Negative Even Grade mKdV Hierarchy and its Soliton Solutions

J. F. Gomes, G. S. França, G.R. de Melo and A. H. Zimerman

Instituto de Física Teórica IFT-Unesp

São Paulo - Brasil

Aim

- Discuss the construction of time evolution equations associated to negative even graded Lie algebraic structure.
- Modify dressing method to construct solutions.
- Construct soliton solutions by deforming vertex operators.

The mKdV Hierarchy

- Start with $sl(2)$ Lie Algebra with generators

$$[h, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}, \qquad [E_\alpha, E_{-\alpha}] = h$$

- Affine Lie algebra $T_a \to T_a^{(m)} \equiv \lambda^m T_a, \quad m \in Z$, i.e.,

$$h \to h^{(m)} = \lambda^m h, \qquad E_{\pm\alpha} \to E_{\pm}^{(m)} = \lambda^m E_{\pm}$$

- Grading operator $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$

- Decomposition of Affine Lie Algebra into graded subspaces $\hat{\mathcal{G}} = \oplus_i \mathcal{G}_i$ such that $[Q, \mathcal{G}_i] = i\mathcal{G}_i$,

$$\begin{aligned} \mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\}, \\ \mathcal{G}_{2m+1} &= \{\lambda^m (E_\alpha + \lambda E_{-\alpha}), \ \lambda^m (E_\alpha - \lambda E_{-\alpha})\} \end{aligned}$$

  $m = 0, \pm 1, \pm 2, \ldots$ where $[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}$.

- Choose element *semi-simple* $E = E^{(1)} = E_\alpha + \lambda E_{-\alpha}$, such that $\mathcal{K} = \text{Kernel} = \{x, /[x, E] = 0\}$, and $\mathcal{G} = \mathcal{K} + \mathcal{M}$, *where* $\mathcal{K} = \mathcal{K}_{2n+1} = \{\lambda^n (E_\alpha + \lambda E_{-\alpha})\}$ has grade $2n + 1$ and $\mathcal{M}$ is the complement. We assume that E is *semi-simple* in the sense that this second decomposition is such that

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}.$$

- Define Lax operator $\partial_x + E^{(1)} + A_0$, where $A_0 = v(x)h \in \mathcal{M} \in \mathcal{G}_0$ (Image)
- Zero Curvature Equation for *Positive Hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + \cdots + D^{(0)}] = 0,$$

  where $D^{(j)} \in \mathcal{G}_j$ which can be decomposed and solved grade by grade. In particular, highest grade eqn. i.e., $[E, D^{(N)}] = 0$ implies $D^{(N)} \in \mathcal{K}_{2n+1}$ and therefore $N = 2n + 1$.

- Examples

$$N = 3 \qquad 4v_{t_3} = v_{3x} - 6v^2 v_x, \qquad mKdV$$

$$N = 5 \qquad 16v_{t_5} = v_{5x} - 10v^2 v_{3x} - 40vv_x v_{2x} - 10v_x^3 + 30v^4 v_x,$$

$$\begin{aligned} N = 7 \qquad 64v_{t_7} &= v_{7x} - 182v_x v_{2x}^2 - 126v_x^2 v_{3x} - 140vv_{2x}v_{3x} \\ &- 84vv_x v_{4x} - 14v^2 v_{5x} + 420v^2 v_{3x} + 560v^3 v_x v_{2x} \\ &+ 70v^4 v_{3x} - 140v^6 v_x \end{aligned}$$

$\cdots etc$

Negative mKdV Hierarchy

- Zero Curvature representation for the *negative hierarchy*

$$[\partial_x + E^{(1)} + A_0, \partial_{t_{-n}} + D^{(-n)} + D^{(-n+1)} + \cdots + D^{(-1)}] = 0.$$

- Lowest grade projection,

$$\partial_x D^{(-n)} + [A_0, D^{(-n)}] = 0$$

yields a nonlocal equation for $D^{(-n)}$. No condition upon n.

- The second lowest projection of grade $-n+1$ leads to

$$\partial_x D^{(-n+1)} + [A_0, D^{(-n+1)}] + [E^{(1)}, D^{(-n)}] = 0$$

determines $D^{(-n+1)}$.

- The same mechanism works recursively until we reach the zero grade equation

$$\partial_{t_{-n}} A_0 + [E^{(1)}, D^{(-1)}] = 0$$

which gives the *time evolution* for the field in A_0 according to time t_{-n}.

