

Long-time/long-distance asymptotics of the two-point functions in the non-linear Schrödinger model

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In collaboration with : *N. Kitanine, J.M. Maillet, N. A. Slavnov and V. Terras* .

"On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain", JMP **50** 095209, (2009).

"Thermodynamic limit of particle-hole form factors in the massless XXZ Heisenberg chain", ArXiv: 1003.4557.

N. Kitanine, K. K. Kozłowski, J.M. Maillet, N. A. Slavnov and V. Terras

"Riemann–Hilbert approach to the time-dependent generalized sine kernel".

"Multidimensional Natte series for the time-dependent correlation functions in the non-linear Schrödinger model".

in preparation

Outline

- 1 Motivations
- 2 The non-linear Schrodinger model
- 3 Large size behavior of form factors
- 4 From form factor expansion to asymptotics
- 5 Conclusion

Correlation functions and asymptotics

- ★ 1D *gapless* models at $T = 0K$ are critical \rightsquigarrow Algebraic in distance decay of correlators.
 \Rightarrow predictions of critical exponents by Luttinger liquid/CFT from finite size corrections
 ([Batchelor](#), [Destri](#), [DeVega](#), [Klumper](#), [Pearce](#), [Wojnarowich](#), [Wehner](#), [Zittartz](#))

Confirmation by exact results of free fermion equivalent models

- Correlators for free fermionic models \equiv Fredholm determinants
- ★ Sine kernel satisfies PV: $\langle \psi^\dagger(x) \psi(0) \rangle$ for imp. bosons ('80 [Jimbo](#), [Miwa](#), [Mori](#), [Sato](#))
- ★ transverse Ising, imp. bosons ('83-'86 [McCoy](#), [Perk](#), [Schrock](#), [Tang](#)).
- ★ $T > 0K$, large x , large t by RHP ('90 [Its](#), [Izergin](#), [Korepin](#), [Varzugin](#), [Slavnov](#),...)

Beyond free fermion point

- ★ Exact representation for correlators [Leningrad](#), [Kyoto](#), [Lyon](#), [Wuppertal](#),...
 - Correlators given by series of multiple integrals depending on large parameter.
 - Method to obtain long-distance asymptotics ('08 [Kitanine](#), [K.](#), [Maillet](#), [Slavnov](#), [Terras](#))

Today's main message

- ★ Correlators in interacting models \equiv multidimensional deformations of Fredholm determinants.
- ★ Construction of multidimensional Natte series allowing to read off asymptotics.

The non-linear Schrödinger model

- NLSE \equiv 1D limit of 3D Bose gas. Test of Bogoliubov theory.
- Eigenfunctions and spectrum ('63 Lieb, Liniger).
- Simplest possible *interacting massless* integrable model.

$$H = \int_0^L \left\{ \partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y) \Psi^\dagger(y) \Psi(y) \Psi(y) - h \Psi^\dagger(y) \Psi(y) \right\} dy$$

L : length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

- ★ N -body eigenstates constructed by algebraic Bethe Ansatz $|\{\lambda_j\}\rangle$

The parameters $\{\lambda_j\}$, $\lambda_j \in \mathbb{R}$, are the unique solutions to the Bethe Ansatz equations

$$\frac{L}{2\pi} p_0(\lambda_j) - \frac{1}{2\pi} \sum_{a=1}^N \theta(\lambda_j - \lambda_a) = n_j - \frac{N+1}{2} \quad \text{with} \quad \begin{cases} p_0(\lambda) = \lambda \\ \theta(\lambda) = i \ln \left(\frac{ic + \lambda}{ic - \lambda} \right) \end{cases}$$

- ★ All choices of $n_1 < \dots < n_N$ yield the complete set of eigenstates of H ('90 Dorlas).

The form factor approach

Form factor expansion for finite L of $j(x, t) \equiv e^{iHt} \Psi^\dagger(x) \Psi(x) e^{-iHt}$

$$\begin{aligned} \langle G.S. | j(x, t) j(0, 0) | G.S. \rangle &= \sum_{\{\mu\}_{\text{ex}}} \langle G.S. | e^{-ixP + itH} j(0, 0) e^{ixP - itH} | \{\mu\}_{\text{ex}} \rangle \langle \{\mu\}_{\text{ex}} | j(0, 0) | G.S. \rangle \\ &= \sum_{\{\mu\}_{\text{ex}}} e^{ix(P_{G.S.} - P_{\text{ex}}) - it(\mathcal{E}_{G.S.} - \mathcal{E}_{\text{ex}})} \left| \langle G.S. | j(0, 0) | \{\mu\}_{\text{ex}} \rangle \right|^2 \end{aligned}$$

presumed steps of the computation

- Characterize the excitations above the ground state;
- Get a determinant representation for $\langle G.S. | j(0, 0) | \{\mu\}_{\text{ex}} \rangle$;
- Asymptotic in size L formula for $\langle G.S. | j(0, 0) | \{\mu\}_{\text{ex}} \rangle$;
- Perform the summation over $\{\mu\}_{\text{ex}}$;
- Re-cast in a form fit for the asymptotic analysis.

