CITY UNIVERSITY LONDON

Antilinear deformations of integrable systems

Andreas Fring

Recent Advances in Quantum Integrable Systems
International workshop, Annecy-le-Vieux, 15-18 June 2010

based on collaborations with Carla Figueira de Morisson Faria (UCL), Miloslav Znojil (Prague), Bijan Bagchi (Kolkata), Paulo Assis (City $\rightarrow$ Kent), Olalla Castro-Alvardo (City), Monique Smith (City), Frederik Scholtz (NITheP), Laure Gouba (NITheP)

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Why is Hermiticity a good property to have?

- Hermiticity ensures real energies
  Schrödinger equation $H\psi = E\psi$

$$\left.\begin{array}{r} \langle\psi| H |\psi\rangle = E \langle\psi|\psi\rangle \\ \langle\psi| H^\dagger |\psi\rangle = E^* \langle\psi|\psi\rangle \end{array}\right\} \Rightarrow 0 = (E - E^*) \langle\psi|\psi\rangle$$

- Hermiticity ensures conservation of probability densities

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$

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- Thus when $H \neq H^\dagger$ one usually thinks of dissipation.
- However, these systems are open and do not possess a self-consistent description.

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Hermiticity is not essential

- Operators $\mathcal{O}$ which are left invariant under an antilinear involution $\mathcal{I}$ and whose eigenfunctions Φ also respect this symmetry,
$$[\mathcal{O},\mathcal{I}]=0 \quad \wedge \quad \mathcal{I}\Phi=\Phi$$
have a real eigenvalue spectrum.
[E. Wigner, *J. Math. Phys.* 1 (1960) 409]
- By defining a new metric also a consistent quantum mechanical framework has been developed for theories involving such operators.
[F. Scholtz, H. Geyer, F. Hahne, *Ann. Phys.* 213 (1992) 74, C. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243, A. Mostafazadeh, *J. Math. Phys.* 43 (2002) 2814]

In particular this also holds for $\mathcal{O}$ being non-Hermitian.

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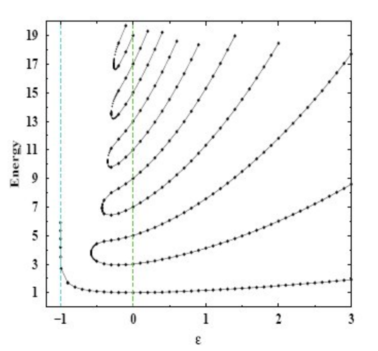
There are plenty of well studied examples of non-Hermitian systems in the literature

"Recent" classical example

$$\mathcal{H} = \frac{1}{2}p^2 + x^2(ix)^\varepsilon \qquad \text{for } \varepsilon \geq 0$$



[C.M. Bender, S. Boettcher, *Phys. Rev. Lett.* 80 (1998) 5243]

A more classical example

- Lattice Reggeon field theory:

$$\mathcal{H} = \sum_{\vec{\imath}} \left[\Delta a^{\dagger}_{\vec{\imath}} a_{\vec{\imath}} + i g a^{\dagger}_{\vec{\imath}} (a_{\vec{\imath}} + a^{\dagger}_{\vec{\imath}}) a_{\vec{\imath}} + \tilde{g} \sum_{\vec{\jmath}} (a^{\dagger}_{\vec{\imath}+\vec{\jmath}} - a^{\dagger}_{\vec{\imath}})(a_{\vec{\imath}+\vec{\jmath}} - a_{\vec{\imath}}) \right]$$

- $a^{\dagger}_{\vec{\imath}}, a_{\vec{\imath}}$ are creation and annihilation operators, $\Delta, g, \tilde{g} \in \mathbb{R}$

[J.L. Cardy, R. Sugar, *Phys. Rev.* D12 (1975) 2514]

- for one site this is almost ix^3

$$\begin{aligned} \mathcal{H} &= \Delta a^{\dagger} a + i g a^{\dagger} \left(a + a^{\dagger} \right) a \\ &= \frac{1}{2} \left(\hat{p}^2 + \hat{x}^2 - 1 \right) + i \frac{g}{\sqrt{2}} (\hat{x}^3 + \hat{p}^2 \hat{x} - 2\hat{x} + i\hat{p}) \end{aligned}$$

with $a = (\omega \hat{x} + i\hat{p})/\sqrt{2\omega}$, $a^{\dagger} = (\omega \hat{x} - i\hat{p})/\sqrt{2\omega}$

[P. Assis and A.F., J. Phys. A41 (2008) 244001]

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- quantum spin chains: (c=-22/5 CFT)

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^x + \lambda\sigma_i^z\sigma_{i+1}^z + ih\sigma_i^z \quad \lambda, h \in \mathbb{R}$$

[G. von Gehlen, J. Phys. A24 (1991) 5371]

- strings on $AdS_5 \times S^5$-background
  [A. Das, A. Melikyan, V. Rivelles, *JHEP* 09 (2007) 104]
- affineToda field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{\beta^2}\sum_{k=a}^{\ell} n_k \exp(\beta\alpha_k \cdot \phi)$$

$a = 1 \equiv$ conformal field theory (Lie algebras)
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- deformed space-time structure
  - deformed Heisenberg canonical commutation relations

$$aa^\dagger - q^2 a^\dagger a = q^{g(N)}, \qquad \text{with } N = a^\dagger a$$

$$X = \alpha a^\dagger + \beta a, \quad P = i\gamma a^\dagger - i\delta a, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

$$[X, P] = i\hbar q^{g(N)}(\alpha\delta + \beta\gamma) + \frac{i\hbar(q^2 - 1)}{\alpha\delta + \beta\gamma}\left(\delta\gamma X^2 + \alpha\beta\, P^2 + i\alpha\delta XP - i\beta\gamma PX\right)$$

- limit: $\beta \to \alpha$, $\delta \to \gamma$, $g(N) \to 0$, $q \to e^{2\tau\gamma^2}$, $\gamma \to 0$

$$[X, P] = i\hbar\left(1 + \tau P^2\right)$$

- representation: $X = (1 + \tau p_0^2)x_0$, $P = p_0$, $[x_0, p_0] = i\hbar$

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- with the standard inner product X is not Hermitian

$$X^\dagger = X + 2\tau i\hbar P \quad \text{and} \quad P^\dagger = P$$

- $\Rightarrow H(X,P)$ is in general not Hermitian
- example harmonic oscillator:

$$\begin{aligned} H_{ho} &= \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}(1+\tau p_0^2)x_0(1+\tau p_0^2)x_0, \\ &= \frac{p_0^2}{2m} + \frac{m\omega^2}{2}\left[(1+\tau p_0^2)^2 x_0^2 + 2i\hbar\tau p_0(1+\tau p_0^2)x_0\right]. \end{aligned}$$

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- dynamical noncommutative space-time
  Replace

$$[x_0, y_0] = i\theta, \qquad [x_0, p_{x_0}] = i\hbar, \qquad [y_0, p_{y_0}] = i\hbar,$$
$$[p_{x_0}, p_{y_0}] = 0, \qquad [x_0, p_{y_0}] = 0, \qquad [y_0, p_{x_0}] = 0,$$

with $\theta \in \mathbb{R}$, by

$$[X, Y] = i\theta(1 + \tau Y^2) \qquad [X, P_x] = i\hbar(1 + \tau Y^2)$$
$$[Y, P_y] = i\hbar(1 + \tau Y^2) \qquad [X, P_y] = 2i\tau Y(\theta P_y + \hbar X)$$
$$[P_x, P_y] = 0 \qquad [Y, P_x] = 0$$

$\Rightarrow$ Non-Hermitian representation

$$X = (1 + \tau y_0^2)x_0 \quad Y = y_0 \quad P_x = p_{x_0} \quad P_y = (1 + \tau y_0^2)p_{y_0}$$

$$X^\dagger = X + 2i\tau\theta Y \quad Y^\dagger = Y \quad P_y^\dagger = P_y - 2i\tau\hbar Y \quad P_x^\dagger = P_x$$

[A.F., L. Gouba and F. Scholtz, arXiv:1003.3025]
[A.F., L. Gouba and B. Bagchi, arXiv:1006.2065]

Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues (QM)

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:
  $$\mathcal{PT}(\lambda\Phi + \mu\Psi) = \lambda^*\mathcal{PT}\Phi + \mu^*\mathcal{PT}\Psi \qquad \lambda, \mu \in \mathbb{C}$$
- Real eigenvalues from unbroken $\mathcal{PT}$-symmetry:
  $$[\mathcal{H}, \mathcal{PT}] = 0 \quad \wedge \quad \mathcal{PT}\Phi = \Phi \qquad \Rightarrow \varepsilon = \varepsilon^* \text{ for } \mathcal{H}\Phi = \varepsilon\Phi$$
- *Proof*:
  $\varepsilon\Phi = \mathcal{H}\Phi = \mathcal{HPT}\Phi = \mathcal{PTH}\Phi = \mathcal{PT}\varepsilon\Phi = \varepsilon^*\mathcal{PT}\Phi = \varepsilon^*\Phi$

