Matrix elements of spin operator in superintegrable quantum chiral Potts chain

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Superintegrable $\mathbb{Z}_N$-symmetric chiral quantum Potts chain

Superintegrable $\mathbb{Z}_N$-symmetric chiral quantum Potts chain of length L is a generalization (from the case $N = 2$) of quantum Ising chain in a transverse field:

$$\mathcal{H} = -\sum_{k=1}^{L} (\hat{\sigma}_k^z \hat{\sigma}_{k+1}^z + \mathrm{k}' \hat{\sigma}_k^x) .$$

The Hamiltonian (G. von Gehlen, V. Rittenberg, 1985) of the chain is

$$\mathcal{H} \;=\; \mathcal{H}_0 \;+\; \mathrm{k}' \, \mathcal{H}_1 \, ,$$

where k' - temperature-like parameter, $\omega = e^{2\pi \mathrm{i}/N}$ and

$$\mathcal{H}_0 \;=\; -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{Z_j^n \, Z_{j+1}^{-n}}{1 - \omega^{-n}} \, , \qquad \mathcal{H}_1 \;=\; -2 \sum_{j=1}^{L} \sum_{n=1}^{N-1} \frac{X_j^n}{1 - \omega^{-n}} \, .$$

Z and X are $\mathbb{Z}_N$-generalization of Pauli matrices $\hat{\sigma}^z$ and $\hat{\sigma}^x$. We consider the periodic boundary condition $Z_{L+1} = Z_1$.

$\mathbb{Z}_N$-charge and corresponding sectors

It is convenient to label the states of the system by spin variables $\sigma_j \in \mathbb{Z}_N$, $j = 1 \dots L$, writing the vectors of states as $|\sigma_1, \sigma_2, \dots, \sigma_L\rangle$. The matrix elements of Z and X are

$$(Z_j)_{\sigma_j,\sigma_j'} = \delta_{\sigma_j,\sigma_j'} \omega^{\sigma_j} \,, \qquad (X_j)_{\sigma_j,\sigma_j'} = \delta_{\sigma_j,\sigma_j'+1} \,.$$

They satisfy the Weyl relations $Z\,X = \omega\,X\,Z$ and $Z^N = X^N = \mathbf{1}$. Both $\mathcal{H}_0$ and $\mathcal{H}_1$ commute with the spin rotation operator $R = X_1 X_2 \cdots X_L$. Since $R^N = 1$, the space of states of the model is separated into sectors with fixed $\mathbb{Z}_N$-charge Q, where $Q = 0, 1, \dots, N-1$, corresponds to eigenvalue ω^Q of operator R.

The quantities to be calculated

- The matrix elements of Z_j^r, $r = 1, 2, \ldots, N-1$, between the eigenvectors of the Hamiltonian $\mathcal{H}$. Such matrix elements allow, in principle, to calculate arbitrary correlation functions (to write down the form-factor expansion of such corr. functions)
- In the thermodynamic limit $L \to \infty$ at $0 \leq \mathrm{k}' < 1$ (ferromagnetic phase) the operators Z_j^r acquire non-zero vacuum expectation value (Albertini,McCoy,Perk,Tang'89):

$$\langle Z^r \rangle = (1 - \mathrm{k}'^2)^{r(N-r)/(2N^2)}.$$

Baxter (2005) used functional relations for $\tau^{(2)}$-model and natural analytical assumptions to prove it.
Our initial motivation was to prove this formula in an algebraic way.

Onsager algebra and the spectrum of the Hamiltonian

G. von Gehlen, V. Rittenberg (1985) showed, that

$$A_0 = -2\mathcal{H}_1/N\,, \qquad A_1 = 2\mathcal{H}_0/N$$

satisfy Dolan–Grady relations which are equivalent to relations of Onsager algebra which is an infinite-dimensional Lie algebra with generators $A_m,\ m \in \mathbb{Z},\ G_n,\ n \in \mathbb{N}$:

$$[A_m, A_n] = 4G_{m-n}\,, \quad [G_m, A_n] = 2A_{m+n} - 2A_{n-m}\,, \quad [G_m, G_n] = 0\,.$$

The space of states decomposes into irreducible representations of Onsager algebra and the spectrum of restriction of Hamiltonian to such subspace is as of m_E free fermions:

$$E = A + \mathrm{k}'B - N \sum_{j=1}^{m_E} \pm\varepsilon(\theta_j)\,,$$

where $\varepsilon(\theta) = \sqrt{1 - 2\mathrm{k}'\cos\theta + \mathrm{k}'^2}$. The constants A, B, m_E and θ_j are not fixed by the representation theory but have to be found from certain functional relations for the transfer-matrix of chiral Potts model.

