# Trigonometric SOS model with DWBC and spin chains with non-diagonal boundaries

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# The XXZ spin-1/2 Heisenberg chain

**1.** Periodic chain.

$$H_{\mathsf{bulk}} = \sum_{m=1}^{M} \Big(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\Big) - \frac{h}{2} \sum_{m=1}^{M} \sigma_m^z$$

$\Delta$ - anisotropy $h$- external magnetic field. Periodic boundary conditions: $\sigma_{M+1} = \sigma_1$.

**2.** Open chain

$$\begin{aligned} H = & \sum_{m=1}^{M-1} \Big(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\Big) \\ & - h_- \sigma_1^z + C_-(e^{i\phi_-}\sigma_1^+ + e^{-i\phi_-}\sigma_1^-) \\ & - h_+ \sigma_M^z + C_+(e^{i\phi_+}\sigma_M^+ + e^{-i\phi_+}\sigma_M^-) \end{aligned}$$

$h_\pm$, $C_\pm$, $\phi_\pm$ - boundary paprameters.

# Bethe Ansatz

**1.** Periodic case:

- **coordinate** Bethe ansatz: H. Bethe 1931, R. Orbach 1958
- **algebraic** Bethe ansatz L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan 1979.

**2.** Open chain, diagonal boundary terms:

- **coordinate** Bethe ansatz: F.C. Alcaraz, M.N. Barber, M.T. Batchelor, R.J. Baxter and G.R.W. Quispel
- **algebraic** Bethe ansatz E.K. Sklyanin 1988.

**3.** Open chain, non-diagonal boundary terms

- Different forms of Bethe ansatz: J. Cao, H.-Q. Lin, K.-J. Shi, Y. Wang, 2003, R. I. Nepomechie 2003
- Algebraic BA, Vertex-IRF correspondence: W.-L. Yang, Y.-Z. Zhang 2007

# Algebraic Bethe Ansatz, periodic case

L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan (1979):

**1.** Yang-Baxter equation:

$$R_{12}(\lambda_{12})\ R_{13}(\lambda_{13})\ R_{23}(\lambda_{23}) = R_{23}(\lambda_{23})\ R_{13}(\lambda_{13})\ R_{12}(\lambda_{12}).$$

Trigonometric solution:

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) & 0 & 0 & 0 \\ 0 & \sinh\lambda & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh\lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda+\eta) \end{pmatrix}, \qquad \Delta = \cosh\eta.$$

Properties:

- $R(0) = \sinh\eta\,\mathcal{P}$
- Unitarity $R(\lambda)R(-\lambda) = f_1(\lambda)\,I$,

- Crossing symmetry $\sigma_1^y\, R^{t_1}(\lambda-\eta)\,\sigma_1^y R(\lambda) = f_2(\lambda)\, I.$

**2.** Monodromy matrix. $T_a(\lambda) = R_{aM}(\lambda-\xi_M)\ldots R_{a1}(\lambda-\xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[a]}$

$\xi_j$ are arbitrary inhomogeneity parameters.

$\hookrightarrow$ **Yang-Baxter algebra**: ∘ generators $A,\ B,\ C,\ D$

∘ commutation relations given by the R-matrix of the model

$$R_{ab}(\lambda-\mu)\, T_a(\lambda)T_b(\mu) = T_b(\mu)T_a(\lambda)\, R_{ab}(\lambda-\mu)$$

- Transfer matrix: $t(\lambda) = \mathrm{tr}_a\, T_a(\lambda) = A(\lambda) + D(\lambda)$
- Commuting charges: $[t(\lambda), t(\mu)] = 0$
- periodic Hamiltonian:

$$H_{\text{bulk}} = c\,\frac{\partial}{\partial\lambda}\log t(\lambda)\Big|_{\lambda=0} - hS_z + \text{const.}$$

- Conserved quantities: $[H_{\text{bulk}}, t(\lambda)] = 0$

# Reflection equation

Cherednik 1984

$$R_{12}(\lambda-\mu)\,K_1(\lambda)\,R_{12}(\lambda+\mu)\,K_2(\mu)=K_2(\mu)\,R_{12}(\lambda+\mu)\,K_1(\lambda)\,R_{12}(\lambda-\mu).$$