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PT-symmetry is only an example of an antilinear involution

Unbroken $\mathcal{PT}$-symmetry guarantees real eigenvalues (QM)

- $\mathcal{PT}$-symmetry: $\mathcal{PT}: \quad x \to -x \quad p \to p \quad i \to -i$
  $(\mathcal{P}: x \to -x, p \to -p; \ \mathcal{T}: x \to x, p \to -p, i \to -i)$
- $\mathcal{PT}$ is an anti-linear operator:

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Pseudo/Quasi-Hermiticity

$$h = \eta H \eta^{-1} = h^\dagger = (\eta^{-1})^\dagger H^\dagger \eta^\dagger \quad \Leftrightarrow \quad H^\dagger \rho = \rho H \qquad \rho = \eta^\dagger \eta \quad (*)$$

	$H^\dagger = \rho H \rho^{-1}$	$H^\dagger \rho = \rho H$	$H^\dagger = \rho H \rho^{-1}$
positivity of ρ	$\checkmark$	$\checkmark$	$\times$
ρ Hermitian	$\checkmark$	$\checkmark$	$\checkmark$
ρ invertible	$\checkmark$	$\times$	$\checkmark$
terminology	(*)	quasi-Herm.	pseudo-Herm.
spectrum of H	real	could be real	real
definite metric	guaranteed	guaranteed	not conclusive

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$\mathcal{CPT}$-metric

[Bender, Brody, Jones, Phys. Rev. Lett. 89 (2002) 270401]

$$\langle\Psi|\Phi\rangle_{\mathcal{CPT}} := (\mathcal{CPT}\,|\Psi\rangle)^{T} \cdot |\Phi\rangle$$

- In position space: $\mathcal{C}(x,y) = \sum_n \Phi_n(x)\Phi_n(y)$
  Very formal as normally one does not know $\Phi_n(x)\ \forall n$
- Algebraic approach: Solve
  $\mathcal{C}^2 = \mathbb{I} \qquad [\mathcal{H},\mathcal{C}] = 0 \qquad [\mathcal{C},\mathcal{PT}] = 0 \qquad [\mathcal{H},\mathcal{PT}] = 0$
- Relation $\mathcal{C}$ and metric (same as pseudo-Hermiticity)

$$\mathcal{C} = \rho^{-1}\mathcal{P}$$

Proof: $\quad \mathcal{P}^2 = \mathbb{I} \qquad \mathcal{H}^\dagger = \mathcal{THT}$

$$\rho\mathcal{H} = \mathcal{PCH} = \mathcal{PHC} = \mathcal{PPTHPTC} = \mathcal{THTPC} = \mathcal{H}^\dagger\mathcal{PC} = \mathcal{H}^\dagger\rho$$

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$\mathcal{CPT}$-metric

[Bender, Brody, Jones, Phys. Rev. Lett. 89 (2002) 270401]

$$\langle \Psi | \Phi \rangle_{\mathcal{CPT}} := (\mathcal{CPT}\, |\Psi\rangle)^T \cdot |\Phi\rangle$$

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Observables and ambiguities

- Observables are Hermitian with respect to the new metric

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  - $\mathcal{O}$ is an observable in the non-Hermitian system

- Ambiguities:
  Given H the metric is not uniquely defined for unknown h.
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  - Fixing one more observable achieves uniqueness.
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General technique:

- Given $H \begin{cases} \text{either} & \text{solve } \eta H \eta^{-1} = h \quad \text{for} \quad \eta \Rightarrow \rho = \eta^\dagger \eta \\ \text{or} & \text{solve } H^\dagger = \rho H \rho^{-1} \quad \text{for} \quad \rho \Rightarrow \eta = \sqrt{\rho} \end{cases}$
- involves complicated commutation relations
- often this can only be solved perturbatively

Note:

- Thus, this is not re-inventing or disputing the validity of quantum mechanics.
- We only give up the restrictive requirement that Hamiltonians have to be Hermitian.

[C. Bender, *Rep. Prog. Phys.* 70 (2007) 947]
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Calogero-Moser-Sutherland models (extended)

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2} \sum\nolimits_i q_i^2 + \frac{g^2}{2} \sum\nolimits_{i \neq k} \frac{1}{(q_i - q_k)^2} + i\tilde{g} \sum\nolimits_{i \neq k} \frac{1}{(q_i - q_k)} p_i$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$
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- From real fields to complex particle systems

  i) No restrictions

  e.g. Benjamin-Ono equation

$$u_t + uu_x + \lambda H u_{xx} = 0 \qquad (*)$$

$H \equiv$ Hilbert transform, i.e. $Hu(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x)}{z-x} dz$

Then

$$u(x,t) = \frac{\lambda}{2} \sum_{k=1}^{\ell} \left(\frac{i}{x - z_k} - \frac{i}{x - z_k^*} \right) \in \mathbb{R}$$

satisfies (*) iff z_k obeys the A_n-Calogero equ. of motion

$$\ddot{z}_k = \frac{\lambda^2}{2} \sum_{k \neq j} (z_j - z_k)^{-3}$$

[H. Chen, N. Pereira, Phys. Fluids 22 (1979) 187]
[talk by J. Feinberg, PHHQP workshop VI, 2007, London]

ii) restrict to submanifold

Theorem: [Airault, McKean, Moser, CPAM, (1977) 95]

Given a Hamiltonian $H(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)$ with flow

$$x_i = \partial H / \partial \dot{x}_i \qquad \text{and} \qquad \ddot{x}_i = -\partial H / \partial x_i \qquad i = 1, \dots, n$$

and conserved charges I_j in involution with H,i.e. $\{I_j, H\} = 0$. Then the locus of $\operatorname{grad} I = 0$ is invariant.

Example: Boussinesq equation

$$v_{tt} = a(v^2)_{xx} + b v_{xxxx} + v_{xx} \qquad (**)$$

Then

$$v(x,t) = c \sum\nolimits_{k=1}^{\ell} (x - z_k)^{-2}$$

satisfies (**) iff b=1/12, c=-a/2 and z_k obeys

$$\begin{aligned} \ddot{z}_k &= 2 \sum\nolimits_{j \neq k} (z_j - z_k)^{-3} \qquad \Leftrightarrow \qquad \ddot{z}_k = -\frac{\partial H}{\partial z_i} \\ \dot{z}_k &= 1 - \sum\nolimits_{j \neq k} (z_j - z_k)^{-2} \qquad \Leftrightarrow \qquad \operatorname{grad}(I_3 - I_1) = 0 \end{aligned}$$

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and conserved charges I_j in involution with H,i.e. $\{I_j, H\} = 0$. Then the locus of $\operatorname{grad} I = 0$ is invariant.

Example: Boussinesq equation

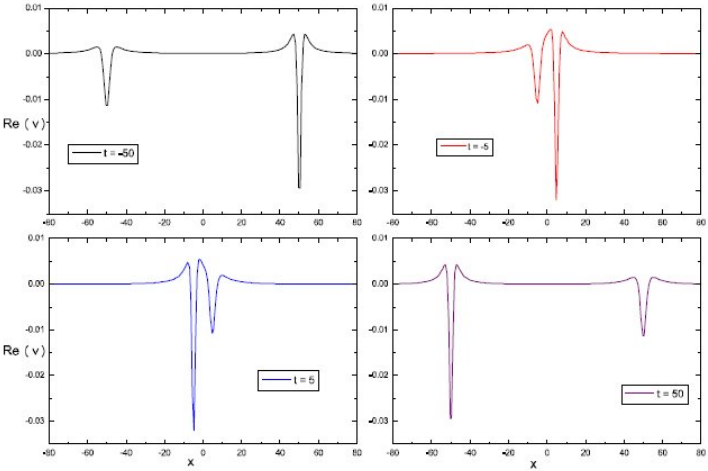
$$v_{tt} = a(v^2)_{xx} + b v_{xxxx} + v_{xx} \qquad (**)$$

Then

$$v(x,t) = c \sum\nolimits_{k=1}^{\ell} (x - z_k)^{-2}$$

satisfies (**) iff b=1/12, c=-a/2 and z_k obeys

$$\begin{aligned} \ddot{z}_k &= 2 \sum\nolimits_{j \neq k} (z_j - z_k)^{-3} && \Leftrightarrow \quad \ddot{z}_k = -\frac{\partial H}{\partial z_i} \\ \dot{z}_k &= 1 - \sum\nolimits_{j \neq k} (z_j - z_k)^{-2} && \Leftrightarrow \quad \operatorname{grad}(I_3 - I_1) = 0 \end{aligned}$$