Extension of Onsager algebra by spin operator

In arXiv:0906.3551 R. Baxter proposed to extend the Onsager algebra by spin operator $S = Z_1^r$. He introduced two operations on operators X on the space of states

$$f_0(X) = \frac{[\mathcal{H}_0, X]}{2N} \qquad \text{and} \qquad f_1(X) = \frac{[\mathcal{H}_1, X] + 2rX}{2N}$$

and has shown that

$$f_0(S) = 0 \qquad \text{and} \qquad f_1(f_1(S)) = f_1(S)\,.$$

Repeated commutation of S with $\mathcal{H}_0$ and $\mathcal{H}_1$ gives many new elements. However Baxter conjectured that these two relations uniquely (up to a multiplier since the relations are homogeneous) define the matrix elements of S. We proved this conjecture and found explicit formulas for the matrix elements.

Matrix elements of spin operator in the intermediate basis

Let us consider the matrix elements of spin operator $S = Z_1^r$ between vectors from two irreducible representations of Onsager algebra with $\mathbb{Z}_N$-charges P and Q. The selection rule on $\mathbb{Z}_N$-charge requires $r = Q - P$ for $P < Q$ and $r = Q - P + N$ for $P > Q$.

In these subspaces there exist bases such that the restriction of the Hamiltonian onto these subspaces is

$$H^P = A^P + \mathrm{k}' B^P - N \sum_{j=1}^{m} \begin{pmatrix} 1 - \mathrm{k}' \cos\theta_j & -\mathrm{k}' \sin\theta_j \\ -\mathrm{k}' \sin\theta_j & -1 + \mathrm{k}' \cos\theta_j \end{pmatrix}_j ,$$

$$H^Q = A^Q + \mathrm{k}' B^Q - N \sum_{j=1}^{n} \begin{pmatrix} 1 - \mathrm{k}' \cos\theta'_j & -\mathrm{k}' \sin\theta'_j \\ -\mathrm{k}' \sin\theta'_j & -1 + \mathrm{k}' \cos\theta'_j \end{pmatrix}_j ,$$

Selection rules for spin matrix elements

From the analysis of the relations of extended Onsager algebra we found selection rules for spin matrix elements which are more fine than the selection rule by $\mathbb{Z}_N$-charge. Let $r = Q - P$ for $P < Q$, and $r = Q - P + N$ for $P > Q$.

If none of two cases

- $\mu = m + n$ is even and $A^Q - A^P = 0$, $B^P - B^Q = N \pm N - 2r$.
- $\mu = m + n$ is odd and $A^Q - A^P = \pm N$, $B^P - B^Q = N - 2r$.

is satisfied then the spin matrix elements are vanishing.

In general it is unknown if these equalities are sufficient to give all non-zero matrix elements. Numerical test shows that these rules are complete for $N = 3$, $L = 3$.

Matrix elements of spin operator in the intermediate basis

For the subspaces, which contain states with the lowest energies at fixed $\mathbb{Z}_N$-charge (ground states sectors), the matrix elements in the intermediate basis are

$$S_{V,V'} = \delta_{|W|,|W'|} \prod_{i\in W} \tan\theta_i/2 \prod_{i\in W'} \cot\theta_i'/2$$

$$\times \frac{\prod_{i\in W, j\in V'}(\cos\theta_i - \cos\theta_j') \prod_{i\in V, j\in W'}(\cos\theta_i - \cos\theta_j')}{\prod_{i\in V, j\in W}(\cos\theta_i - \cos\theta_j) \prod_{i\in W', j\in V'}(\cos\theta_i' - \cos\theta_j')},$$

where V (resp. V') – a subset of $\{1,\ldots,m\}$ (resp. of $\{1,\ldots,n\}$) of the labels of tensor components, in which the basis element $(1,0)^T$ is chosen. The subset W (resp. W') is the supplementary to V (resp. V').

This formula was conjectured by Baxter arXiv:0906.3551. We generalized it for arbitrary pair of Onsager irreps and proved it. In general case the formula is given up to a scalar factor $\mathcal{N}_{PQ}$ depending only on this pair of irreps.

The eigenvectors in an Onsager sector

The eigenvector of H^P in an Onsager sector with the lowest energy

$$E = A^P + \mathrm{k}' B^P - N \sum_{j=1}^{m} \varepsilon(\theta_j)\,,$$

where

$\varepsilon(\theta) = \sqrt{1 - 2\mathrm{k}' \cos\theta + \mathrm{k}'^2}$, $\quad \varepsilon(0) = 1 - \mathrm{k}'$, $\quad \varepsilon(\pi) = 1 + \mathrm{k}'$, has the form

$$\begin{pmatrix} a_0(\theta_1) \\ a_1(\theta_1) \end{pmatrix} \otimes \begin{pmatrix} a_0(\theta_2) \\ a_1(\theta_2) \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} a_0(\theta_m) \\ a_1(\theta_m) \end{pmatrix}\,,$$

$$a_0(\theta) = \frac{\mathrm{k}' \sin\theta}{\sqrt{\mathrm{k}'^2 \sin^2\theta + (1 - \mathrm{k}' \cos\theta - \varepsilon(\theta))^2}}\,,$$

$$a_1(\theta) = \frac{1 - \mathrm{k}' \cos\theta - \varepsilon(\theta)}{\sqrt{\mathrm{k}'^2 \sin^2\theta + (1 - \mathrm{k}' \cos\theta - \varepsilon(\theta))^2}}\,,$$

$$a_0^2(\theta) + a_1^2(\theta) = 1\,.$$

Matrix elements of spin operator in the basis of eigenvectors

The matrix element of spin operator between the eigenvectors with the lowest energy and fixed $\mathbb{Z}_N$-charge is

$$_P\langle g.s.|Z_1^r|g.s.\rangle_Q = \sum_{V,V'} \prod_{i\in V} a_0(\theta_i) \prod_{i\in W} a_1(\theta_i) \prod_{i\in V'} a_0(\theta_i') \prod_{i\in W'} a_1(\theta_i') S_{V,V'} \, .$$

After summation we get the factorized formulas. For example if $m+n$ is an even we get:

$$_P\langle g.s.|Z_1^r|g.s.\rangle_Q \; =$$

$$= \mathcal{N}_{PQ} \prod_{i=1}^{m} \sqrt{\frac{(\varepsilon(\theta_i)+\varepsilon(0))}{2\varepsilon(\theta_i)(\varepsilon(\pi)+\varepsilon(\theta_i))}} \prod_{i=1}^{n} \sqrt{\frac{(\varepsilon(\theta_i')-\varepsilon(0))}{2\varepsilon(\theta_i')(\varepsilon(\pi)-\varepsilon(\theta_i'))}} \times$$

$$\times \frac{(-1)^{(m+n)(m+n-2)/8+n(n+1)/2}(2k')^{(m+n)^2/4}}{\prod_{i<j}^{m}(\varepsilon(\theta_i)+\varepsilon(\theta_j)) \prod_{i<j}^{n}(\varepsilon(\theta_i')+\varepsilon(\theta_j')) \prod_{i=1}^{m}\prod_{j=1}^{n}(\varepsilon(\theta_i)-\varepsilon(\theta_j'))} \, .$$

Matrix elements of spin operator for ground states sectors

For the ground states sectors the constants $\mathcal{N}_{PQ}$ are known. Therefore we get exact formulas for the corresponding matrix elements

$$
{}_P\langle g.s.|Z_1^r|g.s.\rangle_Q = \prod_{i=1}^{m}\sqrt{\frac{(\varepsilon(\theta_i)+\varepsilon(0))}{2\varepsilon(\theta_i)(\varepsilon(\pi)+\varepsilon(\theta_i))}}\prod_{i=1}^{n}\sqrt{\frac{(\varepsilon(\pi)+\varepsilon(\theta_i'))}{2\varepsilon(\theta_i')(\varepsilon(\theta_i')+\varepsilon(0))}}
$$

$$
\times\frac{\prod_{i=1}^{m}\prod_{j=1}^{n}(\varepsilon(\theta_i)+\varepsilon(\theta_j'))}{\prod_{i<j}^{m}(\varepsilon(\theta_i)+\varepsilon(\theta_j))(\varepsilon(\theta_i')+\varepsilon(\theta_j'))}\,.
$$

For the ground state sectors such formulas were obtained independently by Baxter, arXiv: 0912.4549.

The thermodynamic limit. Spontaneous magnetization.

For the superintegrable $\mathbb{Z}_N$-symmetric chiral quantum Potts chain the parameters θ_i are defined by the relation $\cos\theta_j = (1+e^{Nu_j})/(1-e^{Nu_j})$ in terms of zeros e^{Nu_j}, $j = 1, 2, \ldots, m$, of the polynomial

$$\Psi_P(u) = e^{-Pu} \sum_{k=0}^{N-1} \omega^{Pk} \left(\frac{1-e^{Nu}}{1-\omega^{-k} e^{u}} \right)^L$$

in e^{Nu} of degree m.

In the thermodynamic limit $L \to \infty$ $(m \to \infty)$ we use

$$\lim_{L\to\infty} \left(\sum_{i=1}^{m} \log(\lambda + \varepsilon(\theta_i)) - \sum_{i=1}^{m} \log(\lambda + \varepsilon(\theta_i')) \right) = \frac{r}{N} \log \frac{\lambda + 1 - \mathrm{k}'}{\lambda + 1 + \mathrm{k}'}$$

and get the formula for spontaneous magnetization

$$\lim_{L\to\infty} {}_P\langle g.s.|Z_1^r|g.s.\rangle_{P+r} = (1-\mathrm{k}'^2)^{r(N-r)/(2N^2)} .$$

Quantum Ising chain in a transverse field

The Hamiltonian for quantum Ising chain in a transverse field corresponds to $N = 2$ case:

$$\hat{\mathcal{H}} = -\sum_{l=1}^{L} (\hat{\sigma}_l^z \hat{\sigma}_{l+1}^z + \mathrm{k}' \hat{\sigma}_l^x) \,.$$

Its spectrum is $\mathcal{E} = -\sum_q \pm\varepsilon(q)$.
The energies of fermionic excitations are:

$$\varepsilon(q) = (1 - 2\mathrm{k}' \cos q + \mathrm{k}'^2)^{1/2}, \qquad q \neq 0, \pi \,,$$

$$\varepsilon(0) = 1 - \mathrm{k}', \quad \varepsilon(\pi) = 1 + \mathrm{k}' \,.$$

Subspaces with $\mathbb{Z}_2$-charges $Q = 0$ and $Q = 1$ are called NS-sector and R-sector.

Matrix element of spin operator for quantum Ising chain of finite length L are (J.Phys.A,2008):

$$|_{\mathrm{NS}}\langle q_1, q_2, \ldots, q_K|\hat{\sigma}_k^z|p_1, p_2, \ldots, p_M\rangle_{\mathrm{R}}|^2 = \mathrm{k}'^{\frac{(K-M)^2}{2}} \xi \xi_T \prod_{k=1}^{K} \frac{e^{\eta(q_k)}}{L\varepsilon(q_k)} \prod_{m=1}^{M} \frac{e^{-\eta(p_m)}}{L\varepsilon(p_m)} \times$$

$$\times \prod_{k<k'}^{K} \left(\frac{2\sin\frac{\mathrm{q}_k - \mathrm{q}_{k'}}{2}}{\varepsilon(\mathrm{q}_k) + \varepsilon(\mathrm{q}_{k'})} \right)^2 \prod_{m<m'}^{M} \left(\frac{2\sin\frac{\mathrm{p}_m - \mathrm{p}_{m'}}{2}}{\varepsilon(\mathrm{p}_m) + \varepsilon(\mathrm{p}_{m'})} \right)^2 \prod_{k=1}^{K} \prod_{m=1}^{M} \left(\frac{\varepsilon(\mathrm{p}_m) + \varepsilon(\mathrm{q}_k)}{2\sin\frac{\mathrm{p}_m - \mathrm{q}_k}{2}} \right)^2 ,$$

where

$$\xi = \left(1 - \mathrm{k}'^2\right)^{\frac{1}{4}} , \quad \xi_T = \frac{\prod_{\mathrm{q}}^{\mathrm{NS}} \prod_{\mathrm{p}}^{\mathrm{R}} (\varepsilon(\mathrm{q}) + \varepsilon(\mathrm{p}))^{\frac{1}{2}}}{\prod_{\mathrm{q},\mathrm{q}'}^{\mathrm{NS}} (\varepsilon(\mathrm{q}) + \varepsilon(\mathrm{q}'))^{\frac{1}{4}} \prod_{\mathrm{p},\mathrm{p}'}^{\mathrm{R}} (\varepsilon(\mathrm{p}) + \varepsilon(\mathrm{p}'))^{\frac{1}{4}}}$$

and

$$e^{\eta(\mathrm{q})} = \frac{\prod_{\mathrm{q}'}^{\mathrm{NS}} (\varepsilon(\mathrm{q}) + \varepsilon(\mathrm{q}'))}{\prod_{\mathrm{p}}^{\mathrm{R}} (\varepsilon(\mathrm{q}) + \varepsilon(\mathrm{p}))} .$$

The formula is a limiting case of the formula for finite 2D Ising model (Bugrij,Lisovyi'03,'04). Such formulas allow to write down form-factor expansion of any correlation function of Ising model.

Results and perspectives

- From the analysis of the relations of extended Onsager algebra we found selection rules for spin matrix elements which are more fine than the selection rule by $\mathbb{Z}_N$-charge.
- The matrix elements of spin operators are found up to certain unknown constants $\mathcal{N}_{PQ}$ which are common for all the vectors from a pair of irreducible representations of Onsager algebra. These constants are independent of k'.
- The constants $\mathcal{N}_{PQ}$ are known for the ground states sectors (irreps). It gives exact formulas for the matrix elements of spin operators. In the thermodynamic limit we get the formula for spontaneous magnetization. Such matrix elements can be used to find the spontaneous magnetization on a cylinder.
- An open problem: to find unknown constants $\mathcal{N}_{PQ}$ for the other pairs of irreps of Onsager algebra.

Thank you for your attention!