General $2\times 2$ solution (Ghoshal Zamolodchikov 1994):

$$K^-(\lambda)=f(\lambda)\begin{pmatrix}\cosh(\theta+2\xi_-)e^{-\lambda}-e^{\lambda}\cosh\theta & e^{-\tau}\sinh 2\lambda\\ -e^{\tau}\sinh 2\lambda & \cosh(\theta+2\xi_-)e^{\lambda}-\cosh\theta e^{-\lambda}\end{pmatrix}$$

Second boundary: $K^+(\lambda)=K^-(-\lambda-\eta),\ \xi_-\to\xi_+$

We could replace $\theta_-\to\theta_+$, $\tau_-\to\tau_+$ and get commuting charges. However to construct the eigenstates we use a constrain and keep **4 independent** boundary parameters (instead of 6) $\theta,\tau,\xi_\pm$

Algebraic Bethe Ansatz, Sklyanin 1988

$$\mathcal{U}_-(\lambda)=T(\lambda)\,K_-(\lambda)\,\sigma_0^y\,T^{t_0}(-\lambda)\,\sigma_0^y=\begin{pmatrix}\mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda)\\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda)\end{pmatrix},$$

**1.** Reflection algebra

$$R_{12}(\lambda-\mu)\,(\mathcal{U}_-)_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,(\mathcal{U}_-)_2(\mu)$$
$$=(\mathcal{U}_-)_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,(\mathcal{U}_-)_1(\lambda)\,R_{12}(\lambda-\mu)$$

**2.** Transfer matrix:

$$\mathcal{T}(\lambda)=\mathrm{tr}_0\{K_+(\lambda)\,\mathcal{U}_-(\lambda)\}.$$

$$[\mathcal{T}(\lambda),\mathcal{T}(\mu)]=0$$

**3.** Hamiltonian (homogeneous limit):

$$H=C_1\;\frac{d}{d\lambda}\mathcal{T}(\lambda)\Big|_{\lambda=\eta/2}+\text{constant}.$$

No reference state! $|0\rangle$, such that $\mathcal{C}_-(\lambda)|0\rangle=0,\ \forall\lambda$

# Vertex-IRF transformation

Cao et al: 8-vertex like scheme. **Gauge transformation** to diagonalize the boundary matrices.
non diagonal $K(\lambda) \longrightarrow \mathcal{K}(\lambda)$ diagonal, six vertex $R(\lambda) \longrightarrow \mathcal{R}(\lambda,\theta)$ SOS, dynamical

$$S(\lambda;\theta) = \begin{pmatrix} \exp^{-(\lambda+\theta+\tau)} & \exp^{-(\lambda-\theta+\tau)} \\ 1 & 1 \end{pmatrix}$$

$R$-matrix

$$S(\xi,\theta)_1 S(\lambda,\theta-\eta\sigma_1^z)_0 \mathcal{R}_{01}(\lambda-\xi,\theta) = R_{01}(\lambda-\xi) S(\lambda,\theta)_0 S(\xi,\theta-\eta\sigma_0)_1$$

$K$-matrix

$$S^{-1}(\lambda,\theta) K_-(\lambda,\theta) S(-\lambda,\theta) = \mathcal{K}_-(\lambda,\theta) = \begin{pmatrix} \frac{\sinh(\theta-\lambda+\xi_-)}{\sinh(\theta+\lambda+\xi_-)} & 0 \\ 0 & \frac{\sinh(-\lambda+\xi_-)}{\sinh(\lambda+\xi_-)} \end{pmatrix}$$

# Dynamical Yang-Baxter equation

$$\mathcal{R}(\lambda,\theta)=\begin{pmatrix}\sinh(\lambda+\eta) & 0 & 0 & 0\\ 0 & \frac{\sinh\lambda\sinh(\theta-\eta)}{\sinh\theta} & \frac{\sinh\eta\sinh(\theta-\lambda)}{\sinh\theta} & 0\\ 0 & \frac{\sinh\eta\sinh(\theta+\lambda)}{\sinh\theta} & \frac{\sinh\lambda\sinh(\theta+\eta)}{\sinh\theta} & 0\\ 0 & 0 & 0 & \sinh(\lambda+\eta)\end{pmatrix}$$

satisfies the dynamical Yang Baxter equation (DYBE)

$$\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta-\eta\sigma_3^z)\mathcal{R}_{13}(\lambda_1-\lambda_3,\theta)\mathcal{R}_{23}(\lambda_2-\lambda_3,\theta-\eta\sigma_1^z)$$
$$=\mathcal{R}_{23}(\lambda_2-\lambda_3,\theta)\mathcal{R}_{13}(\lambda_1-\lambda_3,\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta)$$