[P. Assis and A.F., J. Phys. A42 (2009) 425206]

Calogero-Moser-Sutherland models (deformed)

$$\mathcal{H}_{\text{CMS}} = \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot q) \qquad m, g_\alpha \in \mathbb{R}$$

- invariance with respect to Coxter group $\mathcal{W}$

$$\begin{aligned} \mathcal{H}_{\text{CMS}} &= \frac{\sigma_i p \cdot \sigma_i p}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\alpha \cdot \sigma_i q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\alpha \cdot \sigma_i q) \\ &= \frac{p^2}{2} + \frac{m^2}{16} \sum_{\alpha \in \Delta_s} (\sigma_i^{-1} \alpha \cdot q)^2 + \frac{1}{2} \sum_{\alpha \in \Delta} g_\alpha V(\sigma_i^{-1} \alpha \cdot q) \end{aligned}$$

- aim: construct new models which are invariant under $\mathcal{W}^{\mathcal{PT}}$

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A_2, G_2: [A. F., M. Znojil, J. Phys. A41 (2008) 194010]
all Coxeter groups: [A. F., Monique Smith, arXiv:1004.0916]

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Construction of antilinear deformations

- Involution $\in \mathcal{W} \equiv$ Coxeter group $\Rightarrow$ deform in antilinear way
- Find a linear deformation map:

$$\delta: \ \Delta \rightarrow \tilde{\Delta}(\varepsilon) \qquad \alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha$$

$$\alpha_i \in \Delta \subset \mathbb{R}^n, \quad \tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n, \quad \varepsilon \in \mathbb{R}$$

- $\tilde{\Delta}(\varepsilon)$ remains invariant under an antilinear transformation ω

(i) $\omega : \tilde{\alpha} = \mu_1\alpha_1 + \mu_2\alpha_2 \mapsto \mu_1^*\omega\alpha_1 + \mu_2^*\omega\alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$
(ii) $\omega^2 = \mathbb{I}$
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$$\sigma_i(x) := x - 2\frac{x \cdot \alpha_i}{\alpha_i^2}\alpha_i, \quad \text{with} \quad 1 \leq i \leq \ell \equiv \mathrm{rank}\,\mathcal{W},$$

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$$\sigma = \prod\nolimits_{i=1}^{\ell} \sigma_i,$$

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$$\sigma := \sigma_-\sigma_+, \quad \sigma_\pm := \prod_{i \in V_\pm} \sigma_i, \quad [\sigma_i, \sigma_j] = 0 \ \text{ for } \ i,j \in V_\pm$$

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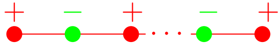

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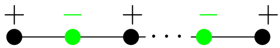

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Summary: properties of θ_ε

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$\because [\sigma, \theta_\varepsilon] = 0$, make Ansatz:

$$\theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon)\sigma^k, \qquad \lim_{\varepsilon\to 0} c_k(\varepsilon) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}, \quad c_k(\varepsilon) \in \mathbb{C}$$

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Now case-by-case

$\tilde{\Delta}(\varepsilon)$ for A_3

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^3\right)$$

with explicit representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix},$$

$\sigma_- = \sigma_1\sigma_3$, $\sigma_+ = \sigma_2$, $\sigma = \sigma_-\sigma_+$

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix}$$

all constraints require

$$
\begin{aligned}
(r_0+r_2)\left[(r_0+r_2)^2-4r_1^2\right] &= 1 \\
r_0-r_2+2r_1 &= (r_0-r_2+2r_1)(r_0+r_2) \\
(r_0+r_2) &= (r_0-r_2)^2-4r_1^2
\end{aligned}
$$

these are solved by

$$
r_0(\varepsilon)=\cosh\varepsilon, \quad r_1(\varepsilon)=\pm\sqrt{\cosh^2\varepsilon-\cosh\varepsilon}, \quad r_2(\varepsilon)=1-\cosh\varepsilon
$$

$\Rightarrow$ simple deformed roots

$$
\tilde{\alpha}_1=\cosh\varepsilon\alpha_1+(\cosh\varepsilon-1)\alpha_3-\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+2\alpha_2+\alpha_3),
$$

$$
\tilde{\alpha}_2=(2\cosh\varepsilon-1)\alpha_2+2\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+\alpha_2+\alpha_3),
$$

$$
\tilde{\alpha}_3=\cosh\varepsilon\alpha_3+(\cosh\varepsilon-1)\alpha_1-\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1+2\alpha_2+\alpha_3).
$$

remaining positive roots

$\tilde{\alpha}_4:=\tilde{\alpha}_1+\tilde{\alpha}_2$, $\tilde{\alpha}_5:=\tilde{\alpha}_2+\tilde{\alpha}_3$, $\tilde{\alpha}_6:=\tilde{\alpha}_1+\tilde{\alpha}_2+\tilde{\alpha}_3$.

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}-subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm\sqrt{r_0^2 - r_0}$
- useful choice $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for E_6

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}$$

$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}-subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm\sqrt{r_0^2 - r_0}$
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$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh \varepsilon$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}-subseries

closed solution

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n}\right),$$

- with $r_{2n} = 1 - r_0$, $r_n = \pm\sqrt{r_0^2 - r_0}$
- useful choice $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for E_6

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0 + \imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0 + 2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0 - 5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0 + 2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}$$

$r_2 = \pm 1/\sqrt{3}\sqrt{r_0^2 - 1}$, $r_0 = \cosh\varepsilon$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1}-subseries

no solution

Solutions from folding

for instance: $B_n \hookrightarrow A_{2n}$
-deformed A_6-roots:

$$
\begin{aligned}
\tilde{\alpha}_1 &= \cosh\varepsilon\alpha_1 - \imath/\sqrt{7}\sinh\varepsilon(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_6) \\
\tilde{\alpha}_2 &= \cosh\varepsilon\alpha_2 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4) \\
\tilde{\alpha}_3 &= \cosh\varepsilon\alpha_3 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6) \\
\tilde{\alpha}_4 &= \cosh\varepsilon\alpha_4 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 2\alpha_6) \\
\tilde{\alpha}_5 &= \cosh\varepsilon\alpha_5 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6), \\
\tilde{\alpha}_6 &= \cosh\varepsilon\alpha_6 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6)
\end{aligned}
$$

$\Rightarrow$ deformed simple B_3-roots as ($\alpha_{ij} := \alpha_i - \alpha_j$)

$$\tilde{\beta}_1 = \tilde{\alpha}_1 + \tilde{\alpha}_6 = \cosh\varepsilon(\alpha_1 + \alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[3(\alpha_1 - \alpha_6) + 2(\alpha_2 - \alpha_5)]$$
$$\tilde{\beta}_2 = \tilde{\alpha}_2 + \tilde{\alpha}_5 = \cosh\varepsilon(\alpha_2 + \alpha_5) + \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_{16} + \alpha_{34}) + \alpha_{25}]$$
$$\tilde{\beta}_3 = \tilde{\alpha}_3 + \tilde{\alpha}_4 = \cosh\varepsilon(\alpha_1 + \alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_2 - \alpha_5) + \alpha_3 - \alpha_4]$$

- longest element:

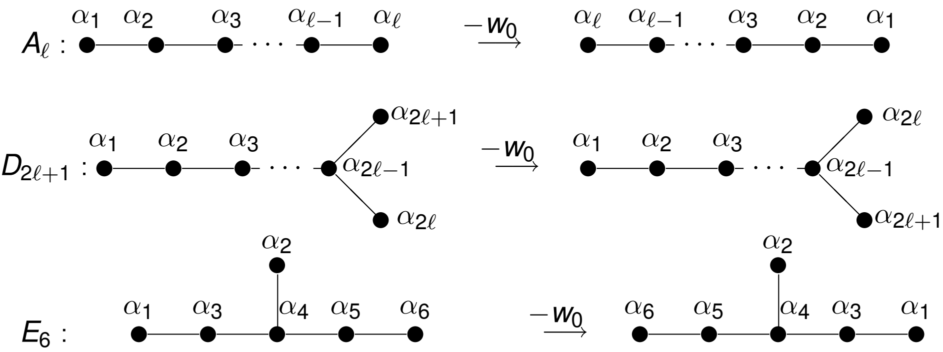
$$w_0: \quad \Delta_\pm \to \Delta_\mp, \quad w_0^2 = \mathbb{I}, \quad \alpha_i \mapsto -\alpha_{\bar{\imath}} = (w_0\alpha)_i$$

- representation in terms of Coxeter transformations

$$w_0 = \begin{cases} \sigma^{h/2} & \text{for } h \text{ even,} \\ \sigma_+\sigma^{(h-1)/2} & \text{for } h \text{ odd.} \end{cases}$$

