Spectral properties of quantum spin chains and Chalker-Coddington-networks of Temperley-Lieb type

A. Klümper

University of Wuppertal - Department of Physics

with: Aufgebauer, Brockmann, Nuding, Sedrakyan

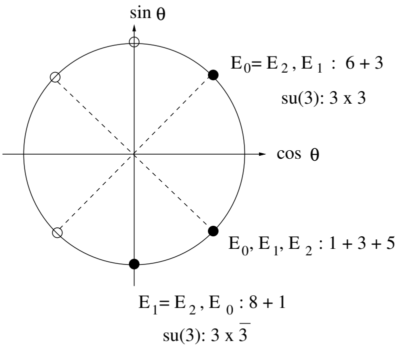
18.06.10

Contents

Biquadratic Spin-1 Quantum Chain

Most general $su(2)$ isotropic quantum $s = 1$ spin chain with nearest-neighbour interaction:

$$H = \sum_{j=1}^{L} \left[\cos\theta \, \vec{S}_j \vec{S}_{j+1} + \sin\theta \left(\vec{S}_j \vec{S}_{j+1} \right)^2 \right]$$



Special/integrable points: $\cot\theta = 3, \pm 1, 0$

Here of particular interest:
purely biquadratic case, representation of Temperley-Lieb algebra

Parkinson 87/88; AK 89, 93; Barber, Batchelor 89; Albertini 00; Alcaraz, Malvezzi 92; Köberle, Lima-Santos 94, 96; Kulish 2003; ...

Spin models of Temperley-Lieb type

Hamiltonian $H = \sum_{i=1} \mathrm{b}_i$ Hilbert space $\mathcal{H}_N = (V)^{\otimes N}$

local interactions proportional to projectors onto $su(2)$-singlets $b_i = |\Psi\rangle\langle\Psi|_{i,i+1}$. For $V = \mathbb{C}^3$

$$|\Psi\rangle = \sum_{\sigma,\mu=-1,0,1} \psi_{\sigma,\mu} |\sigma,\mu\rangle\,, \quad \psi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \psi\cdot\psi^+ = id$$

'accidentally higher' $su(3)$ symmetry of type $[3]\otimes[\bar{3}]$:

$$b_i \text{ commutes with } g\otimes\bar{g}\otimes g\otimes\bar{g}\ldots \text{ where } \bar{g} := \psi(g^{-1})^T\psi$$

for any invertible 3×3 matrix g, (if g is unitary: $\bar{g} = \psi g^* \psi$).

Generalization to $\dim(V) = 2s+1 =: n$ for arbitrary spin ($s = 1/2, 1, 3/2, 2 \ldots$) and $su(n)$ symmetry.

Temperley-Lieb relations

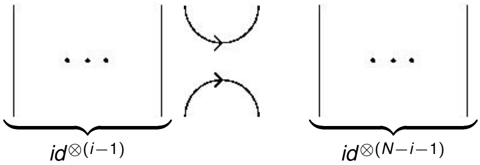
$TL_N(\lambda)$: (open boundary)
Generators b_i for $i = 1, ..., N-1$ with relations:

$$\begin{aligned} \mathrm{b}_i^2 &= \lambda\, \mathrm{b}_i \quad \text{for } i = 1, 2, ..., N-1 \\ \mathrm{b}_i\, \mathrm{b}_{i\pm1}\, \mathrm{b}_i &= \mathrm{b}_i \\ \mathrm{b}_i\, \mathrm{b}_j &= \mathrm{b}_j\, \mathrm{b}_i \quad \text{for } |i-j| > 1 \end{aligned}$$