Dynamical monodromy matrix

$$T(\lambda,\theta)=\mathcal{R}_{01}(\lambda-\xi_1,\theta-\eta\sum_{i=2}^{N}\sigma_i^z)\ldots\mathcal{R}_{0N}(\lambda-\xi_N,\theta)=\begin{pmatrix}A(\lambda,\theta) & B(\lambda,\theta)\\ C(\lambda,\theta) & D(\lambda,\theta)\end{pmatrix}$$

Dynamical Yang-Baxter algebra

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta-\eta\sum_{i=1}^{N}\sigma_i^z)T_1(\lambda_1,\theta)T_2(\lambda_2,\theta-\eta\sigma_1^z)\\ =T_2(\lambda_2,\theta)T_1(\lambda_1,\theta-\eta\sigma_2^z)\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta)\end{aligned}$$

Reflection equation

$$\begin{aligned}\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta)\mathcal{K}_1(\lambda_1)\mathcal{R}_{21}(\lambda_1+\lambda_2,\theta)\mathcal{K}_2(\lambda_2)\\ =\mathcal{K}_2(\lambda_2)\mathcal{R}_{12}(\lambda_1+\lambda_2,\theta)\mathcal{K}_1(\lambda_1,\theta)\mathcal{R}_{21}(\lambda_1-\lambda_2,\theta)\end{aligned}$$

# Dynamical reflection algebra

Double row monodromy matrix

$$\mathcal{U}(\lambda,\theta) \equiv \begin{pmatrix} \mathcal{A}(\lambda,\theta) & \mathcal{B}(\lambda,\theta) \\ \mathcal{C}(\lambda,\theta) & \mathcal{D}(\lambda,\theta) \end{pmatrix} = T(\lambda,\theta)\mathcal{K}(\lambda;\theta)\widehat{T}(\lambda,\theta),$$

$$\widehat{T}(\lambda,\theta) \ =\mathcal{R}_{N0}(\lambda+\xi_N,\theta)\ldots\mathcal{R}_{10}(\lambda+\xi_1,\theta-\eta\sum_{i=2}^{N}\sigma_i^z)$$

Reflection Algebra

$$\mathcal{R}_{12}(\lambda_1-\lambda_2,\theta-\eta\sum_{i=1}^{N}\sigma_i^z)\mathcal{U}_1(\lambda_1,\theta)\mathcal{R}_{21}(\lambda_1+\lambda_2,\theta-\eta\sum_{i=1}^{N}\sigma_i^z)\mathcal{U}_2(\lambda_2,\theta)$$

$$=\mathcal{U}_2(\lambda_2,\theta)\mathcal{R}_{12}(\lambda_1+\lambda_2,\theta-\eta\sum_{i=1}^{N}\sigma_i^z)\mathcal{U}_1(\lambda_1,\theta)\mathcal{R}_{21}(\lambda_1-\lambda_2,\theta-\eta\sum_{i=1}^{N}\sigma_i^z)$$

# Algebraic Bethe Ansatz

After the Vertex-IRF transformation there are two reference states: $|0\rangle$, $|\bar{0}\rangle$ (all the spins up, all the spins down): eigenstates of $\mathcal{A}(\lambda,\theta)$, $\mathcal{D}(\lambda,\theta)$ and $\mathcal{C}(\lambda,\theta)|0\rangle = 0$, $\mathcal{B}(\lambda,\theta)|\bar{0}\rangle = 0$. Then we can construct states:

$$|\Psi^1_{\mathrm{SOS}}(\{\lambda\})\rangle = \mathcal{B}(\lambda_1,\theta)\dots\mathcal{B}(\lambda_{\frac{M}{2}},\theta)|0\rangle,\ |\Psi^2_{\mathrm{SOS}}(\{\mu\})\rangle = \mathcal{C}(\mu_1,\theta)\dots\mathcal{C}(\mu_{\frac{M}{2}},\theta)|\bar{0}\rangle,$$