The thermodynamic limit I

Chemical potential $h \rightsquigarrow N$. Thermodynamic limit $N, L \rightarrow +\infty, N/L \rightarrow D$.

The ground state

$n_j = j \Rightarrow$ Bethe roots for GS *densify* on $[-q; q]$ with density $\rho(\lambda) \equiv \lim_{\substack{L \rightarrow +\infty \\ \lambda_j \rightarrow \lambda}} \frac{1}{L(\lambda_j - \lambda_{j-1})}$

$$\rho(\lambda) - \int_{-q}^q K(\lambda - \mu) \rho(\mu) d\mu = \frac{\rho'_0(\lambda)}{2\pi} \quad \text{and} \quad D = \int_{-q}^q \rho(\lambda) d\lambda$$

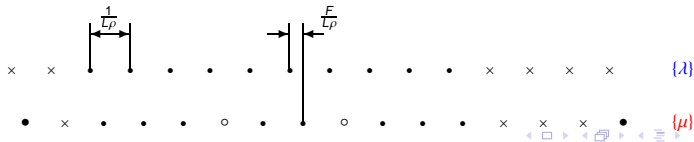
"particle-hole" excitations \rightsquigarrow new choice of integers n_j :

$$n_j = j \text{ for } j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\} \quad \text{and} \quad n_{h_a} = p_a \text{ for } a \in \{1, \dots, N\}$$

\Rightarrow Excited state's roots μ shifted infinitesimally in respect to GS roots λ

$$\mu_j - \lambda_j = F(\lambda_j) / L\rho(\lambda_j) + O(L^{-2}) \quad j \in \{1, \dots, N\} \setminus \{h_1, \dots, h_n\} .$$

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \dots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \dots, \mu_{p_n}$



The thermodynamic limit II

The *shift function* F solves a linear integral equation

$$F(\lambda) - \int_{-q}^q K(\lambda - \mu) F(\mu) d\mu = -\frac{1}{2\pi} \sum_{p=1}^n \theta(\lambda - \mu_{p_a}) - \theta(\lambda - \mu_{h_a}),$$

With leading position of particles/holes $\xi(\mu_j) = \frac{j}{L}$ $\xi(\lambda) = \frac{p}{2\pi} - \frac{N+1}{2L}$.

Additive structure in "particles" and "holes" rapidities.

$$F(\lambda) = \sum_{p=1}^n \phi(\lambda, \mu_{h_a}) - \phi(\lambda, \mu_{p_a}) \quad \text{where} \quad \phi(\lambda, \tau) - \int_{-q}^q K(\lambda - \mu) \phi(\mu, \tau) d\mu = \frac{1}{2\pi} \theta(\lambda - \tau).$$

↪ Additive excitation spectrum.

$$\mathcal{P}_{\text{ex}} - \mathcal{P}_{\text{G.S.}} = \sum_{a=1}^n p(\mu_{p_a}) - p(\mu_{h_a}) \quad \mathcal{E}_{\text{ex}} - \mathcal{E}_{\text{G.S.}} = \sum_{a=1}^n \epsilon(\mu_{p_a}) - \epsilon(\mu_{h_a})$$

$$p(\lambda) = 2\pi \int_0^\lambda \rho(s) ds \quad \left(1 - \frac{K}{2\pi}\right) \cdot \epsilon(\lambda) = \epsilon_0(\lambda) \quad \epsilon_0(\lambda) = \lambda^2 - h.$$

The main result

T=0K long-time & distance asymptotics of currents: $j(x, t) \equiv e^{iHt} \Psi^\dagger(x) \Psi(x) e^{-iHt}$

x/t fixed $\rightarrow p(\lambda) - t\epsilon(\lambda)/x$ has a unique simple saddle-point λ_0 : $p'(\lambda_0) - t\epsilon'(\lambda_0)/x = 0$.