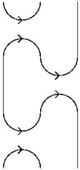
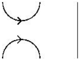
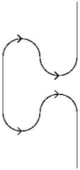
$PTL_N(\lambda)$: (closed boundary) additional generator b_N:

$$\begin{aligned} \mathrm{b}_N^2 &= \lambda \mathrm{b}_N \\ \mathrm{b}_j \mathrm{b}_N \mathrm{b}_j &= \mathrm{b}_j; \quad \mathrm{b}_N \mathrm{b}_j \mathrm{b}_N = \mathrm{b}_N \quad \text{for} \quad j = 1, N-1 \\ \mathrm{b}_j \mathrm{b}_N &= \mathrm{b}_N \mathrm{b}_j \quad \text{for} \quad j \neq 1, N-1 \end{aligned}$$

graphical notation for $\mathrm{b}_i = id^{\otimes(i-1)} \otimes |\Psi\rangle \langle\Psi|_{i,i+1} \otimes id^{\otimes(N-i-1)}$

$\mathrm{b}_i =$ 

Temperley-Lieb condition : $\mathrm{b}_i \mathrm{b}_{i+1} \mathrm{b}_i = \mathrm{b}_i$

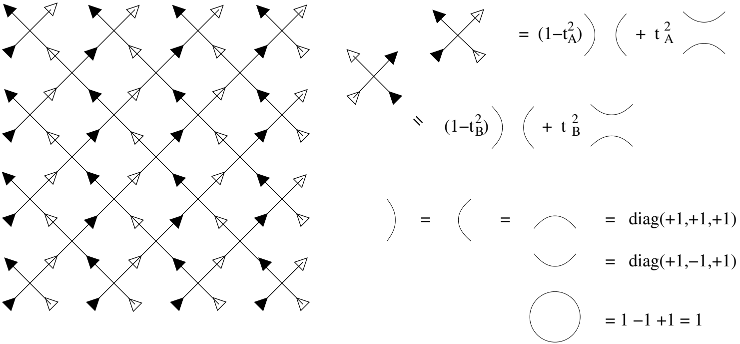
 $=$  $\Leftrightarrow$  $= \quad id$

Temperley-Lieb condition : $\mathrm{b}_i^2 = \lambda\, \mathrm{b}_i$

 $= \quad \lambda$

Chalker-Coddington network

3-state model with staggered $su(2|1)$ symmetry for spin quantum Hall effect



boundary conditions/super-trace: closed loops evaluate to 1

Gruzberg, Ludwig, Read 1999

Temperley-Lieb Models: Summary of Examples

i) Biquadratic chains and generalizations

$$V \otimes V = \bigoplus_{J=0}^{2s} V^J, \qquad |\Psi\rangle \; su(2) \text{ singlet}, \qquad \lambda = \dim(V)$$

singlet of two spin-s reps $\equiv$ quark–anti-quark singlet of $su(n)$, $n = 2s + 1$

ii) generalizing model (arbitrary anisotropy and spin)

$$|\Psi\rangle \; U_q(su(2)) \text{ singlet} ; \quad \lambda = \sum_{l=-s}^{s} (q^2)^l \quad \text{for } q \in \mathbb{R}$$

here most important: $q = 1$, i $\leftrightarrow$ $su(n)$, $su(s+1, s) \leftrightarrow \lambda = n, 1$.

iii) reference system: XXZ-model $s = 1/2 \rightarrow$ Bethe-Ansatz

$$|\Psi\rangle \langle\Psi| = \left(q^{-1/2} |{+-}\rangle - q^{1/2} |{-+}\rangle\right)\left(q^{-1/2} \langle{+-}| - q^{1/2} \langle{-+}|\right)$$

hermitian for $q \in \mathbb{R}$, TL-parameter $\lambda = q + q^{-1}$, interaction $\Delta = \frac{\lambda}{2}$