- Simplest Example $t_{-n} = t_{-1}$.

$$\begin{aligned} \partial_x D^{(-1)} + [A_0, D^{(-1)}] &= 0, \\ \partial_{t_{-1}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned}$$

Solution is

$$D^{(-1)} = B^{-1}E^{(-1)}B, \qquad A_0 = B^{-1}\partial_x B, \qquad B = \exp(\mathcal{G}_0)$$

The time evolution is then given by the Leznov-Saveliev equation,

$$\partial_{t_{-1}}\left(B^{-1}\partial_x B\right) = [E^{(1)}, B^{-1}E^{(-1)}B]$$

which for $\hat{sl}(2)$ with principal gradation $Q = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$, yields the sinh-Gordon equation (*relativistic*)

$$\partial_{t_{-1}}\partial_x\phi = e^{2\phi} - e^{-2\phi}, \qquad B = e^{\phi h}.$$

where $t_{-1} = z, x = \bar{z}, A_0 = vh = \partial_x\phi h$.

- No restriction for even Negative Hierarchy
- Next simplest example $t = t_{-2}$

Z.Qiao and W. Strampp *Physica* **A 313**, (2002), 365 ;

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$$\begin{aligned} \partial_x D^{(-2)} + [A_0, D^{(-2)}] &= 0, \\ \partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0, \\ \partial_{t_{-2}} A_0 - [E^{(1)}, D^{(-1)}] &= 0. \end{aligned}$$

Propose solution of the form

$$\begin{aligned} D^{(-2)} &= c_{-2}\lambda^{-1}h, \\ D^{(-1)} &= a_{-1}\left(\lambda^{-1}E_\alpha + E_{-\alpha}\right) + b_{-1}\left(\lambda^{-1}E_\alpha - E_{-\alpha}\right). \end{aligned}$$

Get $c_{-2} = const$ and

$$a_{-1} + b_{-1} = 2c_{-2}\exp(-2d^{-1}v)d^{-1}\left(\exp(2d^{-1}v)\right),$$
$$a_{-1} - b_{-1} = -2c_{-2}\exp(2d^{-1}v)d^{-1}\left(\exp(-2d^{-1}v)\right),$$

Equation of motion is

$$\partial_{t_{-2}}v + 2c_{-2}e^{-2d^{-1}v}d^{-1}\left(e^{2d^{-1}v}\right) + 2c_{-2}e^{2d^{-1}v}d^{-1}\left(e^{-2d^{-1}v}\right) = 0.$$

where $d^{-1}f = \int^x f(x')dx'$.

- Vacuum for $t = t_{-2}$ equation

1) $v = 0$

$$0 + 2c_{-2}e^{-2\alpha}\int e^{2\alpha} + 2c_{-2}e^{2\alpha}\int e^{-2\alpha} \neq 0,$$

$c_{-2} \neq 0, \quad \alpha = const = d^{-1}0.$

2) $v = v_0$

$$0 + 2c_{-2}e^{-2v_0x}\int e^{2v_0x} + 2c_{-2}e^{2v_0x}\int e^{-2v_0x} = 0$$

Notice that, for $c_{-2} \neq 0$, $v = 0$ is not solution of the evolution equation and therefore $A_0 = 0$ does not satisfy the zero curvature representation for $t = t_{-2}$.

Dressing Solutions for Negative Hierarchy

- Propose the simplest non trivial vacuum configuration

$$\begin{aligned} A_{x,vac} &= \left(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}\right) + v_0 h^{(0)} - \frac{1}{v_0} t_{-2m} \delta_{m-1,0} \hat{c}, \\ A_{t_{-2m},vac} &= \frac{1}{v_0}\left(E_{\alpha}^{(-m)} + E_{-\alpha}^{(1-m)}\right) + h^{(-m)} \end{aligned}$$

with $v_0 = const. \neq 0$.

It is straightforward to verify the zero curvature equation

$$[\partial_x + A_{x,vac}, \partial_{t_{-2m}} + A_{t_{-2m},vac}] = 0.$$

This *nontrivial vacuum* leads to the following deformation, i.e., $A_{x,vac} = T_0^{-1}\partial_x T_0$ and $A_{t_k,vac} = T_0^{-1}\partial_{t_k} T_0$,

$$T_0 = \exp\left\{x\left(E_{\alpha}^{(0)} + E_{-\alpha}^{(1)} + v_0 h^{(0)}\right)\right\} \cdot \exp\left\{\frac{t_{-2m}}{v_0}\left(E_{\alpha}^{(-m)} + E_{-\alpha}^{(1-m)} + v_0 h^{(-1)}\right)\right\}.$$

Observe that consistency of the zero curvature representation with nontrivial vacuum configuration requires terms with mixed gradation in constructing T_0

The solution is then given by

$$
\begin{aligned}
e^{-\nu} &= <\lambda_0|T_0^{-1} g T_0|\lambda_0> \equiv \tau_0, \\
e^{-\phi+xv_0-\nu} &= <\lambda_1|T_0^{-1} g T_0|\lambda_1> \equiv \tau_1
\end{aligned}
$$

and hence,

$$
v = v_0 - \partial_x \ln\left(\frac{\tau_0}{\tau_1}\right), \qquad v = \partial_x \phi.
$$