In the space-like regime ($\lambda_0 > q$):

$$\begin{aligned} \langle j(x, t) j(0, 0) \rangle &= \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2} \frac{x^2 + t^2 v_F^2}{(x^2 - t^2 v_F^2)^2} + \frac{2 \cos(2xp_F)}{[-i(x - v_F t)]^{\mathcal{Z}^2} [i(x + v_F t)]^{\mathcal{Z}^2}} \left| \mathcal{F} \left(\begin{matrix} q \\ -q \end{matrix} \right) \right|^2 (1 + o(1)) \\ &+ \frac{p'(\lambda_0) \sqrt{2\pi}}{[i(t\epsilon''(\lambda_0) - xp''(\lambda_0))]^{\frac{1}{2}}} \frac{\exp\{i(xp(\lambda_0) - p_F - t\epsilon(\lambda_0))\}}{[-i(x - v_F t)]^{(F_+(q)-1)^2} [i(x + v_F t)]^{(F_+(-q))^2}} \left| \mathcal{F} \left(\begin{matrix} \lambda_0 \\ q \end{matrix} \right) \right|^2 (1 + o(1)) \\ &+ \frac{p'(\lambda_0) \sqrt{2\pi}}{[i(t\epsilon''(\lambda_0) - xp''(\lambda_0))]^{\frac{1}{2}}} \frac{\exp\{i(xp(\lambda_0) + p_F - t\epsilon(\lambda_0))\}}{[-i(x - v_F t)]^{(F_-(q))^2} [i(x + v_F t)]^{(F_-(-q)+1)^2}} \left| \mathcal{F} \left(\begin{matrix} \lambda_0 \\ -q \end{matrix} \right) \right|^2 (1 + o(1)) \end{aligned}$$

$$F_{\pm}(\lambda) = \phi(\lambda, \pm q) - \phi(\lambda, \lambda_0) \quad \mathcal{Z} = 1 + \phi(q, -q) - \phi(q, q).$$

v_F : Fermi velocity.

Asymptotic behavior of form factors: The result

NLSE, **Slavnov '90**, **XX Arikawa, Kabrach, Miller, Wiele '06**
 6-Vertex R matrix **Kitanine, K, Maillet, Slavnov, Terras '09-'10**

- $\{\mu\}$: excited state with particles $\mu_{p_1}, \dots, \mu_{p_n}$ and holes $\mu_{h_1}, \dots, \mu_{h_n}$.
- $\{\lambda\}$ GS roots.
- F shift function associated with $\{\mu\}$ and $\{\lambda\}$.
- Power-law behavior in L with exponent $\alpha_c[F]$.

$$\left| \frac{\langle \{\mu\} | j(0,0) | \{\lambda\} \rangle}{\|\{\mu\}\| \cdot \|\{\lambda\}\|} \right|^2 = \left(\frac{2\pi}{L} \right)^{\alpha_c[F]} \cdot \mathcal{F}_n \left(\begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) \cdot \left(1 + O\left(\frac{\ln L}{L}\right) \right).$$

$$\mathcal{F}_n \left(\begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) = \mathcal{R}_n \left(\begin{matrix} \{p_a\}; \{\mu_{p_a}\} \\ \{h_a\}; \{\mu_{h_a}\} \end{matrix} \right) [F] \cdot \mathcal{S}_n \left(\begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) [F]$$

- Particle-hole excitations **on** Fermi boundary $\alpha_c[F] \equiv$ **critical exponent**

eg Particle at q / hole at $-q \rightsquigarrow \Delta E = 0 \quad \Delta P = 2p_F$ - excitation, $\alpha_c[F] = 2Z^2$

The generating function

$$\frac{1}{2} \frac{\partial^2}{\partial \kappa^2} \frac{\partial^2}{\partial X^2} \mathcal{L}(\kappa) = \langle j(x, t) j(0, 0) \rangle \quad \text{with} \quad \mathcal{L}_N(\kappa) = \sum_{\substack{n_1 < \dots < n_N \\ n_i \in \mathbb{Z}}} e^{ix\mathcal{P}_{\text{ex}} - it\mathcal{E}_{\text{ex}}} \cdot \left| \frac{\langle \{\mu\}_\kappa | \{\lambda\} \rangle}{\|\{\mu\}_\kappa\| \cdot \|\{\lambda\}\|} \right|^2$$

$$\mathcal{P}_{\text{ex}} = \sum_{a=1}^n p(\mu_{p_a}) - p(\mu_{h_a}) + O(L^{-1}) \quad \mathcal{E}_{\text{ex}} = \sum_{a=1}^n \epsilon(\mu_{p_a}) - \epsilon(\mu_{h_a}) + O(L^{-1})$$