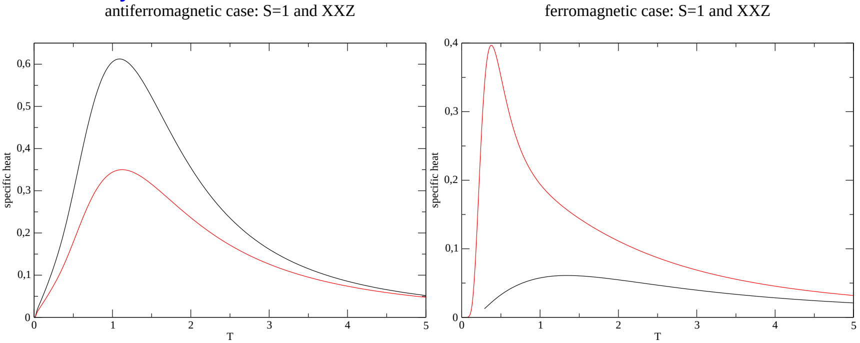
Temperley-Lieb Equivalence at $T > 0$?

Two different models being representations of the same TL-algebra (same λ!) have same spectrum, but multiplicities may differ!

TL equivalent are:
3-state biquadratic chain and 2-state *XXZ* chain with $\Delta = 3/2$

Thermodynamics differ!

Aufgebauer, AK 2010



Temperley-Lieb Equivalence and CFT?

Two different models being representations of the same TL-algebra (same λ!) have same spectrum for open boundary conditions, but central charge and scaling dimensions may differ!

TL equivalent are:
3-state $su(2|1)$ invariant chain and 2-state XXZ chain with $\Delta = 1/2$

Scaling dimensions for $\Delta = \cos\gamma$ Heisenberg chain

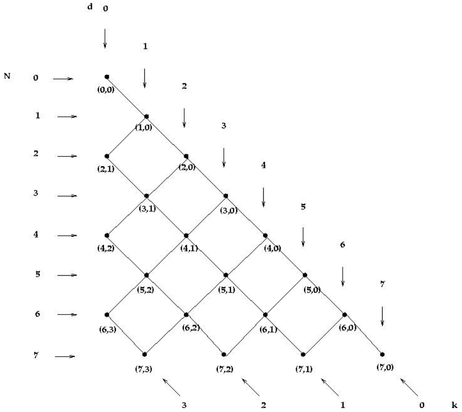
$$
\begin{aligned}
x &= \frac{1-\gamma/\pi}{2} S^2 + \frac{1}{2(1-\gamma/\pi)} m^2 \\
&\to \frac{1}{3} S^2 + \frac{3}{4} m^2
\end{aligned}
$$

with central charge $c = 1$.

$su(2|1)$ quantum chain has different characteristics, in particular $c = 0$.

Open boundary conditions

Decomposition of Hilbert space into irreducibles of $TL_N(\lambda)$ (generic λ)



Equiclasses of irred. indexed by N and k ; $0 \le k \le \left[\frac{N}{2}\right]$

$O(N,k)$

decomposition rule:

$O(N,k) \downarrow_{TL_{N-1}(\lambda)}$

(P. Martin: Potts models and related problems in statistical mechanics)

Reference States alias 1d-TL-reps

Bethe-ansatz reference states (by definition) are annihilated by all b_i's

$$\Omega_N := \{\omega \mid \mathrm{b}_i\omega = 0 \ : \ 1 \le i \le N-1\} \equiv O(N,0)$$

General representation isomorphic to $O(N,k)$

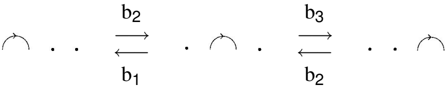
constructed from reference state on chain with $N-2k$ sites and k many local TL-singlets

$$v^1 := \Psi^{\otimes k} \otimes \omega \quad \text{with} \quad \omega \in \Omega_{N-2k}$$

spans TL_N-invariant subspace
$\rightarrow$ multiplicity of $O(N,k)$ in $\mathcal{H}_N$ is equal to $\dim(\Omega_{N-2k})$

example $k = 1$ $N = 4$

$$\Psi \otimes \omega \quad \underset{b_1}{\overset{b_2}{\rightleftarrows}} \quad b_2(\Psi \otimes \omega) \quad \underset{b_2}{\overset{b_3}{\rightleftarrows}} \quad b_3\, b_2(\Psi \otimes \omega)$$



orthogonal-basis:

$$v^1 = \Psi \otimes \omega, \quad v^2 = \frac{\lambda}{\lambda^2 - 1}\left(b_2(\Psi \otimes \omega) - \frac{1}{\lambda}(\Psi \otimes \omega)\right)$$

$$v^3 = \frac{\lambda^2 - 1}{\lambda^3 - 2\lambda}\left(b_3\ v^2 - \frac{\lambda}{\lambda^2 - 1} v^2\right)$$

polynomials : $P_0(\lambda) = 1, \quad P_1(\lambda) = \lambda, \quad P_k(\lambda) = \lambda\ P_{k-1} - P_{k-2}$

Dimension of Ω_N (sits in Ω_{N-1}) is kernel of

$$\text{map} \quad \mathfrak{b}_{N-1} : \Omega_{N-1} \otimes h \rightarrow \Omega_{N-2} \otimes \Psi$$

which is surjectiv!

$\Big($Proof: For given $\omega \in \Omega_{N-2}$ consider the $k = 1$ TL_N-sector

$$v^{N-1}(\omega) \stackrel{\mathfrak{b}_{N-1}}{\longrightarrow} \frac{P_1}{P_{N-2}} \omega \otimes \Psi \ \Big)$$

Dimension formula for kernel and image yields recursion relation

$$\rightarrow \dim(\Omega_N) = n \dim(\Omega_{N-1}) - \dim(\Omega_{N-2}) \qquad n = \dim(h)$$

$$\rightarrow \dim(\Omega_N) = P_N(n) = \frac{\left(n + \sqrt{n^2 - 4}\right)^{N+1} - \left(n - \sqrt{n^2 - 4}\right)^{N+1}}{2^{N+1}\sqrt{n^2 - 4}}$$

Multiplicities for open boundaries derived earlier by 'double centralizer property' Kulish, Manojlovic and Nagy 2008.

Temperley-Lieb Equivalence of Partition Functions

The spectra of two Hamiltonian in equivalent representations $O(N,k)$ are identical. Asymptotically, there are z^{N-2k} many of such representations/sectors, with 'fugacity'

$$z = \left(\frac{n + \sqrt{n^2 - 4}}{2} \right)$$

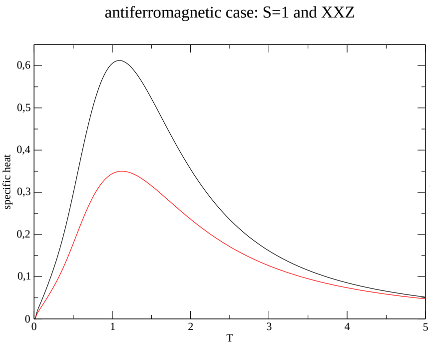
We sum over all sectors and in each one over all energies

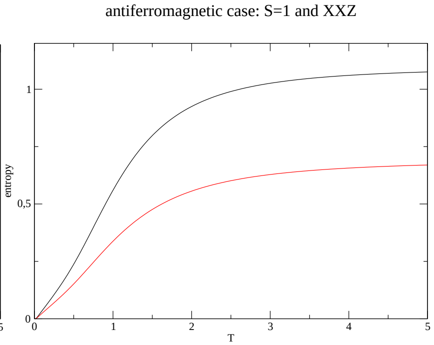
$$Z(T, h=0) = \sum_{k=0}^{N} \sum_{\mathrm{all} E_k} \mathrm{e}^{-\beta \mathrm{E_k}} \cdot \mathrm{z}^{\mathrm{N}-2\mathrm{k}}.$$

which gives the grand-canonical partition function of the *XXZ* reference model with magnetic field

$$N-2k = M \longrightarrow Z(T, h=0) = \mathrm{Tr}\, \mathrm{e}^{-\beta(\mathrm{H_{XXZ}} - (\mathrm{T} \ln \mathrm{z})\mathrm{M})} = \mathrm{Z_{XXZ}}(\mathrm{T}, \mathrm{T} \ln \mathrm{z})$$