- action

$$\begin{aligned}
A_\ell &: \quad \alpha_{\bar{\imath}} = \alpha_{\ell+1-i}, \\
D_\ell &: \quad \begin{cases} \alpha_{\bar{\imath}} = \alpha_i \quad \text{for } 1 \leq i \leq \ell, & \text{when } \ell \text{ even} \\ \alpha_{\bar{\imath}} = \alpha_i \quad \text{for } 1 \leq i \leq \ell - 2,\ \alpha_{\bar{\ell}} = \alpha_{\ell-1}, & \text{when } \ell \text{ odd,} \end{cases} \\
E_6 &: \quad \alpha_{\bar{1}} = \alpha_6, \alpha_{\bar{3}} = \alpha_5, \alpha_{\bar{2}} = \alpha_2, \alpha_{\bar{4}} = \alpha_4, \\
B_\ell, C_\ell &: \quad \alpha_{\bar{\imath}} = \alpha_i \\
E_7, E_8, F_4 &: \quad \alpha_{\bar{\imath}} = \alpha_i \\
G_2 &: \quad \alpha_{\bar{\imath}} = \alpha_i
\end{aligned}$$



- two cases:

$$[\sigma, \theta_\varepsilon] = 0 \Rightarrow \begin{cases} \text{no solution} & \text{for } h \text{ even} \\ \text{previous solutions} & \text{for } h \text{ odd} \end{cases}$$

$$[\sigma, \theta_\varepsilon] \neq 0, \quad \theta_\varepsilon^* w_0 = w_0 \theta_\varepsilon, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1 \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}$$

Solutions:

$\tilde{\Delta}(\varepsilon)$ **for** A_3

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon & 0 & \imath\sinh\varepsilon \\ (-\sinh^2\frac{\varepsilon}{2} + \frac{\imath}{2}\sinh\varepsilon) & 1 & (-\sinh^2\frac{\varepsilon}{2} - \frac{\imath}{2}\sinh\varepsilon) \\ -\imath\sinh\varepsilon & 0 & \cosh\varepsilon \end{pmatrix}$$

$\tilde{\Delta}(\varepsilon)$ **for** E_6

$$\theta_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \theta_\varepsilon^{A_3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Deformed Weyl reflections

This construction only works for groups of rank 2.

For **any** model based on roots, these deformed roots can be used to define new invariant models simply by

$$\alpha \rightarrow \tilde{\alpha}.$$

For instance Calogero models:

- generalized Calogero Hamiltionian (undeformed)

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

- define the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot q) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2,$$

$\hat{h} \equiv$ dual Coxeter number, $t_\ell \equiv \ell$-th symmetrizer of I

- Ansatz:

$$\psi(q) \to \psi(z,r) = z^{\kappa+1/2}\varphi(r)$$

$\Rightarrow$ solution for $\kappa = 1/2\sqrt{1+4g}$.

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}}\frac{\omega}{2}r^2\right) L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right).$$

$L_n^a(x) \equiv$ Laguerre polynomial, $a = \left(2 + h + h\sqrt{1+4g}\right) l/4 - 1$

Generalization of Calogero's solution, undeformed case

- generalized Calogero Hamiltionian (undeformed)

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

- define the variables

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$L_n^a(x) \equiv$ Laguerre polynomial, $a = \left(2+h+h\sqrt{1+4g}\right) l/4 - 1$

• eigenenergies

$$E_n = \frac{1}{4}\left[\left(2+h+h\sqrt{1+4g}\right)l+8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega$$

• anyonic exchange factors

$$\psi(q_1,\ldots,q_i,q_j,\ldots q_n) = e^{\imath\pi s}\psi(q_1,\ldots,q_j,q_i,\ldots q_n), \quad \text{for } 1 \leq i,j \leq n,$$

with

$$s = \frac{1}{2}+\frac{1}{2}\sqrt{1+4g}$$

$\because$ r is symmetric and z antisymmetric

The construction is based on the identities:

$$\begin{aligned}
\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} &= \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} &= \frac{\hat{h}h\ell}{2} t_\ell, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) &= \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2, \\
\sum_{\alpha\in\Delta^+} \alpha^2 &= \ell\hat{h}t_\ell.
\end{aligned}$$

Strong evidence on a case-by-case level, but no rigorous proof.

- antilinearly deformed Calogero Hamiltionian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2$$

- Ansatz

$$\psi(q) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^s \varphi(\tilde{r})$$

when identies still hold $\Rightarrow$

$$\psi(q) = \psi(\tilde{z},r) = \tilde{z}^s \varphi_n(r)$$

with same eigenenergies

- antilinearly deformed Calogero Hamiltionian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\tilde{\alpha}\in\tilde{\Delta}^+}(\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+}\frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

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- antilinearly deformed Calogero Hamiltonian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$$

- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2$$

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- define the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell} \sum_{\tilde{\alpha}\in\tilde{\Delta}^+} (\tilde{\alpha}\cdot q)^2$$

- Ansatz

$$\psi(q) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^s \varphi(\tilde{r})$$

when identies still hold $\Rightarrow$

$$\psi(q) = \psi(\tilde{z},r) = \tilde{z}^s \varphi_n(r)$$

with same eigenenergies

Deformed A_3-models

- potential from deformed Coxeter group factors

  $\alpha_1 = \{1, -1, 0, 0\}$, $\alpha_2 = \{0, 1, -1, 0\}$, $\alpha_3 = \{0, 0, 1, -1\}$

$$
\begin{aligned}
\tilde{\alpha}_1 \cdot q &= q_{43} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_2 \cdot q &= q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{14} \\
\tilde{\alpha}_3 \cdot q &= q_{21} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_4 \cdot q &= q_{42} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_5 \cdot q &= q_{31} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_6 \cdot q &= q_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{23}
\end{aligned}
$$

notation $q_{ij} = q_i - q_j$, No longer singular for $q_{ij} = 0$

Deformed A_3-models

- potential from deformed Coxeter group factors
  $\alpha_1 = \{1, -1, 0, 0\}$, $\alpha_2 = \{0, 1, -1, 0\}$, $\alpha_3 = \{0, 0, 1, -1\}$

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\tilde{\alpha}_1 \cdot q &= q_{43} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_2 \cdot q &= q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{14} \\
\tilde{\alpha}_3 \cdot q &= q_{21} + \cosh\varepsilon(q_{12} + q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13} + q_{24}) \\
\tilde{\alpha}_4 \cdot q &= q_{42} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_5 \cdot q &= q_{31} + \cosh\varepsilon(q_{13} + q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12} + q_{34}) \\
\tilde{\alpha}_6 \cdot q &= q_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{23}
\end{aligned}
$$

notation $q_{ij} = q_i - q_j$, **No longer singular for $q_{ij} = 0$**

- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \ \tilde{\alpha}_1 \to -\tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_6, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5, \tilde{\alpha}_5 \to \tilde{\alpha}_4, \tilde{\alpha}_6 \to \tilde{\alpha}_2$$
$$\sigma_+^\varepsilon : \tilde{\alpha}_1 \to \tilde{\alpha}_4, \tilde{\alpha}_2 \to -\tilde{\alpha}_2, \tilde{\alpha}_3 \to \tilde{\alpha}_5, \tilde{\alpha}_4 \to \tilde{\alpha}_1, \tilde{\alpha}_5 \to \tilde{\alpha}_3, \tilde{\alpha}_6 \to \tilde{\alpha}_6$$

- $\mathcal{PT}$-symmetry in dual space

$$\sigma_-^\varepsilon : q_1 \to q_2, q_2 \to q_1, q_3 \to q_4, q_4 \to q_3, \imath \to -\imath$$
$$\sigma_+^\varepsilon : q_1 \to q_1, q_2 \to q_3, q_3 \to q_2, q_4 \to q_4, \imath \to -\imath$$

$\Rightarrow$

$$\sigma_-^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) = \tilde{z}^*(q_2, q_1, q_4, q_3) = \tilde{z}(q_1, q_2, q_3, q_4)$$
$$\sigma_+^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) = \tilde{z}^*(q_1, q_3, q_2, q_4) = -\tilde{z}(q_1, q_2, q_3, q_4)$$

$$\psi(q_1, q_2, q_3, q_4) = e^{\imath\pi s}\psi(q_2, q_4, q_1, q_3).$$

- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \tilde{\alpha}_1 \to -\tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_6, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5, \tilde{\alpha}_5 \to \tilde{\alpha}_4, \tilde{\alpha}_6 \to \tilde{\alpha}_2$$
$$\sigma_+^\varepsilon : \tilde{\alpha}_1 \to \tilde{\alpha}_4, \tilde{\alpha}_2 \to -\tilde{\alpha}_2, \tilde{\alpha}_3 \to \tilde{\alpha}_5, \tilde{\alpha}_4 \to \tilde{\alpha}_1, \tilde{\alpha}_5 \to \tilde{\alpha}_3, \tilde{\alpha}_6 \to \tilde{\alpha}_6$$