- Only implicit description of Bethe roots;
- each form factor introduce high couplings in the summation variables;
- Exact resummation possible with use of multidimensional residues;

reasonable simplifications

- Only states having the same per-site energy as GS contribute in $L \rightarrow +\infty$
- Only the leading in L behavior of individual form factors contributes
- The roots of excited states can be approximated by their leading position
 $\xi(\mu_{p_a}) = p_a/L$ and $\xi(\mu_{h_a}) = h_a/L$.

The effective form factor series I

$$\mathcal{L}_N(\kappa) = \sum_{n=0}^N \sum_{p_1 < \dots < p_n} \sum_{h_1 < \dots < h_n} \left(\frac{2\pi}{L}\right)^{\alpha_c[F]} \prod_{a=1}^n \left\{ \frac{e^{i\exp(\mu_{p_a}) - it\epsilon(\mu_{p_a})}}{e^{i\exp(\mu_{h_a}) - it\epsilon(\mu_{h_a})}} \right\} \cdot \mathcal{R}_n \left(\begin{matrix} \{p_a\}; \{\mu_{p_a}\} \\ \{h_a\}; \{\mu_{h_a}\} \end{matrix} \right) [F] \cdot \mathcal{S}_n \left(\begin{matrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{matrix} \right) [F]$$

- Summation dependent shift function $F(\lambda) = \frac{\ln \kappa}{2i\pi} Z + \sum_{a=1}^n \phi(\lambda; \mu_{h_a}) - \phi(\lambda; \mu_{p_a})$.
- Divergent sums not replaceable by integrals because of discrete part.
- Smooth part highly coupled, symmetric function with reduction properties

$$\mathcal{S}_n \left(\begin{matrix} \{\mu_{p_a}\}_{a=1}^n \\ \{\mu_{h_a}\}_{a=1}^n \end{matrix} \right) [G] \Big|_{\mu_{p_k} = \mu_{h_\ell}} = \mathcal{S}_{n-1} \left(\begin{matrix} \{\mu_{p_a}\}_{a=1}^n \\ \{\mu_{h_a}\}_{a=1}^n \\ \neq k \\ \neq \ell \end{matrix} \right) [G] .$$

$$\mathcal{G}[F] = \prod_{a=1}^n \exp \left\{ \int_{\mathbb{R}} \phi(\lambda, \mu_{h_a}) - \phi(\lambda, \mu_{p_a}) \frac{\delta}{\delta \rho(\lambda)} \right\} \cdot \mathcal{G}[\nu_\rho] \quad \text{with} \quad \nu_\rho = \frac{\beta Z}{2i\pi} + \rho .$$

The effective form factor series II

Archeologist theorem

Every symmetric holomorphic function of n variables $\{\lambda_j\}_1^n$ and n variables $\{\mu_j\}_1^n$ that has the reduction property admits the decomposition with holomorphic functions g_r

$$S_n \left(\begin{array}{c} \{\lambda_j\}_{j=1}^n \\ \{\mu_j\}_{j=1}^n \end{array} \right) = \sum_{r \geq 0} \alpha_r \prod_{a=1}^n e^{g_r(\lambda_a) - g_r(\mu_a)}.$$

$$\mathcal{L}_N(\kappa) =: \sum_{r \geq 0} \alpha_r \exp \left\{ - \int_{-q}^q \ln' [\widehat{E}_-^2(\lambda)] \nu_\rho(\lambda) \right\} \cdot X_N[\nu_\rho, \widehat{E}_-^2]: \quad \widehat{E}_-^2(\lambda) = e^{i x \rho(\lambda) - i t \epsilon(\lambda) + g(\lambda)}$$

$$g(\lambda) = g_r(\lambda) + \int_{\mathbb{R}} d\mu \phi(\mu, \lambda) \frac{\delta}{\delta \rho(\mu)}$$

⊗ : \cdot : operator order "all functional derivative to the left"

$$X_N[\nu, E_-^2] = \sum_{n=0}^N \sum_{\rho_1 < \dots < \rho_n} \sum_{h_1 < \dots < h_n} \left(\frac{2\pi}{L} \right)^{\alpha_c[\nu]} \prod_{a=1}^n \frac{E_-^2(\mu_{h_a})}{E_-^2(\mu_{\rho_a})} \cdot \mathcal{R}_n \left(\begin{array}{c} \{p_a\}; \{\mu_{p_a}\} \\ \{h_a\}; \{\mu_{h_a}\} \end{array} \right) [\nu]$$

where $\xi(\mu_{p_a}) = p_a/L$.