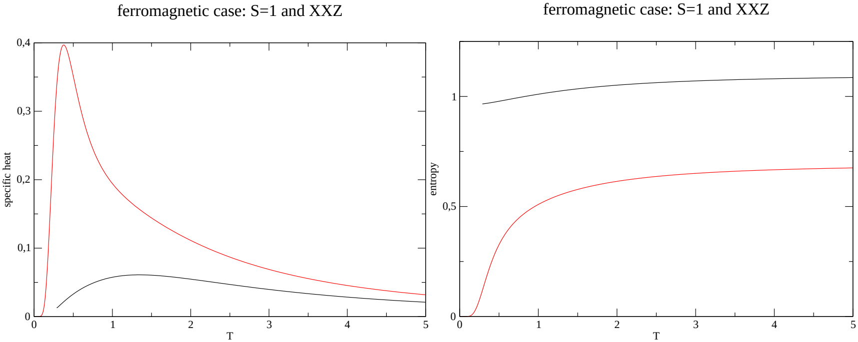
Antiferromagnetic Biquadratic Spin-1 Quantum Chain: Specific Heat and Entropy





(Note the thermodynamically activated behaviour with very small gap)

Ferromagnetic Biquadratic Spin-1 Quantum Chain: Specific Heat and Entropy



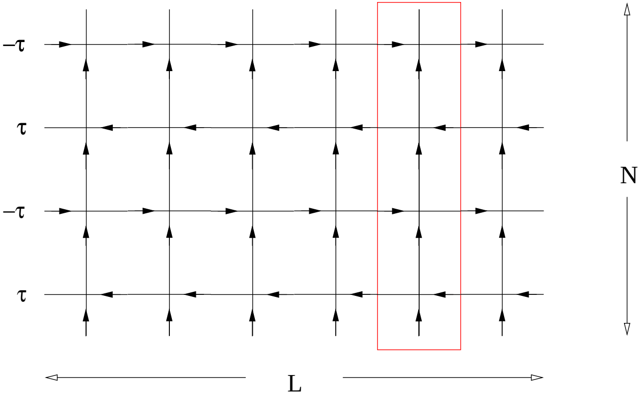
There is residual entropy!

$$S(T=0) = \ln z = \ln\left(\frac{n+\sqrt{n^2-4}}{2}\right) \to_{n=3} 0.9624...$$

and activated behaviour with noticeable gap.

Partition Function with external magnetic field

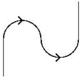
lattice path integral approach ⟶ classical model based on TL



with periodic boundary conditions of column-to-column transfer matrix
magnetic field leads to twisted boundary conditions

Periodic and twisted boundary conditions

Consider models –like biquadratic chain– with

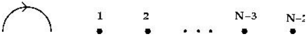


$= \pm id$

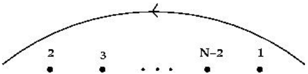
on periodically closed chain.

For $k = 1$:

$\Psi \otimes \omega_{N-2} =$



$\mathrm{b}_N \, \mathrm{b}_{N-1} \cdots \mathrm{b}_2 \, \Psi \otimes \omega_{N-2} =$



$\mathrm{b}_1 \, (\mathrm{b}_N \, \mathrm{b}_{N-1} \cdots \mathrm{b}_2 \, \Psi \otimes \omega_{N-2}) =$

$= \Psi \otimes T^2(\omega_{N-2})$

$\rightarrow$ Take $\omega \in \Omega^p_{N-2}$ and ω eigenstate of $T = e^{-iP}$

Momentum eigenvalue of translationally invariant reference state enters!

Periodic and twisted boundary conditions

Consequence:
TL-equivalence of periodic biquadratic chain to twisted *XXZ*

Goal: Decompose the Hilbert space into *PTL*-representations (see Martin, Saleur 93) isomorphic to twisted XXZ-PTL_N-representations.

$$H_N = \sum_{i=1}^{N-1} \frac{1}{2}\Big(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + (q+q^{-1}) S_i^z S_{i+1}^z \Big) + \frac{1}{2}\Big(e^{-2i\varphi} S_N^+ S_1^- + e^{2i\varphi} S_N^- S_1^+ + (q+q^{-1}) S_N^z S_1^z \Big).$$

$\tilde{b}_N$ acts as projector $|\tilde{\Psi}\rangle\langle\tilde{\Psi}|$ in $h_N \otimes h_1$

$$|\tilde{\Psi}\rangle = e^{-i\varphi} q^{-1/2} |+-\rangle - e^{i\varphi} q^{1/2} |-+\rangle$$

$PTL_N(\lambda)$-relations fulfilled for all $\varphi \in \mathbb{C}$

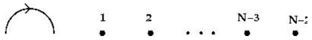
Bethe-Ansatz for twisted XXZ in S^z =constant:

$\longrightarrow$ eigenstates in PTL_N-subrepresentation

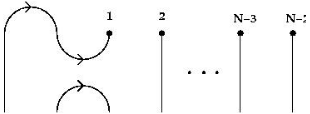
$$|\Psi\rangle \otimes |+\rangle^{\otimes(N-2)} \xrightarrow{b_2} -|+\rangle \otimes |\Psi\rangle \otimes |+\rangle^{\otimes(N-3)}$$

graphical notation:

$\Psi \otimes \omega_{N-2} \quad =$



$\xrightarrow{b_2}$



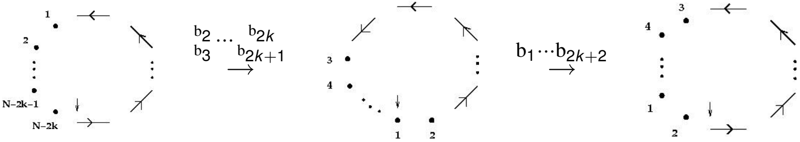
$$b_1\, \tilde{b}_N b_{N-1} \cdots b_2\, |\Psi\rangle \otimes |+\rangle^{\otimes(N-2)} = (-1)^N\, e^{-i2\varphi}\, |\Psi\rangle \otimes |+\rangle^{\otimes(N-2)}$$

Construction for arbitrary k

$$\omega \in \Omega^{p}_{N-2k} \qquad T\omega = e^{-i\varphi}\omega$$

$$(\mathrm{b}_1\mathrm{b}_N\cdots\mathrm{b}_{2k+2})(\mathrm{b}_3\mathrm{b}_2)\cdots(\mathrm{b}_{2k+1}\mathrm{b}_{2k})\Psi^{\otimes k}\otimes\omega = (\pm)^N(e^{2i\varphi})\,\Psi^{\otimes k}\otimes\omega$$

graphically:



sector $P(N,k,\varphi)$:
$\Psi^{\otimes k}\otimes\omega \rightarrow$ construct PTL_N-invariant subspace.
$\dim(P(N,k,\varphi)) \leq \binom{N}{k}$

Special extremal case $P(N, N/2, \varphi)$

Even N

Sector $k = \frac{N}{2}$: appended reference state trivial
$\rightarrow$ imaginary twist-angle, i.e. magnetic field

$$\varphi = i \ln\left(\frac{1}{2}\left(n + \sqrt{n^2 - 4}\right)\right) \qquad n = \dim(h)$$