- $\mathcal{PT}$-symmetry in dual space

$$\sigma_-^\varepsilon : q_1 \to q_2, q_2 \to q_1, q_3 \to q_4, q_4 \to q_3, \imath \to -\imath$$
$$\sigma_+^\varepsilon : q_1 \to q_1, q_2 \to q_3, q_3 \to q_2, q_4 \to q_4, \imath \to -\imath$$

$\Rightarrow$

$$\begin{aligned} \sigma_-^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) &= \tilde{z}^*(q_2, q_1, q_4, q_3) = \tilde{z}(q_1, q_2, q_3, q_4) \\ \sigma_+^\varepsilon \tilde{z}(q_1, q_2, q_3, q_4) &= \tilde{z}^*(q_1, q_3, q_2, q_4) = -\tilde{z}(q_1, q_2, q_3, q_4) \end{aligned}$$

$$\psi(q_1, q_2, q_3, q_4) = e^{\imath \pi s} \psi(q_2, q_4, q_1, q_3).$$

- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$\sigma_-^\varepsilon : \tilde{\alpha}_1 \to -\tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_6, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_5, \tilde{\alpha}_5 \to \tilde{\alpha}_4, \tilde{\alpha}_6 \to \tilde{\alpha}_2$$
$$\sigma_+^\varepsilon : \tilde{\alpha}_1 \to \tilde{\alpha}_4, \tilde{\alpha}_2 \to -\tilde{\alpha}_2, \tilde{\alpha}_3 \to \tilde{\alpha}_5, \tilde{\alpha}_4 \to \tilde{\alpha}_1, \tilde{\alpha}_5 \to \tilde{\alpha}_3, \tilde{\alpha}_6 \to \tilde{\alpha}_6$$

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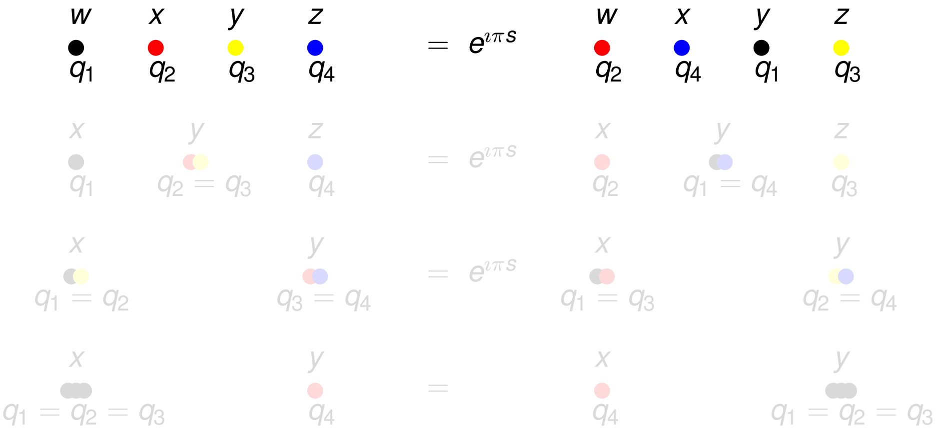
Anyonic exchange factors in the 4-particle scattering process



Anyonic exchange factors in the 4-particle scattering process

$$\begin{array}{cccc} w & x & y & z \\ q_1 & q_2 & q_3 & q_4 \end{array} \;=\; e^{\imath\pi s} \begin{array}{cccc} w & x & y & z \\ q_2 & q_4 & q_1 & q_3 \end{array}$$

$$\begin{array}{ccc} x & y & z \\ q_1 & q_2 = q_3 & q_4 \end{array} \;=\; e^{\imath\pi s} \begin{array}{ccc} x & y & z \\ q_2 & q_1 = q_4 & q_3 \end{array}$$

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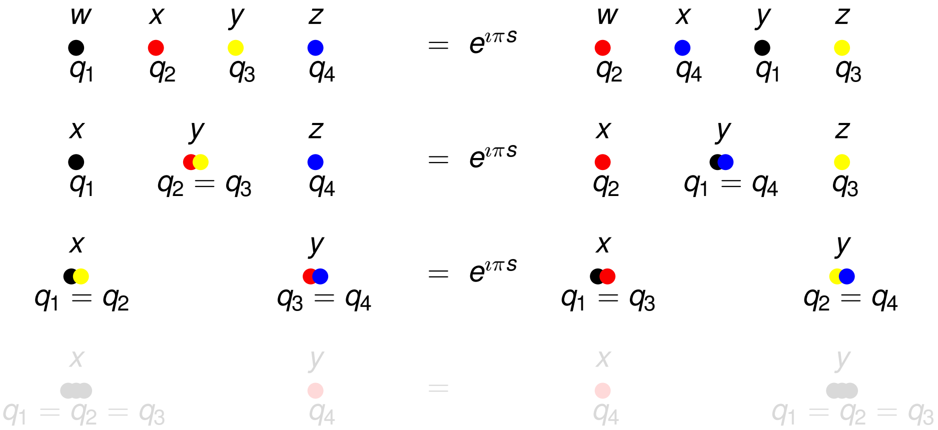

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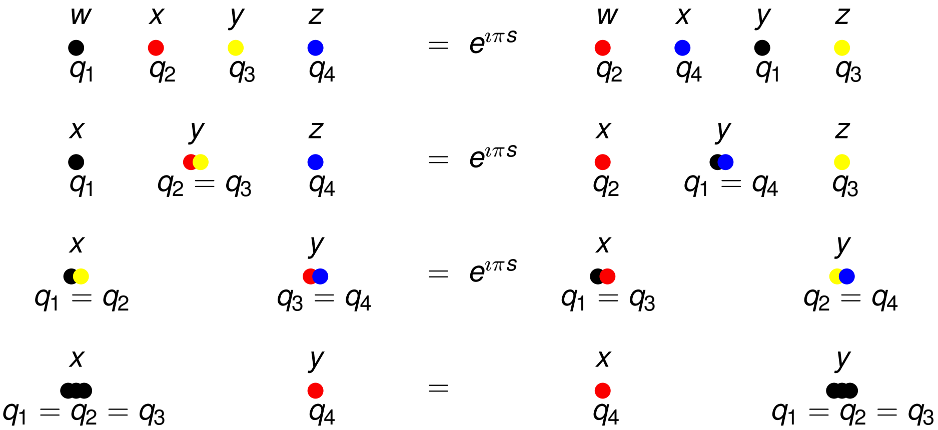

- Physical properties of the A_2, G_2-models:
  - The deformed model can be solved by separation of variables as the undeformed case.
  - Some restrictions cease to exist, as the wavefunctions are now regularized.
  - $\Rightarrow$ modified energy spectrum:

$$E = 2\,|\omega|\,(2n + \lambda + 1)$$

becomes

$$E^{\pm}_{n\ell} = 2|\omega|\left[2n + 6(\kappa^{\pm}_{s} + \kappa^{\pm}_{l} + \ell) + 1\right] \qquad \text{for } n, \ell \in \mathbb{N}_0,$$

with $\kappa^{\pm}_{s/l} = (1 \pm \sqrt{1 + 4g_{s/l}})/4$

Ising quantum spin chain of length N

$$\mathcal{H} = -\frac{1}{2}\sum_{i=1}^{N}(\sigma_i^z + \lambda\sigma_i^x\sigma_{i+1}^x + i\kappa\sigma_i^x) \qquad \kappa, \lambda \in \mathbb{R}$$

in a magnetic field in the z-direction and in a longitudinal imaginary field in the x-direction

- $\mathcal{H}$ acts on the Hilbert space of the form $(\mathbb{C}^2)^{\otimes N}$
- $\sigma_i^{x,y,z} := \mathbb{I} \otimes \mathbb{I} \otimes \ldots \otimes \sigma^{x,y,z} \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I}$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathcal{H}$ is a perturbation of the $\mathcal{M}_{5,2}$-model (c=-22/5) in the $\mathcal{M}_{p,q}$-series of minimal conformal field theories
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[G. von Gehlen, J. Phys. A24 (1991) 5371]

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Pre-$\mathcal{PT}$ spectral analysis

- spin rotation operator:

$$\mathcal{R} = e^{\frac{i\pi}{4} S_z^N} \quad \text{with} \quad S_z^N = \sum_{i=1}^{N} \sigma_i^z$$

$$\mathcal{R} : (\sigma_i^x, \sigma_i^y, \sigma_i^z) \to (-\sigma_i^y, \sigma_i^x, \sigma_i^z)$$

- $\Rightarrow$ real matrix

$$\hat{\mathcal{H}}(\lambda, \kappa) = \mathcal{R}\mathcal{H}(\lambda, \kappa)\mathcal{R}^{-1} = -\frac{1}{2}\sum_{i=1}^{N}(\sigma_i^z + \lambda\sigma_i^y\sigma_{i+1}^y - i\kappa\sigma_i^y).$$

