Nonultralocal quantum algebra and 1D anyonic quantum integrable models

Anjan Kundu
Theory Group & CAMCS,
Saha Institute of Nuclear Physics
Calcutta, INDIA
anjan.kundusaha.ac.in

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Plan of Talk:

- Exactly solvable 1D boson & anyon models
- Systematic construction through braided YBE
- Quantum Integrable 1D anyon lattice and anyon field models (NLS, derivative NLS)
- Novel anyon quantum group
- Concluding Remarks

Background

- In 3D- only Boson + Fermion
- In 2D- Boson + Fermion + Anyon
- Anyons are receiving intence attention after

  - Experimental confirmation (Through Quantum Hall Effect, PRB'05)

  - Potential applications of nonabelian Anyons (Kitaev, AnnPhys'03,'06) to quantum computation (Due to their *topological stability* against *quantum fluctuation*)

  - Though anyons manifested in 2D

Surprisingly, they retain basic properties, when projected to 1D:
2-particle Anyon wave function in 1D (L= length of chain):
i) **Exchange** :

$$\Phi(x_1, x_2) = e^{-i\theta}\Phi(x_2, x_1)$$

(with $\theta = 0$, Boson & $\theta = \pi$, Fermion)
ii) **Sensitive BC** : (*1 "passes" 2*) $\neq$ *2 "passes" 1*)

$$\Phi(x_1 + L, x_2) = e^{-2i\theta}\ \Phi(x_1, x_2 + L)$$

Advantage: Anyon models in 1D can be *exactly solvable*
Interesting history : Apart from Anyon-type exchange statstics in Calogero model, there are exactly solvable 1D anyon models.
Exactly solvable Bose gases

1) δ-function Bose gas: (Lieb-Liniger, PR'65)

$$H_N^{(1)} = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>} c\delta(x_k - x_l)$$

2) Derivative-δ-function Bose gas: (Snirman et al PR'94)

$$H_N^{(2)} = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>} i\kappa\delta(x_k - x_l)\left((\partial_{x_k} + \partial_{x_l}\right))$$

3) Can there be solvable bosonic models with higher singular potentials like Double δ-function ??

$$\gamma_1 \sum_{<j,k,l>} \delta(x_j - x_k)\delta(x_l - x_k) + \gamma_2 \sum_{<k,l>} (\delta(x_k - x_l))^2.$$

Such attempts were unsuccessful untill the introduction of

Exactly solvable δ-function Anyon gas

(A Kundu, PRL'99)
Equivalent to Double δ- Bose gas with

$$\gamma_1 = \gamma_2 = \kappa^2$$

Subsequently also proposed:
Exactly solvable derivative-δ-function Anyon gas
(A Kundu + M Batchelor *et al* JPA'08)
Presently These 1D -Anyon models became much popular:
Different research Groups:
Korepin (USA), Batchelor (Australia), Girardeau (France), Wang (China), Calabrese (Italy), Santachiara (France))
are actively enganged.
Note: (Drawback): Anyons behave like bosons/fermions at the coinciding points!
(How to find remedy? We will see !)
Quantum Integrable Field models

Well known : *Nonlinear Schödinger equation (NLS)*:

$$H^{(1f)} = \int dx(\psi_x^\dagger \psi_x + c(\psi^\dagger \psi)^2)$$

in bosonic field :$[\psi(x), \psi^\dagger(y)] = \delta(x-y)$
is *Quantum Integrable* :
N-particle sector $\longrightarrow$ δ-Bose gas.
Similarly,*Derivative NLS* is another *Quantum Integrable Field model* :
N-particle sector $\longrightarrow$ Derivative δ-Bose gas.
Therefore *Unsolved problem* !!

i) What are integrable *Anyon QFT models* ,
s.t. N-particle sector $\rightarrow \delta$ & derivative δ - Anyon gases ?
ii) How to construct in a systematic way such novel Anyon Lattice & Field models
through Yang-Baxter Equation?
iii) How to remedy the existing *drawback*

Find Anyon statistics at all points? (including coinciding points $x = y$!)
Note: Anyon CR $\longrightarrow$ Nonultralocality !
Hence, goes beyond standard QIS and YBE !
Our Result:
We resolve above unsolved problems using
Braided Yang-Baxter Equation(BYBE)
(A Kundu + V Hlavaty , IJMPA'96):

$$R(u-v)Z^{-1}L_{1j}(u)ZL_{2j}(v) = Z^{-1}L_{2j}(v)ZL_{1j}(u)R(u-v),$$

at sites $j = 1, 2, \ldots . N$, with the braiding relation (BR) :

$$L_{2k}(v)Z^{-1}L_{1j}(u) = Z^{-1}L_{1j}(u)ZL_{2k}(v)Z^{-1}$$

for $k > j$: *Nonultralocality* ! (noncommutativity at space separated points)= giving Anyon CR !
Note:
- L_j-Lax operator,

- standard R- matrix

$$R(\lambda) = \begin{pmatrix} a(\lambda) & & & \\ & b(\lambda) & c & \\ & c & b(\lambda) & \\ & & & a(\lambda) \end{pmatrix},$$

I) *Rational*

$$a(\lambda) = \lambda + \alpha,\ b(\lambda) = \lambda,\ c = \alpha$$

II) *Trigonometric*

$$a(\lambda) = \sin(\lambda + \alpha),\ b(\lambda) = \sin\lambda,\ c = \sin\alpha$$

They would generate *two-different classes of models*

- *Z-braiding matrix* satisfies additional relations:

$$R_{21}(u) Z_{13} Z_{23} = Z_{23} