The form factor program - a review and new results - $\mathbf{SU(N)}$ and $\mathbf{O(N)}$ models

H. Babujian, A. Foerster, and M. Karowski

Freie Universität Berlin

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The "Bootstrap Program"

Construct a quantum field theory explicitly in 3 steps

1. S-matrix

   using
   1. general Properties: unitarity, crossing etc
   2. "Yang-Baxter Equation"
   3. "bound state bootstrap"
   4. 'maximal analyticity'

2. "Form factors"

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   2. LSZ-assumptions
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3. "Wightman functions"

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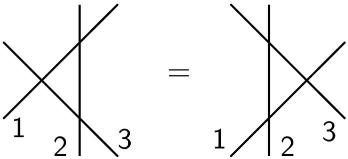
The bootstrap program classifies *integrable* quantum field theories

Integrability

"Yang-Baxter equation" $S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$



"bound state bootstrap equation" $S_{(12)3}\,\Gamma_{12}^{(12)} = \Gamma_{12}^{(12)}\,S_{13}S_{23}$

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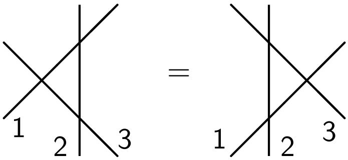

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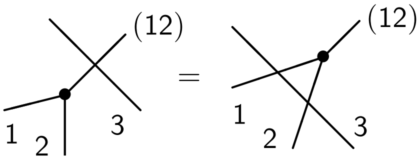
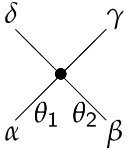


$SU(N)$ S-matrix

Particles $\alpha, \beta, \gamma, \delta = 1, \ldots, N \leftrightarrow$ vector representation of $SU(N)$

$$S_{\alpha\beta}^{\delta\gamma}(\theta) =$$



$$= \delta_{\alpha\gamma}\delta_{\beta\delta}\, b(\theta) + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c(\theta).$$

Yang-Baxter + crossing + unitarity
[Berg Karowski Kurak Weisz 1978
Köberle Kurak Swieca 1979; Abdalla Berg Weisz 1979]

$$a(\theta) = b(\theta) + c(\theta) = -\frac{\Gamma\left(1 - \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} + \frac{\theta}{2\pi i}\right)}{\Gamma\left(1 + \frac{\theta}{2\pi i}\right)\Gamma\left(1 - \frac{1}{N} - \frac{\theta}{2\pi i}\right)}$$

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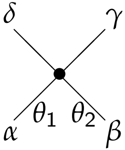

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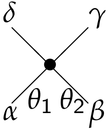

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For $O(3)$

$$a(\theta) = \frac{\theta - i\pi}{\theta + i\pi}$$

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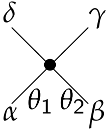
$S_{\alpha\beta}^{\delta\gamma}(\theta) =$ 

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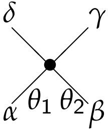

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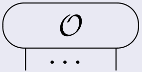
$$\boxed{a(\theta) = \frac{\theta - i\pi}{\theta + i\pi}}$$

Form factors

Definition

Let $\mathcal{O}(x)$ be a local operator

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, p_1, \ldots, p_n \,\rangle^{in}_{\alpha_1 \ldots \alpha_n} = F^{\mathcal{O}}_{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_n)\, e^{-ix\sum p_i}$$

$=$ 

$F^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta}) =$ form factor (co-vector valued function)

LSZ-assumptions
+ 'maximal analyticity' $\Bigg\} \Longrightarrow$ Properties of form factors

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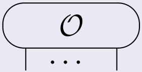
$=$ 

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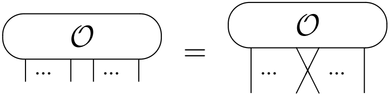
Form factors equations

(i) Watson's equation

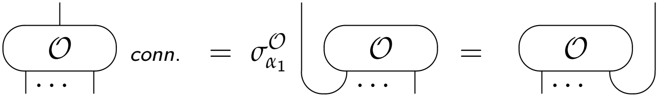
$$F^{\mathcal{O}}_{\dots ij \dots}(\dots,\theta_i,\theta_j,\dots) = F^{\mathcal{O}}_{\dots ji \dots}(\dots,\theta_j,\theta_i,\dots)\, S_{ij}(\theta_i-\theta_j)$$