Fredholm determinant representation for $X_N \left[\nu, E_-^2 \right]$

Introduce two sets of parameters $\widehat{\xi}_\lambda(\lambda_j) = j/L$ and $\widehat{\xi}_\mu(\mu_j) = j/L$:

$$\widehat{\xi}_\mu = \frac{p}{2\pi} + \frac{N+1}{2L}, \quad \text{and} \quad \widehat{\xi}_\lambda = \frac{p}{2\pi} + \frac{\nu}{L} + \frac{N+1}{2L}.$$

$$X_N \left[\nu, E_-^2 \right] = \prod_{j=1}^N \left\{ 2 \sin \left[\pi \nu (\lambda_j) \right] \right\}^2 \sum_{n_1 < \dots < n_N} \prod_{j=1}^N \frac{E_-^2(\lambda_j) / E_-^2(\mu_{n_j})}{4\pi^2 \widehat{\xi}'_\mu(\mu_{n_j}) \widehat{\xi}'_\lambda(\lambda_j)} \cdot \det_N^2 \left[\frac{1}{L(\mu_{n_j} - \lambda_k)} \right].$$

⊗ separation of sums is possible

$$X_N \left[\nu, E_-^2 \right] \xrightarrow{N, L \rightarrow +\infty} \det_{[-q; q]} [I + V]$$

⊗ $I + V$ is an integrable integral operator with kernel

$$V(\lambda, \mu) = 4 \frac{\sin[\pi \nu(\lambda)] \sin[\pi \nu(\mu)]}{2i\pi(\lambda - \mu)} \{ E_+(\lambda) E_-(\mu) - E_+(\mu) E_-(\lambda) \}$$

$$E_+(\lambda) = iE_-(\lambda) \left\{ \oint_{\mathbb{R}} \frac{d\mu}{2\pi} \frac{E_-^{-2}(\mu)}{\mu - \lambda} + \frac{E_-^{-2}(\lambda)}{2} \cot \pi \nu(\lambda) \right\}.$$

Natte series for Fredholm determinants

RHP Its, Izergin, Korepin, Slavnov '90

A.E. Deift, Zhou '92 Its, Izergin, Korepin, Varzugin '92

$$\det [I + V] = \det [I + V]^{(0)} \left\{ 1 + \sum_{n \geq 1} \sum_{\Sigma s \ell_s = n} \sum_{\substack{\Sigma_{t_3} \epsilon_i = 0 \\ \epsilon_i \in \{\pm 1, 0\}}} \int_{\mathcal{C}} \frac{d^n z_t}{(2i\pi)^n} H_n(\{\ell\}, \{\epsilon_t\}, \{z_t\}) [v] \prod_{t \in J_{\{\ell\}}} e^{\epsilon_t g(z_t)} \right\}.$$

$E_-^2 = e^{ixu+g}$ and $J_{\{\ell\}}$ set of triplets $((s, p, j), s \in [1; n], p \in [1; \ell_s], j \in [1; s])$.

$$\det [I + V]^{(0)} = \mathcal{B}_x[v, u] \exp \left\{ \int_{-q}^q \ln' [E_-^2(\lambda)] v(\lambda) \right\} \quad \text{with} \quad \mathcal{B}_x[v, u] = \frac{C[v, u]}{x^{v_+^2 + v_-^2}}$$

- Functionals H_n are subdominant $|H_n(\{\ell\}, \{\epsilon_t\}, \{z_t\}) [v]| = o(1)$ in L^1 sense
- Natte series obtained through *algebraic* manipulations on Fredholm series.
- perfectly fit for computing the asymptotic $x \rightarrow +\infty$ expansion of the determinant.

Conclusion and perspectives

Review of the results

- ✓ Method for computing size-asymptotics of form factors ;
- ✓ Construction of Natte series for Fredholm determinant ;
- ✓ Application of Natte series for asymptotic analysis of correlation functions .

Next possible extensions

- ⊗ Field-conjugated field correlator in NLSE ;
- ⊗ Asymptotic behavior for $T > 0K$ and for massive models ;
- ⊗ Applying the method to models with a more complex description of Bethe roots (XXZ) .