For this boundary constraint: number of operators is always $M/2$ necessary to diagonalize $K_+(\lambda)$ . More precisely, if $\{\lambda\}$ is a solution of the corresponding Bethe equations

$$\mathrm{tr}_0\Big(\mathcal{K}_+(\mu)\mathcal{U}(\mu,\theta)\Big)|\Psi^1_{\mathrm{SOS}}(\{\lambda\})\rangle = \Lambda_1(\mu,(\{\lambda\})|\Psi^1_{\mathrm{SOS}}(\{\lambda\})\rangle$$

with a diagonal boundary matrix:

$$\mathcal{K}^+(\lambda) = \begin{pmatrix} \frac{\sinh(\theta-\eta)\sinh(\theta+\lambda+\eta+\xi_+)}{\sinh\theta\sinh(\theta-\lambda-\eta+\xi_+)} & 0 \\ 0 & \frac{\sinh(\theta+\eta)\sinh(\lambda+\eta+\xi_+)}{\sinh\theta\sinh(-\lambda-\eta+\xi_+)} \end{pmatrix}$$

# Back to XXZ

Eigenstates for the XXZ transfer matrix

$$|\Psi^{1,2}_{\mathrm{XXZ}}(\{\lambda\})\rangle = S_{1\ldots M}(\theta)|\Psi^{1,2}_{\mathrm{SOS}}(\{\lambda\})\rangle$$

$$S_{1\ldots M}(\theta) = S_M(\xi_M, \theta)\ldots S_1(\xi_1, \theta - \eta\sum_{i=2}^{M}\sigma_i)$$

If $|\Psi^{1}_{\mathrm{SOS}}(\{\lambda\})\rangle$ is a SOS eigenstate, then

$$\mathcal{T}(\mu)\,|\Psi^{1}_{\mathrm{XXZ}}(\{\lambda\})\rangle = \Lambda_1(\mu, (\{\lambda\})|\Psi^{1}_{\mathrm{XXZ}}(\{\lambda\})\rangle$$

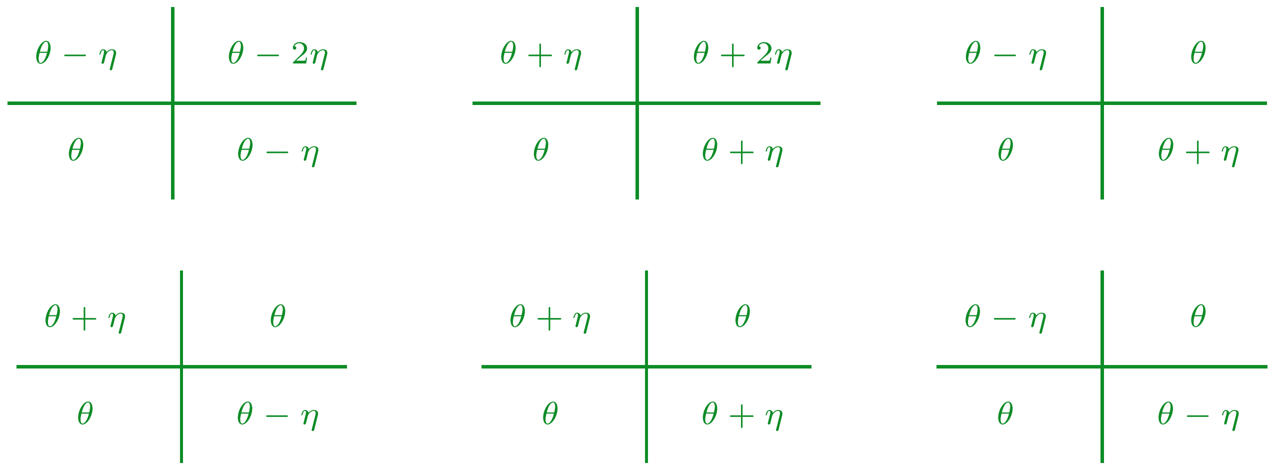
$$H_{\mathrm{XXZ}}\,|\Psi^{1}_{\mathrm{XXZ}}(\{\lambda\})\rangle = \sum_{j=1}^{M/2}\epsilon_1(\lambda_j)|\Psi^{1}_{\mathrm{XXZ}}(\{\lambda\})\rangle$$

# Partition function

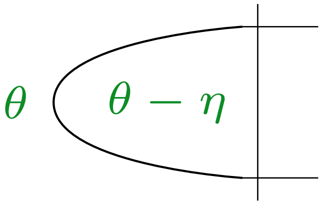
To compute the **scalar products** and then **correlation functions** a necessary step is to obtain the partition function of the corresponding two-dimensional model with domain wall boundary conditions (DWBC).