(ii) Crossing

$$^{\bar\alpha_1}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, \dots, p_n \,\rangle^{in,conn.}_{\dots\alpha_n} =$$

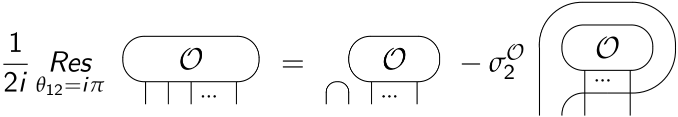
$$\mathbf{C}^{\bar\alpha_1\alpha_1}\sigma^{\mathcal{O}}_{\alpha_1} F^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1+i\pi,\dots,\theta_n) = F^{\mathcal{O}}_{\dots\alpha_n\alpha_1}(\dots,\theta_n,\theta_1-i\pi)\mathbf{C}^{\alpha_1\bar\alpha_1}$$



Form factors equations

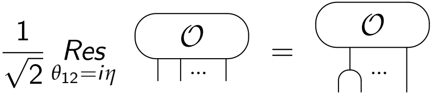
(iii) Annihilation recursion relation

$$\frac{1}{2i} \mathop{Res}_{\theta_{12}=i\pi} F^{\mathcal{O}}_{1\ldots n}(\theta_1,..) = \mathbf{C}_{12}\; F^{\mathcal{O}}_{3\ldots n}(\theta_3,..)\left(\mathbf{1} - \sigma_2^{\mathcal{O}} S_{2n}\ldots S_{23}\right)$$



(iv) Bound state form factors

$$\frac{1}{\sqrt{2}} \mathop{Res}_{\theta_{12}=i\eta} F^{\mathcal{O}}_{123\ldots n}(\underline{\theta}) = F^{\mathcal{O}}_{(12)3\ldots n}(\theta_{(12)},\underline{\theta}')\,\Gamma^{(12)}_{12}$$



(v) Lorentz invariance

$$F^{\mathcal{O}}_{1\ldots n}(\theta_1+u,\ldots,\theta_n+u) = e^{su}\,F^{\mathcal{O}}_{1\ldots n}(\theta_1,\ldots,\theta_n)$$

2-particle form factor

$$\langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \rangle^{in/out} = F\left((p_1 + p_2)^2 \pm i\varepsilon\right) = F\left(\pm\theta_{12}\right)$$

where $p_1 p_2 = m^2 \cosh\theta_{12}$.

"Watson's equations"

$$\begin{cases} F(\theta) = F(-\theta)\, S(\theta) \\ F(i\pi - \theta) = F(i\pi + \theta) \end{cases}$$

"maximal analyticity" $\Rightarrow$ unique solution

Examples:

The highest weight $SU(N)$ minimal 2-particle form factor

$$F(\theta) = \exp \int_0^\infty dt \frac{e^{\frac{t}{N}} \sinh t \left(1 - \frac{1}{N}\right)}{t \sinh^2 t} \left(1 - \cosh t \left(1 - \theta/(i\pi)\right)\right)$$

The highest weight $O(N)$ minimal 2-particle form factor

$$F(\theta) = \exp \int_0^\infty \frac{dt}{t \sinh t} \frac{1 - e^{-2t/(N-2)}}{1 + e^{-t}} \left(1 - \cosh t \left(1 - \theta/(i\pi)\right)\right)$$

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General form factor formula

$$F^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\theta_1,\dots,\theta_n) = K^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij})$$

"Nested off-shell Bethe Ansatz"

$$K^{\mathcal{O}}_{\alpha_1\dots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, L^{\mathcal{O}}_{\beta_1\dots\beta_m}(\underline{z}) \Psi^{\beta_1\dots\beta_m}_{\alpha_1\dots\alpha_n}(\underline{\theta},\underline{z})$$

$$h(\underline{\theta},\underline{z}) = \prod_{i=1}^{n}\prod_{j=1}^{m}\phi(\theta_i - z_j) \prod_{1\le i<j\le m}\tau(z_i - z_j)\,,$$

$$\tau(z) = \frac{1}{\phi(z)\phi(-z)}$$

depend only on $F(\theta)$ i.e. on the S-matrix (see below),

$p^{\mathcal{O}}(\underline{\theta},\underline{z})$ = simple function of e^{z_i}, depends on the operator $\mathcal{O}$

