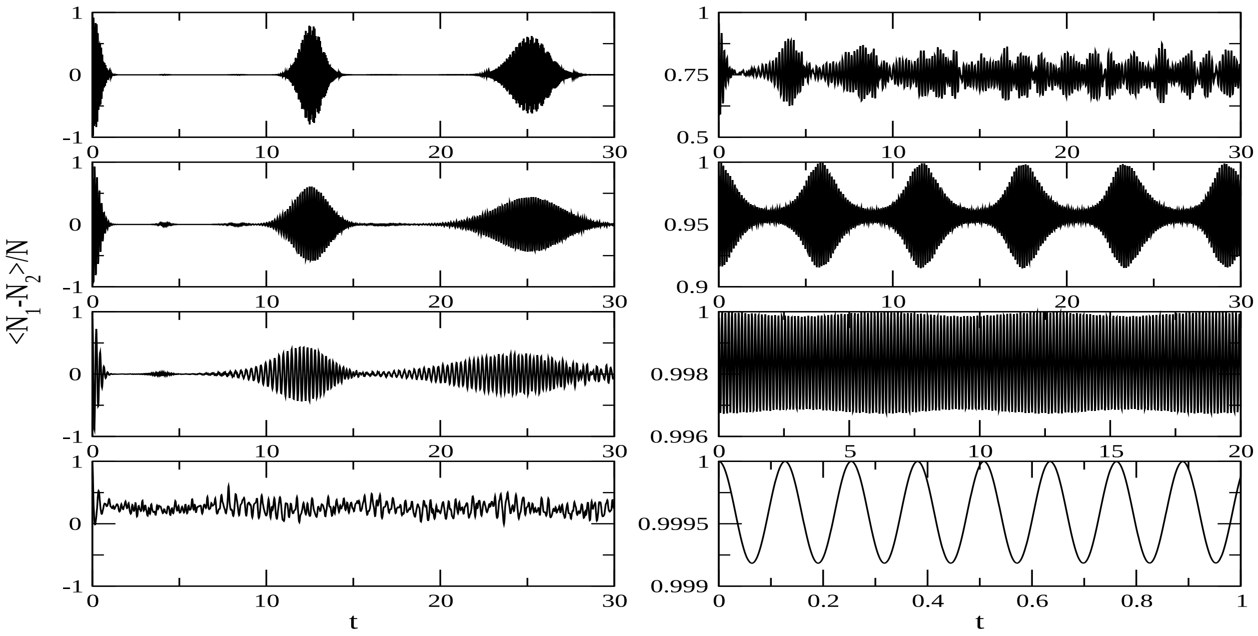
- Temporal evolution of any state: $|\psi(t)\rangle = U|\phi\rangle = \sum_{n=0}^{N} a_n e^{-i\lambda_n t}|\psi_n\rangle$,
  $a_n = \langle\psi_n|\phi\rangle$ and $|\phi\rangle$: initial state
- Expectation value of any operator A
  $\langle A\rangle = \langle\psi(t)|A|\psi(t)\rangle$
- Imbalance population
  $A = (N_1 - N_2)/N$

Plot the time evolution of the expectation value of the imbalance population for different ratios of the coupling $K/\mathcal{E}_J$

Appendix: Dynamical regimes:



$$\frac{K}{\mathcal{E}_J} = \frac{\mathbf{1}}{\mathbf{N^2}}, \quad \frac{\mathbf{1}}{\mathbf{N}}, \quad \mathbf{1}, \quad \mathbf{N}, \quad \mathbf{N^2}$$



$$\frac{K}{\mathcal{E}_J} = \frac{1}{N},\ \frac{2}{N},\ \frac{3}{N},\ \frac{\mathbf{4}}{\mathbf{N}},\ \frac{5}{N},\ \frac{10}{N},\ \frac{50}{N},\ \mathbf{1}$$

$$\lambda_t = 2 \Rightarrow \frac{K}{\mathcal{E}_J} = \frac{4}{N}, ;\Delta\mu = 0$$

Tunneling X Self-trapping

Appendix: 3 wells-algebraic construction

- R matrix:
  $R_{12}(x) = I \otimes I - \frac{1}{x} P_{12};\ P_{12} = \sum_{i,j=1}^{3} E_{ij} \otimes E_{ji}$
- Monodromy matrix:
  $\pi(T(u)) = \Lambda^{[1]}(u + w_1)\, \Lambda^{[2]}(u + w_2)$

$$\Lambda^{[1]}(u) = \begin{pmatrix} u + N_1 & a_1^\dagger\, a_2 & \beta_3\, a_1^\dagger \\ a_2^\dagger\, a_1 & u + N_2 & \beta_3\, a_2^\dagger \\ \beta_3\, a_1 & \beta_3\, a_2 & \beta_3^2 \end{pmatrix}$$

$$\Lambda^{[2]}(u) = \begin{pmatrix} -\beta_1^2 & \beta_1\, \beta_2 & \beta_1\, a_3^\dagger \\ \beta_1\, \beta_2 & -\beta_2^2 & \beta_2\, a_3^\dagger \\ \beta_1\, a_3 & \beta_2\, a_3 & u - N_3 \end{pmatrix}$$

Appendix: Bethe ansatz

A solvable or integrable quantum many-body system is one in which N-particle wave function may be explicitly constructed. In general, $N!$ plane waves are N-fold products of individual exponential phase factors $e^{\mathrm{i}k_i x_j}$, where the N distinct wave numbers, k_i, are permuted among the N distinct coordinates, x_j. Each of the $N!$ plane waves have an amplitude coefficient in each of regions. For example, in the domain $0 < x_{Q1} < x_{Q2} < \ldots < x_{QN} < L$, the wave function is written as

$$\psi = \sum_P A_{\sigma_1 \ldots \sigma_N}(P_1, \ldots, P_N | Q_1, \ldots, Q_N) \exp \mathrm{i}(k_{P1} x_{Q1} + \ldots + k_{PN} x_{QN})$$

- Continuity: $\psi_{x_{Qi} = x^-_{Qj}} = \psi_{x_{Qi} = x^+_{Qj}}$
- Schrödinger equation: $\mathcal{H}\psi = E\psi$
- two-body scattering relation:$A_{\sigma_1 \ldots \sigma_N}(P_i P_j | Q_i Q_j) = [Y_{ij}]^{\sigma'_1 \ldots \sigma'_N}_{\sigma_1 \ldots \sigma_N} A_{\sigma'_1 \ldots \sigma'_N}(P_j P_i | Q_i Q_j)$
- boundary conditions: $\psi(x_1, \ldots, x_i, \ldots, x_N) = \psi(x_1, \ldots, x_i + L, \ldots, x_N)$