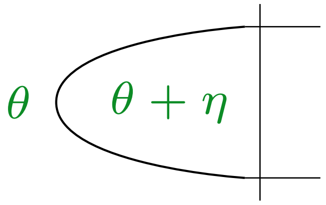
$$Z_M(\lambda_1, \ldots, \lambda_M; \xi_1, \ldots, \xi_M; \theta) = \langle \bar{0} | \mathcal{B}(\lambda_1, \theta) \ldots \mathcal{B}(\lambda_M, \theta) | 0 \rangle$$

This quantity can be interpreted as a partition function of a SOS model with the statistical weights given by the entries of $\mathcal{R}(\lambda, \theta)$:

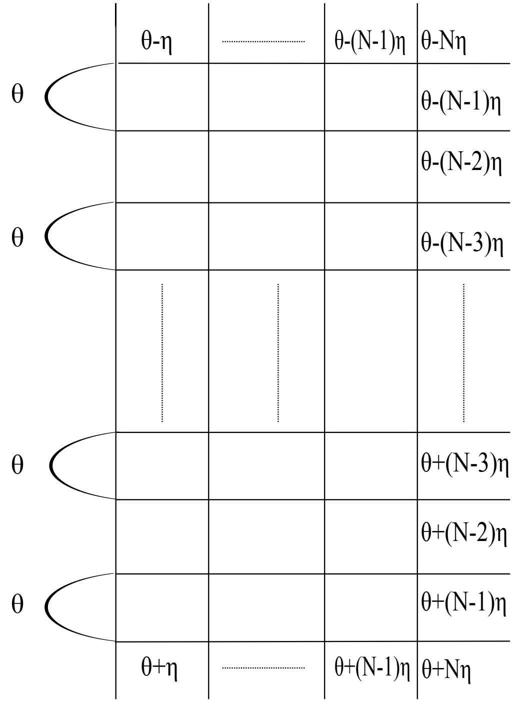


with one reflecting end




and DWBC



# Six vertex and SOS partition functions

The DWBC were first introduced for the **six vertex model** on a square lattice in the framework of the computation of the correlation functions of the periodic XXZ chain in 1982 (Izergin, Korepin)

- Korepin 1982: **properties** defining the partition function of the six vertex model with DWBC
- Izergin 1987: Determinant representation for the partition function of the six vertex model with DWBC (permits to compute correlation functions for the periodic XXZ chain)
- Tsuchiya 1998: Modification of Izergin's determinant for a six vertex model with **reflecting end** (permits to compute correlation functions for the open XXZ chain with diagonal boundaries).
- Rosengren 2009: Representation for the partition function of the **SOS model** on a square lattice. Not a single determinant!

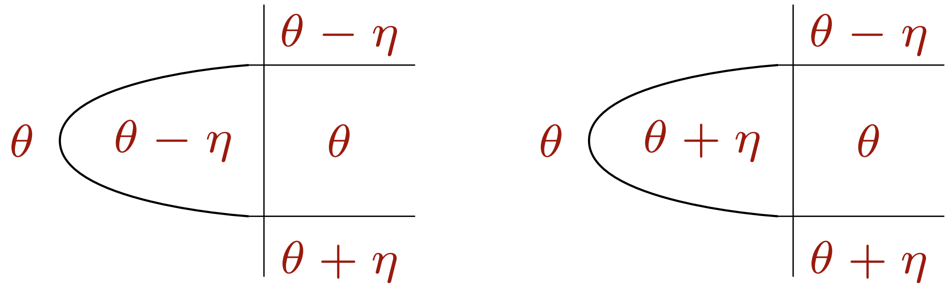
# Properties of the partition function

**1.** For each parameter $\lambda_i$ the normalized partition function

$$\tilde{Z}_N(\{\lambda\},\{\xi\},\theta) = \exp\left((2N+2)\sum_{i=1}^{N}\lambda_i\right)$$
$$\times \sinh(\theta+\xi_- +\lambda_i)\sinh(\theta+\lambda_i) Z_N(\{\lambda\},\{\xi\},\theta)$$

is a **polynomial** of degree at most $2N+2$ in $e^{2\lambda_i}$.

**2.** $Z_1(\lambda,\xi,\theta)$ can be easily computed as there are only two configurations possible.




**3.** $Z_N(\{\lambda\},\{\xi\},\theta)$ is **symmetric in** $\lambda_j$ as $[\mathcal{B}(\lambda,\theta),\mathcal{B}(\mu,\theta)] = 0$.