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Equations for $\tilde{\phi}(z) = a(z)\phi(z)$

Example: $SU(N)$

$$(ii): \tilde{\phi}(z) = -\tilde{b}(z+2\pi i)\tilde{\phi}(z+2\pi i)\,, \qquad \tilde{b}(z) = b(z)/a(z)$$

$$(iii): \prod_{k=0}^{N-2} \tilde{\phi}\left(-z-ki\eta\right) \prod_{k=0}^{N-1} F\left(z+ki\eta\right) = 1\,, \qquad \eta = \frac{2\pi}{N}$$

Solution:

$$\tilde{\phi}(z) = \Gamma\left(-\frac{z}{2\pi i}\right)\Gamma\left(1-\frac{1}{N}+\frac{z}{2\pi i}\right)$$

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$$(iii): \tilde{\phi}(-z)\tilde{\phi}(-z-i\pi+i\eta)F(z)F(i\pi+z) = 1\,, \qquad \eta = \frac{2\pi}{N-2}$$

Solution for $O(3)$

$$\boxed{\tilde{\phi}(z) = \frac{1}{z}}$$

Equations for $\tilde{\phi}(z) = a(z)\phi(z)$

Example: $O(N)$

$$(ii) : \tilde{\phi}(z) = \tilde{b}(z + 2\pi i)\tilde{\phi}(z + 2\pi i)$$

$$(iii) : \tilde{\phi}(-z)\tilde{\phi}(-z - i\pi + i\eta)F(z)F(i\pi + z) = 1 \,, \qquad \eta = \frac{2\pi}{N-2}$$

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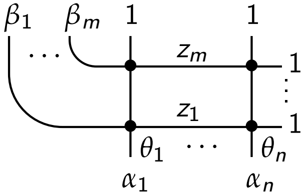
Example: $SU(N)$

$$\Psi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta},\underline{z}) = \Big(\Omega C^{\beta_m}(\underline{\theta},z_m)\dots C^{\beta_1}(\underline{\theta},z_1)\Big)_{\alpha_1\dots\alpha_n}$$

$=$



$$2 \le \beta_i \le N$$
$$1 \le \alpha_i \le N$$

Nesting means:

write for $L^{\mathcal{O}}_{\underline{\beta}}(\underline{z})$ an integral representation as for $K^{\mathcal{O}}_{\underline{\alpha}}(\underline{\theta})$

$\Rightarrow$ Bethe Ansatz of level $1, 2, \dots,$ $\begin{cases} \text{rank}(SU(N)) = N-1 \\ \text{rank}(O(N)) = [N/2] \end{cases}$

“Bethe ansatz” state

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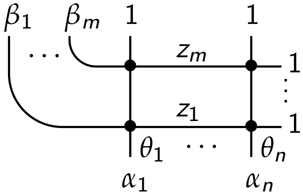
$=$ 

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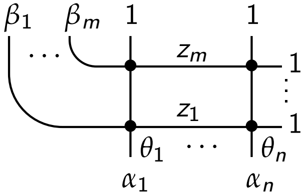

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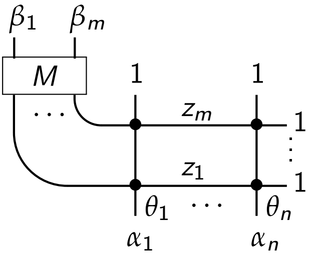

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The matrix M maps

$$M \quad : \quad \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N \quad \rightarrow \quad \mathbb{C}^{N-2} \otimes \cdots \otimes \mathbb{C}^{N-2}$$