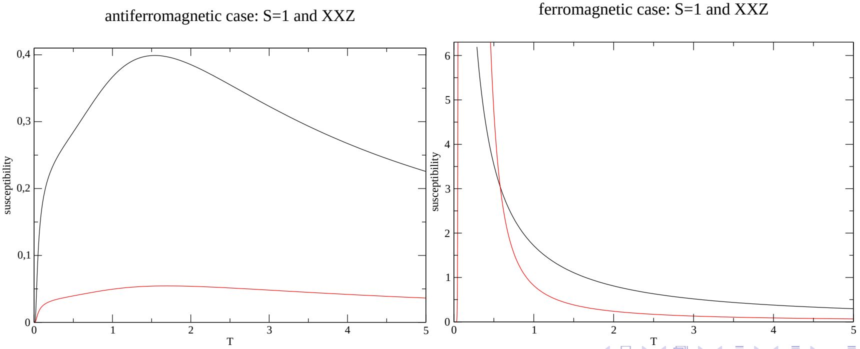
yields thermodynamics of biquadratic quantum spin chain for arbitrary T and h.

Sectors with $k < \frac{N}{2}$: not needed.

Temperley-Lieb Equivalence at finite T and h

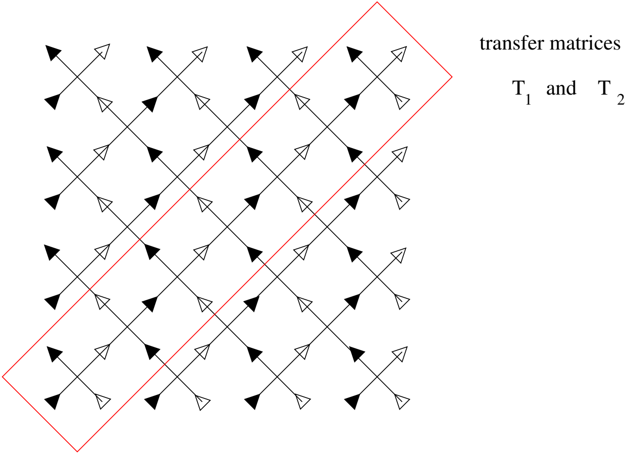
The partition functions for the biquadratic chain and the *XXZ* chain are identical $Z(T,h) = Z_{XXZ}(T,\tilde{h})$ provided

$$1 + 2\cosh\left(\frac{h}{T}\right) = 2\cosh\left(\frac{\tilde{h}}{T}\right)$$



Chalker-Coddington network at critical point

Isotropic case $t_A = t_B$ corresponds to integrable, critical vertex model.



Two commuting families of diagonal-to-diagonal transfer matrices, T_1 and T_2. 'Logarithmic derivative' yields TL-Hamiltonian.

Chalker-Coddington network: conformal properties

Sector $k = \frac{L}{2}$:

appended reference state trivial, $\rightarrow$ real twist-angle $\varphi = \pi/3$.

Central charge: $$c(\varphi) = 1 - \frac{6(\varphi/\pi)^2}{1-\gamma/\pi} \rightarrow 0$$

Sectors with $k < \frac{L}{2}$

$$\varphi = l \cdot \frac{\pi}{L-2k}, \qquad l = 0, 1, ..., L-2k-1,$$

(If magnetization of state is odd: $l = 1/2, 3/2, ..., L-2k-1/2$.)

Scaling dimension: $$x = \frac{4\,S^2 + 9(m-\varphi/\pi)^2 - 1}{12}, \quad s = S\left(m - \frac{\varphi}{\pi}\right)$$

Log-corrections: $$E_{x,s} - E_{x,s}^{CFT} = -2\pi v\left[(1/12-x)^2 - s^2)\right]\frac{\log L}{L^3},$$

Aufgebauer, Brockmann, Nuding, AK, Sedrakyan (2010)

Dimension of Ω_N^p

Multiplicities of scaling dimension identical to dimension of space of reference states.