In order to construct explicit soliton solutions consider the deformed vertex operators

$$
\begin{aligned}
F(\gamma, v_0) = \sum_{n=-\infty}^{\infty} \left(\gamma^2 - v_0^2\right)^{-n} [h^{(n)} &+ \frac{v_0-\gamma}{2\gamma}\delta_{n,0}\hat{c} + E_\alpha^{(n)}(\gamma+v_0)^{-1} \\
&- E_{-\alpha}^{(n+1)}(\gamma-v_0)^{-1}].
\end{aligned}
$$

A direct calculation shows that

$$
[b_{-2m}, F(\gamma, v_0)] = -2\gamma\left(\gamma^2 - v_0^2\right)^{-m} F(\gamma, v_0).
$$

If we now take,

$$
g = \exp\{F(\gamma, v_0)\}
$$

We find the soliton solution of the form

$$
\tau_0 = 1 + C_0 \rho(\gamma, v_0), \qquad \tau_1 = 1 + C_1 \rho(\gamma, v_0)
$$

where,

$$
\rho(\gamma, v_0) = \exp\left\{2\gamma x + \frac{2\gamma t_{-2m}}{v_0\left(\gamma^2 - v_0^2\right)^m}\right\}.
$$

Models allowing constant vacuum solution

- Negative even

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{-2m}} + D_{vac}^{(-2m)} + D_{vac}^{(-2m+1)} + \cdots + D_{vac}^{(-2)} + D_{vac}^{(-1)}] =$$

solution is

$$D_{vac}^{(-2m)} = \lambda^{-m} h^{(0)} \qquad D_{vac}^{(-2m+1)} = \frac{\lambda^{-m} E}{v_0},$$
$$\vdots \qquad \vdots$$
$$D_{vac}^{(-2)} = \lambda^{-1} h^{(0)} \qquad D_{vac}^{(-1)} = \frac{\lambda^{-1} E}{v_0},$$

Notice the r.h.s. of zero curvature equation requires $2m$ **terms which combine together to form**

$$D_{vac}^{(-2i)} + D_{vac}^{(-2i+1)} = \lambda^{-i}(h^{(0)} + \frac{E}{v_0})$$

such that

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{-2m}} + \lambda^{-m}(h^{(0)} + \frac{E}{v_0}) + \cdots + \lambda^{-1}(h^{(0)} + \frac{E}{v_0})] = 0$$

- Positive odd (mKdV, etc)

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{2m+1}} + D^{(2m+1)}_{vac} + D^{(2m)}_{vac} + \cdots + D^{(1)}_{vac} + D^{(0)}_{vac}] = 0$$

solution is

$$D^{(2m+1)}_{vac} = \frac{\lambda^m E}{v_0}\lambda^m h^{(0)} \qquad D^{(2m)}_{vac} = \lambda^m h^{(0)},$$

$$\vdots \qquad \vdots$$

$$D^{(1)}_{vac} = \frac{\lambda E}{v_0} \qquad D^{(0)}_{vac} = h^{(0)},$$

The r.h.s. of zero curvature equation requires $2m$ **terms which combine together to form**

$$D^{(2i+1)}_{vac} + D^{(2i)}_{vac} = \lambda^i (h^{(0)} + \frac{E}{v_0})$$

such that

$$[\partial_x + E + v_0 h^{(0)}, \partial_{t_{2m+1}} + \lambda^m (h^{(0)} + \frac{E}{v_0}) + \cdots + \lambda^0 (h^{(0)} + \frac{E}{v_0})] = 0$$

- Negative Odd

$$
\begin{aligned}
[\partial_x + E + v_0 h^{(0)} \quad &, \quad \partial_{t_{-2m-1}} + D_{vac}^{(-2m-1)} \\
&+ \quad D_{vac}^{(-2m)} + D_{vac}^{(-2m+1)} + \cdots + D_{vac}^{(-2)} + D_{vac}^{(-1)}] = 0
\end{aligned}
$$

Notice the r.h.s. of zero curvature equation has $2m+1$**-terms** that they cannot combine to form combinations proportional to $E + v_0 h^{(0)}$ and **henceforth the negative odd hierarchy does not allow constant vacuum solutions.** E.g. sine-Gordon

Conclusions and Further Developments

- Introduced *Negative Even Sub-Hierarchy* of integrable equations.
- Proposed systematic construction of solutions in terms of deformed vertex operators.
- Generalize to other integrable hierarchies allowing constant vacuum solutions.
- Adapt Dressing method to construct *periodic solutions* (Jacobi Theta functions).

  where

$$\tau_a = \sum_{k=-\infty}^{+\infty} e^{2\pi i \eta k^2} \rho^k, \qquad \eta = \text{deform.} \quad \text{parameter}$$

  c.f. soliton where

$$\tau_0 = 1 + \rho, \qquad \tau_1 = 1 - \rho$$