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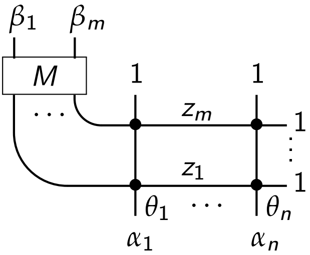
$\Psi^{\underline{\beta}}_{\underline{\alpha}}(\underline{\theta}, \underline{z}) =$ 

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In general the function $p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}})$ depends on the rapidities $\underline{\theta}$ and all integration variables $\underline{z}^{(l)}$

If the p-function $p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}})$ satisfies some simple equations, the form factor $F^{\mathcal{O}}(\theta)$ satisfies the form factor equations (i) - (iii)

e. g.

$$\begin{aligned} p^{\mathcal{O}}(\underline{\theta}, \underline{\underline{z}}) &= p^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \dots, \underline{\underline{z}}) \\ &= p^{\mathcal{O}}(\underline{\theta}, \dots, z_i^{(l)} + 2\pi i, \dots) \end{aligned}$$

(for $SU(N)$ there are some additional phase factors)

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Example: $SU(N)$

The chiral $SU(N)$-Gross-Neveu model

[H. Babujian, A. Fring, M. Karowski, A. Zapletal,1996]

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar\psi_\alpha \, i\gamma\partial \, \psi_\alpha + \tfrac{1}{2}g^2 \left(\left(\sum_{\alpha=1}^{N} \bar\psi_\alpha \psi_\alpha \right)^2 - \left(\sum_{\alpha=1}^{N} \bar\psi_\alpha \gamma^5 \psi_\alpha \right)^2 \right)$$

[H. Babujian, A. Foerster, M. Karowski, 2010]

The p-function for the field $\psi_1(x)$ is

$$p^{\psi^{(\pm)}}(\underline{\theta}, \underline{z}) = \exp \pm \frac{1}{2} \left(\sum_{i=1}^{m} z_i - \left(1 - \frac{1}{N}\right) \sum_{i=1}^{n} \theta_i \right) .$$

The 1-particle form factor is

$$\langle \, 0 \, | \, \psi^{(\pm)}(0) \, | \, \theta \, \rangle_\alpha = \delta_{\alpha 1} \, e^{\mp \frac{1}{2}\left(1 - \frac{1}{N}\right)\theta} .$$

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The 3-particle form factor
[H. Babujian, A. Foerster, M. Karowski, 2010]

$$F^{\psi^{(\pm)}}_{\alpha\beta\bar{\gamma}}(\theta_1,\theta_2,\theta_3) = K^{\psi^{(\pm)}}_{\alpha\beta\bar{\gamma}}(\theta_1,\theta_2,\theta_3)F(\theta_{12})G(\theta_{13})G(\theta_{23})$$

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can be expressed in term of Meijer's G-functions

$$G^{33}_{33}\left(e^{\pm i\pi}\,\middle|\, \begin{matrix} \frac{\theta_1}{2\pi i}+1, \frac{\theta_2}{2\pi i}+1, \frac{\theta_3}{2\pi i}+\frac{3}{2}-\frac{1}{N} \\ \frac{\theta_1}{2\pi i}-\frac{1}{N}, \frac{\theta_2}{2\pi i}-\frac{1}{N}+1, \frac{\theta_3}{2\pi i}+\frac{1}{2} \end{matrix}\right)$$

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Example: $O(N)$

The $O(N)$ σ-model

Lagrangian and constraint

$$\mathcal{L}^{NLS} = \frac{1}{2} \sum_{\alpha=1}^{N} \left(\partial_\mu \varphi_\alpha\right)^2 \qquad \text{with} \quad g \sum_{\alpha=1}^{N} \varphi_\alpha^2 = 1$$

The field $\varphi_\alpha(x)$ transforms as the vector representation of $O(N)$.

The p-function for the field $\varphi_1(x)$ is

$$p^\varphi(\underline{\theta}, \underline{z}) = 1$$

The 1-particle form factor is

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result $\quad F^{\varphi}_{\alpha\beta\gamma}(\underline{\theta}) = g^{\varphi}_{\alpha\beta\gamma}(\underline{\theta}) G(\theta_{12}) G(\theta_{13}) G(\theta_{23})$

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this result agrees with [J.Balog, M.Niedermeyer]

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