Ω_N^p is kernel of the map

$$\mathrm{b}_N \;:\; \Omega_N \longrightarrow \Psi \otimes \Omega_{N-2} \;\subset V_N \otimes V_1 \otimes \cdots \otimes V_{N-1}.$$

Again surjectivity can be proven, dimension formula:

$$\dim(\Omega_N^p) = \dim(\Omega_N) - \dim(\Omega_{N-2}).$$

$$\dim(\Omega_N^p)(n) = \left(\frac{n+\sqrt{n^2-4}}{2}\right)^N + \left(\frac{n-\sqrt{n^2-4}}{2}\right)^N.$$

Spectrum of T in Ω_N^p

$T = e^{-iP}$ diagonalisable in Ω_N^p

Eigenvalues of momentum operator P:

$$\frac{2\pi}{N} l, \quad l \in \{0, 1, ..., N-1\}$$

(1) Spectrum of T in $\mathcal{H}_N$

(2) Spectrum of T in Ω_N^p, multiplicities of k-sectors

ad (i):

If N is prime: multiplicities $M(l=0) = \frac{(2^N - 2)}{N} + 2$, $M(l \neq 0) = \frac{(2^N - 2)}{N}$.

$N = 6$: $M(0) = 14$, $M(\pm 1) = 9$, $M(\pm 2) = 11$, $M(3) = 10$.

(1) $X=\{x_i\}$ basis of $V \to X^{\otimes N}=\left\{x_{i_1}\otimes x_{i_2}\otimes\cdots\otimes x_{i_N}\right\}$
T acts on $X^{\otimes N} \to$ partition of T-orbits
$\sigma(p)$: dimension of p-periodic subspace

$$\sum_{p\,|\,N}\sigma(p)=\nu(N) \qquad \nu(N):=\dim(\mathcal{H}_N)=n^N$$

$$\to \sigma(N)=\sum_{n\,|\,N}\mu(n)\,\nu\left(\frac{N}{n}\right) \qquad \mu:\ \text{Möbius function}$$

Multiplicity of momentum eigenvalues

$$M\left(\frac{2\pi}{N}l;\mathcal{H}_N\right)=\sum_{n\,|\,(l,N)}\frac{\sigma(\frac{N}{n})}{\frac{N}{n}}=\sum_{n\,|\,(l,N)}\frac{n}{N}\sum_{\tilde{n}\,|\,\frac{N}{n}}\mu(\tilde{n})\,\nu\left(\frac{N}{n\tilde{n}}\right).$$

(2) Formula for multiplicities of momentum eigenvalues in Ω_N^p similar to above

$$M\left(\frac{2\pi}{N}l;\Omega_N^p\right)=\sum_{n\,|\,(l,N)}\frac{\sigma(\frac{N}{n})}{\frac{N}{n}}=\sum_{n\,|\,(l,N)}\frac{n}{N}\sum_{\tilde{n}\,|\,\frac{N}{n}}\mu(\tilde{n})\,\nu\left(\frac{N}{n\tilde{n}}\right).$$

where now $\nu(N)=\dim(\Omega_N^p)$.

Summary

direct-sum decomposition of Hilbert space

(1) open boundaries :
irreducible TL_N-representations

(2) periodic boundaries :
PTL_N-representations depend on additional parameter
$\rightarrow$ XXZ-sectors with twist, spectrum via Bethe-Ansatz

(3) applications:
thermodynamics (arbitrary T, h), conformal properties

(i) multiplicities known $\rightarrow$ thermodynamics

(ii) nongeneric Temperley-Lieb parameter?

(iii) correlation functions?

Decomposition of P into O's

decomposition rule for $P(N,k,\varphi)$ into irreducible TL_N-representations: ($\lambda > 2$ and $2k < N$)

$$P(N,k,\varphi)\downarrow_{TL_N} \cong \oplus_{l=0}^{k} O(N,k-l)$$

$$\rightarrow \dim(P(N,k,\varphi)) = \binom{N}{k}$$

proof by construction of orthogonal basis polynomials:

$$D_k^{\varphi,\epsilon}(\lambda) = P_k(\lambda) - P_{k-2}(\lambda) - (-\epsilon)^k \underbrace{(e^{2i\varphi} + e^{-2i\varphi})}_{2\cos(2\varphi)} \quad \text{for } k \geq 2.$$