**4.** $Z_N(\{\lambda\},\{\xi\},\theta)$ is **symmetric in** $\xi_k$ (simple consequence of the DYBE).

**5.** Recursion relations: Setting $\lambda_1 = \xi_1$ we fix the configuration in the lower right corner

| | | |
|---|---|---|
| $\dots$ | $\dots$ | $\theta + (N-2)\eta$ |
| $\dots$ | $\theta + N\eta$ | $\theta + (N-1)\eta$ |
| $\theta + (N-2)\eta$ | $\theta + (N-1)\eta$ | $\theta + N\eta$ |

$\longrightarrow$ fixes configuration of two lines and one column $\longrightarrow$ $Z_N(\{\lambda\},\{\xi\},\theta)$ is reduced to $Z_{N-1}(\lambda_2,\dots\lambda_N,\xi_2\dots\xi_N,\theta)$
Setting $\lambda_N = -\xi_1$ we fix the configuration in the upper right corner

| | | |
|---|---|---|
| $\theta - (N-2)\eta$ | $\theta - (N-1)\eta$ | $\theta - N\eta$ |
| $\dots$ | $\theta - N\eta$ | $\theta - (N-1)\eta$ |
| $\dots$ | $\dots$ | $\theta - (N-2)\eta$ |

$\longrightarrow$ fixes configuration of two lines and one column $\longrightarrow$ $Z_N(\{\lambda\},\{\xi\},\theta)$ is reduced to $Z_{N-1}(\lambda_1,\dots\lambda_{N-1},\xi_2\dots\xi_N,\theta)$

Note: these are general conditions for the partition function, but here we have $2N$ conditions and polynomials of degree $2N+2$. We need more conditions.

**6.** Crossing symmetry:

$\mathcal{R}$-matrix:

$$-\sigma_1^y : \mathcal{R}_{12}^{t_1}(-\lambda-\eta, \theta+\eta\sigma_1^z) : \sigma_1^y \frac{\sinh(\theta-\eta\sigma_2^z)}{\sinh\theta} = \mathcal{R}_{21}(\lambda,\theta)$$

Normal ordering: $\sigma_1^z$ in the argument of the $\mathcal{R}$ matrix (which does not commute with it) is always on the right of all other operators involved in the definition of $\mathcal{R}$. It implies for the $\mathcal{B}$ operators (and hence for the partition function)

$$\mathcal{B}(-\lambda-\eta,\theta) = (-1)^{N+1}\frac{\sinh(\lambda+\zeta)\sinh(2(\lambda+\eta))\sinh(\lambda+\zeta+\theta)}{\sinh(2\lambda)\sinh(\lambda-\zeta+\eta)\sinh(\lambda-\theta-\zeta+\eta)}\mathcal{B}(\lambda,\theta)$$

Now the polynomial of degree $2N+2$ is defined in $4N$ points $\longrightarrow$ Partition function is defined uniquely by these conditions.

# Determinant representation

The unique function satisfying properties **1.** - **6.** is the following determinant

$$Z_{N,2N}(\{\lambda\},\{\xi\},\{\theta\}) = (-1)^N \det M_{ij} \prod_{i=1}^{N} \left( \frac{\sinh(\theta+\eta(N-2i))}{\sinh(\theta+\eta(N-2i+1))} \right)^{N-i+1}$$

$$\times \frac{\prod_{i,j=1}^{N} \sinh(\lambda_i+\xi_j)\sinh(\lambda_i-\xi_j)\sinh(\lambda_i+\xi_j+\eta)\sinh(\lambda_i-\xi_j+\eta)}{\prod_{1\le i<j\le N} \sinh(\xi_j+\xi_i)\sinh(\xi_j-\xi_i)\sinh(\lambda_j-\lambda_i)\sinh(\lambda_j+\lambda_i+\eta)}$$

$$M_{i,j} = \frac{\sinh(\theta+\zeta+\xi_j)}{\sinh(\theta+\zeta+\lambda_i)} \cdot \frac{\sinh(\zeta-\xi_j)}{\sinh(\zeta+\lambda_i)}$$

$$\times \frac{\sinh(2\lambda_i)\sinh\eta}{\sinh(\lambda_i-\xi_j+\eta)\sinh(\lambda_i+\xi_j+\eta)\sinh(\lambda_i-\xi_j)\sinh(\lambda_i+\xi_j)}$$

Very similar to the **six vertex** case.

# Next steps

1. **Scalar products** (Dual reflection algebra + $F$-basis)
2. Boundary inverse problem?
3. Correlation functions
4. Some results for $\Delta = \frac{1}{2}$, three colors model.
5. Weaker constraints for the boundary terms $\longrightarrow$ non-equilibrium systems (ASEP)?