- $\Rightarrow$ eigenvalues are real or complex conjugate pairs

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$\mathcal{PT}$-symmetry for spin chains

- "macro-reflections": [Korff, Weston, J. Phys. A40 (2007)]
  $\mathcal{P}' : \sigma_i^{x,y,z} \rightarrow \sigma_{N+1-i}^{x,y,z}$

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  $\mathcal{P}' : \sigma_i^{x,y,z} \longrightarrow \sigma_{N+1-i}^{x,y,z}$

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- but with $\mathcal{T} : i \rightarrow -i$ $\qquad [\mathcal{P}'\mathcal{T}, \mathcal{H}] \neq 0$

- "site-by-site reflections":
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- XXZ-spin-chain in a magnetic field

$$\mathcal{H}_{XXZ} = \frac{1}{2}\sum_{i=1}^{N-1}\left[(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta_+(\sigma_i^z\sigma_{i+1}^z - 1)\right] + \frac{\Delta_-}{2}(\sigma_1^z - \sigma_N^z),$$

$$\Delta_\pm = (q \pm q^{-1})/2 \qquad \Rightarrow \mathcal{H}_{XXZ}^\dagger \neq \mathcal{H}_{XXZ} \text{ for } q \notin \mathbb{R}$$

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$\mathcal{PT}$-symmetry $\Rightarrow$ domains in the parameter space of λ and κ

Broken and unbroken $\mathcal{PT}$-symmetry

$$[\mathcal{PT}, \mathcal{H}] = 0 \quad \bigwedge \quad \mathcal{PT}\Phi(\lambda,\kappa) \begin{cases} = \Phi(\lambda,\kappa) & \text{for } (\lambda,\kappa) \in U_{\mathcal{PT}} \\ \neq \Phi(\lambda,\kappa) & \text{for } (\lambda,\kappa) \in U_{b\mathcal{PT}} \end{cases}$$

$(\lambda,\kappa) \in U_{\mathcal{PT}} \Rightarrow$ real eigenvalues
$(\lambda,\kappa) \in U_{b\mathcal{PT}} \Rightarrow$ eigenvalues in complex conjugate pairs

- Left and right eigenvectors:

$$H|\Phi_n\rangle = \varepsilon_n |\Phi_n\rangle \qquad \text{and} \qquad H^\dagger |\Psi_n\rangle = \epsilon_n |\Psi_n\rangle \qquad \text{for } n \in \mathbb{N}$$

- Biorthonormal basis:

$$\langle \Psi_n | \Phi_m \rangle = \delta_{nm} \qquad \sum\nolimits_n |\Psi_n\rangle \langle \Phi_n| = \mathbb{I}$$

- Parity $\mathcal{P}$ and signature s:

$$H^\dagger = \mathcal{P} H \mathcal{P} \qquad \text{and} \qquad \mathcal{P}^2 = \mathbb{I}$$

$$\mathcal{P}|\Phi_n\rangle = s_n |\Psi_n\rangle \qquad \text{with } s_n = \pm 1$$

$s := (s_1, s_2, \ldots, s_n)$

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$$\mathcal{C} := \sum\nolimits_n s_n |\Phi_n\rangle \langle \Psi_n| ,$$

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- The two site Hamiltonian

$$\begin{aligned}\mathcal{H} &= -\frac{1}{2}\left[\sigma_1^z + \sigma_2^z + 2\lambda\sigma_1^x\sigma_2^x + i\kappa\left(\sigma_2^x + \sigma_1^x\right)\right] \\ &= -\frac{1}{2}\left[\sigma^z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma^z + 2\lambda\sigma^x \otimes \sigma^x + i\kappa\left(\mathbb{I} \otimes \sigma^x + \sigma^x \otimes \mathbb{I}\right)\right] \\ &= -\begin{pmatrix} -1 & \frac{i\kappa}{2} & \frac{i\kappa}{2} & \lambda \\ \frac{i\kappa}{2} & 0 & \lambda & \frac{i\kappa}{2} \\ \frac{i\kappa}{2} & \lambda & 0 & \frac{i\kappa}{2} \\ \lambda & \frac{i\kappa}{2} & \frac{i\kappa}{2} & -1 \end{pmatrix}\end{aligned}$$

with periodic boundary condition $\sigma_{N+1}^x = \sigma_1^x$

- domain of unbroken $\mathcal{PT}$-symmetry:
  char. polynomial factorises into 1st and 3rd order
  discriminant: $\Delta = r^2 - q^3$

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$$\begin{aligned}\mathcal{H} &= -\frac{1}{2}\left[\sigma_1^z + \sigma_2^z + 2\lambda\sigma_1^x\sigma_2^x + i\kappa\left(\sigma_2^x + \sigma_1^x\right)\right] \\ &= -\frac{1}{2}\left[\sigma^z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma^z + 2\lambda\sigma^x \otimes \sigma^x + i\kappa\left(\mathbb{I} \otimes \sigma^x + \sigma^x \otimes \mathbb{I}\right)\right] \\ &= -\begin{pmatrix} -1 & \frac{i\kappa}{2} & \frac{i\kappa}{2} & \lambda \\ \frac{i\kappa}{2} & 0 & \lambda & \frac{i\kappa}{2} \\ \frac{i\kappa}{2} & \lambda & 0 & \frac{i\kappa}{2} \\ \lambda & \frac{i\kappa}{2} & \frac{i\kappa}{2} & -1 \end{pmatrix}\end{aligned}$$

with periodic boundary condition $\sigma_{N+1}^x = \sigma_1^x$

- domain of unbroken $\mathcal{PT}$-symmetry:
  char. polynomial factorises into 1st and 3rd order
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Deformed quantum spin chains (Exact Results, $N = 2$)

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$$
\begin{aligned}
\mathcal{H} &= -\frac{1}{2}\left[\sigma_1^z + \sigma_2^z + 2\lambda\sigma_1^x\sigma_2^x + i\kappa\left(\sigma_2^x + \sigma_1^x\right)\right] \\
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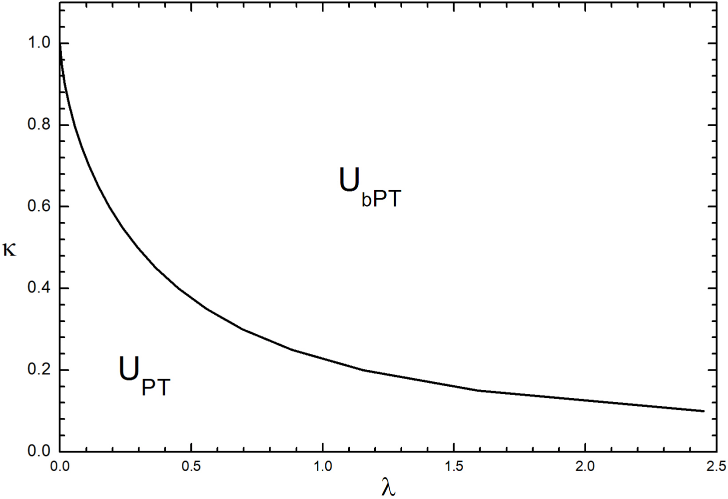
Deformed quantum spin chains (Exact Results, $N = 2$)

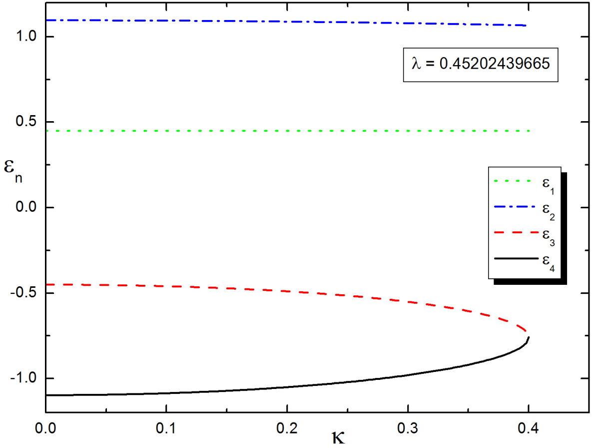
$$U_{\mathcal{PT}} = \left\{ \lambda, \kappa : \Delta = \kappa^6 + 8\lambda^2\kappa^4 - 3\kappa^4 + 16\lambda^4\kappa^2 + 20\lambda^2\kappa^2 + 3\kappa^2 - \lambda^2 \leq 1 \right\}$$



Real eigenvalues: $\left[\theta = \arccos\left(r/q^{3/2}\right)\right]$

$$\varepsilon_1 = \lambda, \quad \varepsilon_2 = 2q^{\frac{1}{2}} \cos\left(\frac{\theta}{3}\right) - \frac{\lambda}{3}, \quad \varepsilon_{3/4} = 2q^{\frac{1}{2}} \cos\left(\frac{\theta}{3} + \pi \mp \frac{1\pi}{3}\right) - \frac{\lambda}{3}$$

Avoided level crossing:



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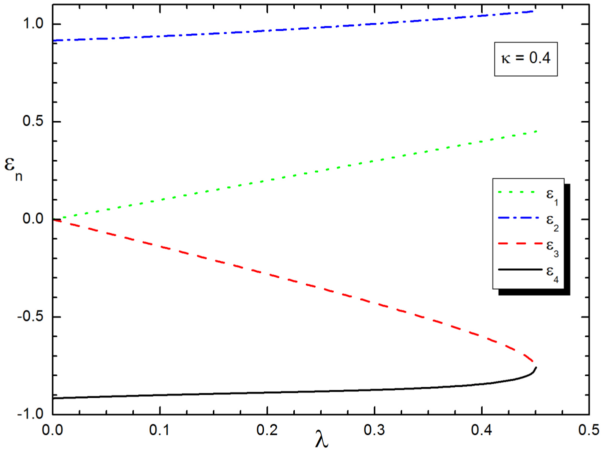

Deformed quantum spin chains (Exact Results, $N = 2$)

- Right eigenvectors of $\mathcal{H}$:

$$|\Phi_1\rangle = (0, -1, -1, 0) \quad |\Phi_n\rangle = (\gamma_n, -\alpha_n, -\alpha_n, \beta_n) \quad n = 2, 3, 4$$

$$\alpha_n = i\kappa(\lambda - \varepsilon_n + 1)$$
$$\beta_n = \kappa^2 + 2\lambda^2 + 2\lambda\varepsilon_n$$
$$\gamma_n = -\kappa^2 - 2\varepsilon_n^2 + 2\lambda - 2\lambda\varepsilon_n + 2\varepsilon_n$$

- signature: $s = (+, -, +, -)$

$$\mathcal{P}\,|\Phi_n\rangle = s_n\,|\Psi_n\rangle$$

from relating left and right eigenvectors

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from relating left and right eigenvectors

- $\mathcal{C}$-operator:

$$
\begin{aligned}
\mathcal{C} &= \sum_n s_n \left|\Phi_n\right\rangle \left\langle\Psi_n\right| \\
&= \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ -C_3 & -C_1 - 1 & -C_1 & C_2 \\ -C_3 & -C_1 & -C_1 - 1 & C_2 \\ C_4 & C_2 & C_2 & 2(C_1+1) - C_5 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \frac{\alpha_4^2}{N_4^2} - \frac{\alpha_2^2}{N_2^2} - \frac{\alpha_3^2}{N_3^2} - \frac{1}{2}, \qquad & C_2 &= \frac{\alpha_4\beta_4}{N_4^2} - \frac{\alpha_2\beta_2}{N_2^2} - \frac{\alpha_3\beta_3}{N_3^2}, \\
C_3 &= \frac{\alpha_2\gamma_2}{N_2^2} + \frac{\alpha_3\gamma_3}{N_3^2} - \frac{\alpha_4\gamma_4}{N_4^2}, \qquad & C_4 &= \frac{\beta_2\gamma_2}{N_2^2} + \frac{\beta_3\gamma_3}{N_3^2} - \frac{\beta_4\gamma_4}{N_4^2}, \\
C_5 &= \frac{\gamma_2^2}{N_2^2} + \frac{\gamma_3^2}{N_3^2} - \frac{\gamma_4^2}{N_4^2}
\end{aligned}
$$

$$
N_1 = \sqrt{2},\ N_n = \sqrt{2\alpha_n^2 + \beta_n^2 + \gamma_n^2} \text{ for } n =, 2, 3, 4
$$

- metric operator:

$$\rho = \mathcal{P}\mathcal{C} = \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ C_3 & 1+C_1 & C_1 & -C_2 \\ C_3 & C_1 & 1+C_1 & -C_2 \\ C_4 & C_2 & C_2 & 2(1+C_1)-C_5 \end{pmatrix}$$

- since $i\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$
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  $\Rightarrow \rho$ is Hermitian $\rho = \rho^\dagger$

- EV of ρ:

$$y_1 = y_2 = 1, \qquad y_{3/4} = 1 + 2C_1 \pm 2\sqrt{C_1(1+C_1)}$$

since $C_1 > 0 \;\Rightarrow \rho$ is positive

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$$\rho = \mathcal{PC} = \begin{pmatrix} C_5 & -C_3 & -C_3 & C_4 \\ C_3 & 1+C_1 & C_1 & -C_2 \\ C_3 & C_1 & 1+C_1 & -C_2 \\ C_4 & C_2 & C_2 & 2(1+C_1)-C_5 \end{pmatrix}$$

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- square root of the metric operator:

$$\eta = \rho^{1/2} = UD^{1/2}U^{-1}$$

where $D = \mathrm{diag}(y_{1,}y_{2,}, y_{3,}, y_4)$, $U = \{r_1, r_2, r_3, r_4\}$

$$|r_1\rangle = (0, -1, 1, 0)$$
$$|r_2\rangle = (C_4, 0, 0, 1 - C_5),$$
$$|r_{3/4}\rangle = (\tilde{\gamma}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\alpha}_{3/4}, \tilde{\beta}_{3/4})$$

$$\tilde{\alpha}_{3/4} = y_{3/4}(C_3C_4 + C_2(-4C_1 + C_5 - 1))/2 - C_3C_4$$
$$\tilde{\beta}_{3/4} = -C_3^2 - C_1 - C_1C_5 + \left(C_3^2 + C_1(4C_1 - C_5 + 3)\right) y_{3/4},$$
$$\tilde{\gamma}_{3/4} = C_1C_4 - C_2C_3 + (C_2C_3 + C_1C_4)y_{3/4}$$

- isospectral Hermitian counterpart:

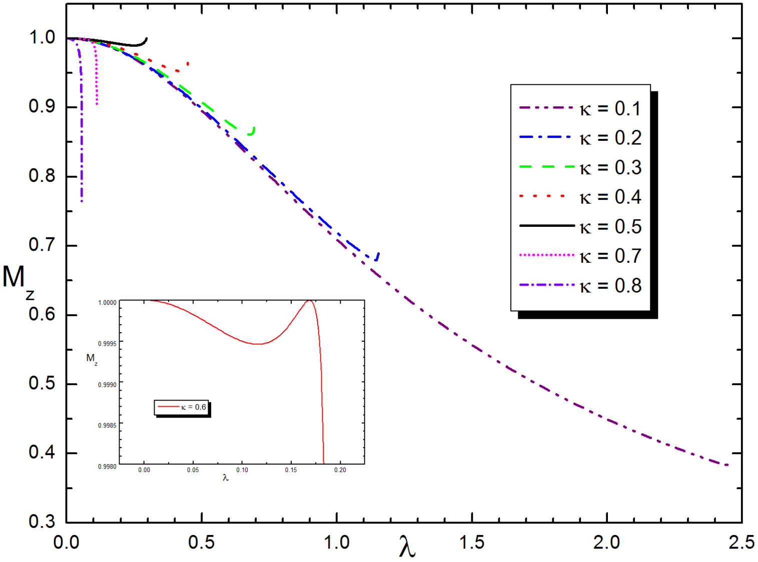
$$\begin{aligned} h &= \eta \mathcal{H} \eta^{-1} \\ &= \mu_1 \sigma_x \otimes \sigma_x + \mu_2 \sigma_y \otimes \sigma_y + \mu_3 \sigma_z \otimes \sigma_z + \mu_4 (\sigma_z \otimes \mathbb{I} + \mathbb{I} \otimes \sigma_z) \end{aligned}$$

$\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$

for $\lambda = 0.1, \kappa = 0.5$:

$$h = \begin{pmatrix} -0.829536 & 0 & 0 & -0.0606492 \\ 0 & -0.0341687 & -0.1341687 & 0 \\ 0 & -0.1341687 & -0.0341687 & 0 \\ -0.0606492 & 0 & 0 & 0.897873 \end{pmatrix}$$

The magnetization in the z-direction for $N = 2$:



Deformed quantum spin chains ($N \neq 2$, perturbation theory)

- Perturbation theory about the Hermitian part

$$H(\lambda,\kappa) = h_0(\lambda) + i\kappa h_1 \qquad h_0 = h_0^\dagger, h_1 = h_1^\dagger \quad \kappa \in \mathbb{R}$$

assume $\eta = \eta^\dagger = e^{q/2} \Rightarrow$ solve for q

$$H^\dagger = e^q H e^{-q} = H + [q,H] + \frac{1}{2}[q,[q,H]] + \frac{1}{3!}[q,[q,[q,H]]] + \cdots$$

for $c_q^{(\ell+1)}(h_0) = [q, \ldots [q,[q,h_0]] \ldots] = 0$ closed formulae:

$$h = h_0 + \sum_{n=1}^{[\frac{\ell}{2}]} \frac{(-1)^n E_n}{4^n (2n)!} c_q^{(2n)}(h_0) \quad H = h_0 - \sum_{n=1}^{[\frac{\ell+1}{2}]} \frac{\kappa_{2n-1}}{(2n-1)!} c_q^{(2n-1)}(h_0)$$

$E_n \equiv$ Euler numbers, e.g. $E_1 = 1$, $E_2 = 5$, $E_3 = 61, \ldots$

$$\kappa_n = \frac{1}{2^n} \sum_{m=1}^{[(n+1)/2]} (-1)^{n+m} \binom{n}{2m} E_m$$

$\kappa_1 = 1/2, \kappa_3 = -1/4, \kappa_5 = 1/2, \kappa_7 = -17/8, \ldots$

[C. F. de Morisson Faria, A.F., J. Phys. A39 (2006) 9269]

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[C. F. de Morisson Faria, A.F., J. Phys. A39 (2006) 9269]

further assumption

$$q = \sum_{k=1}^{\infty} \kappa^{2k-1} q_{2k-1}$$

solve recursively:

$$\begin{aligned}
[h_0, q_1] &= 2ih_1 \\
[h_0, q_3] &= \frac{i}{6}[q_1, [q_1, h_1]] \\
[h_0, q_5] &= \frac{i}{6}[q_1, [q_3, h_1]] + \frac{i}{6}[q_3, [q_1, h_1]] - \frac{i}{360}[q_1, [q_1, [q_1, [q_1, h_1]]]]
\end{aligned}$$

Here

$$h_0(\lambda) = -\sum\nolimits_{i=1}^{N} (\sigma_i^z + \lambda \sigma_i^x \sigma_{i+1}^x)/2, \qquad h_1 = -\sum\nolimits_{i=1}^{N} \sigma_i^x/2$$

- Perturbation theory in λ

$$H(\lambda, \kappa) = h_0(\kappa) + \lambda h_1 \qquad h_0 \neq h_0^\dagger, h_1 = h_1^\dagger \quad \lambda \in \mathbb{R}$$

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exact result for $N = 2$:

$\lambda = 0.1$, $\kappa = 0.5$:

$$h = \begin{pmatrix} -0.829536 & 0 & 0 & -0.0606492 \\ 0 & -0.0341687 & -0.1341687 & 0 \\ 0 & -0.1341687 & -0.0341687 & 0 \\ -0.0606492 & 0 & 0 & 0.897873 \end{pmatrix}$$

$\lambda = 0.9$, $\kappa = 0.1$:

$$h = \begin{pmatrix} -0.985439 & 0 & 0 & -0.890532 \\ 0 & -0.0094167 & -0.909417 & 0 \\ 0 & -0.909417 & -0.0094167 & 0 \\ -0.890532 & 0 & 0 & 1.00427 \end{pmatrix}$$

perturbative result 4th order for $N = 2$:

$\lambda = 0.1$, $\kappa = 0.5$:

$$h = \begin{pmatrix} -0.829534 & 0 & 0 & -0.0606716 \\ 0 & -0.0341688 & -0.134169 & 0 \\ 0 & -0.134169 & -0.0341688 & 0 \\ -0.0606716 & 0 & 0 & 0.897872 \end{pmatrix}$$

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- new notation:

$$S^N_{a_1 a_2 \ldots a_p} := \sum_{k=1}^{N} \sigma_k^{a_1} \sigma_{k+1}^{a_2} \ldots \sigma_{k+p-1}^{a_p}, \qquad a_i = x, y, z, u; i = 1, \ldots, p \leq N$$

with $\sigma^u = \mathbb{I}$ to allow for non-local interactions

- for instance:

$$\begin{aligned} H(\lambda, \kappa) &= -\frac{1}{2} \sum_{j=1}^{N} (\sigma_j^z + \lambda \sigma_j^x \sigma_{j+1}^x + i\kappa \sigma_j^x), \qquad \lambda, \kappa \in \mathbb{R} \\ &= -\frac{1}{2}(S_z^N + \lambda S_{xx}^N) - i\kappa \frac{1}{2} S_x^N \end{aligned}$$

- perturbative result for $N = 3$:

$$\begin{aligned} h &= \mu_{xx}^3(\lambda,\kappa) S_{xx}^3 + \mu_{yy}^3(\lambda,\kappa) S_{yy}^3 + \mu_{zz}^3(\lambda,\kappa) S_{zz}^3 + \mu_z^3(\lambda,\kappa) S_z^3 \\ &\quad + \mu_{xxz}^3(\lambda,\kappa) S_{xxz}^3 + \mu_{yyz}^3(\lambda,\kappa) S_{yyz}^3 + \mu_{zzz}^3(\lambda,\kappa) S_{zzz}^3 \end{aligned}$$

- perturbative result for $N = 4$:

$$
\begin{aligned}
h \;=\;& \mu^4_{xx}(\lambda,\kappa)S^4_{xx} + \nu^4_{xx}(\lambda,\kappa)S^4_{xux} + \mu^4_{yy}(\lambda,\kappa)S^4_{yy} + \nu^4_{yy}(\lambda,\kappa)S^4_{yuy} \\
&+\mu^4_{zz}(\lambda,\kappa)S^4_{zz} + \nu^4_{zz}(\lambda,\kappa)S^4_{zuz} + \mu^4_{z}(\lambda,\kappa)S^4_{z} + \mu^4_{xzx}(\lambda,\kappa)S^4_{xzx} \\
&+\mu^4_{xxz}(\lambda,\kappa)(S^4_{xxz} + S^4_{zxx}) + \mu^4_{yyz}(\lambda,\kappa)(S^4_{yyz} + S^4_{zyy}) \\
&+\mu^4_{yzy}(\lambda,\kappa)S^4_{yzy} + \mu^4_{zzz}(\lambda,\kappa)S^4_{zzz} + \mu^4_{xxxx}(\lambda,\kappa)S^4_{xxxx} \\
&+\mu^4_{yyyy}(\lambda,\kappa)S^4_{yyyy} + \mu^4_{zzzz}(\lambda,\kappa)S^4_{zzzz} + \mu^4_{xxyy}(\lambda,\kappa)S^4_{xxyy} \\
&+\mu^4_{xyxy}(\lambda,\kappa)S^4_{xyxy} + \mu^4_{zzyy}(\lambda,\kappa)S^4_{zzyy} + \mu^4_{zyzy}(\lambda,\kappa)S^4_{zyzy} \\
&+\mu^4_{xxzz}(\lambda,\kappa)S^4_{xxzz} + \mu^4_{xzxz}(\lambda,\kappa)S^4_{xzxz}
\end{aligned}
$$

non-local terms

- perturbative result for $N = 4$:

$$\begin{aligned} h \;=\; & \mu^4_{xx}(\lambda,\kappa)S^4_{xx} + \nu^4_{xx}(\lambda,\kappa)S^4_{xux} + \mu^4_{yy}(\lambda,\kappa)S^4_{yy} + \nu^4_{yy}(\lambda,\kappa)S^4_{yuy} \\ & + \mu^4_{zz}(\lambda,\kappa)S^4_{zz} + \nu^4_{zz}(\lambda,\kappa)S^4_{zuz} + \mu^4_{z}(\lambda,\kappa)S^4_{z} + \mu^4_{xzx}(\lambda,\kappa)S^4_{xzx} \\ & + \mu^4_{xxz}(\lambda,\kappa)(S^4_{xxz} + S^4_{zxx}) + \mu^4_{yyz}(\lambda,\kappa)(S^4_{yyz} + S^4_{zyy}) \\ & + \mu^4_{yzy}(\lambda,\kappa)S^4_{yzy} + \mu^4_{zzz}(\lambda,\kappa)S^4_{zzz} + \mu^4_{xxxx}(\lambda,\kappa)S^4_{xxxx} \\ & + \mu^4_{yyyy}(\lambda,\kappa)S^4_{yyyy} + \mu^4_{zzzz}(\lambda,\kappa)S^4_{zzzz} + \mu^4_{xxyy}(\lambda,\kappa)S^4_{xxyy} \\ & + \mu^4_{xyxy}(\lambda,\kappa)S^4_{xyxy} + \mu^4_{zzyy}(\lambda,\kappa)S^4_{zzyy} + \mu^4_{zyzy}(\lambda,\kappa)S^4_{zyzy} \\ & + \mu^4_{xxzz}(\lambda,\kappa)S^4_{xxzz} + \mu^4_{xzxz}(\lambda,\kappa)S^4_{xzxz} \end{aligned}$$

non-local terms

Some general conclusions

- Non-Hermitian Hamiltonians describe physical systems within a self-consistent quantum mechanical framework.
- One can use this possibility to explore deformations of well studied models, e.g. integrable systems.

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Thank you for your attention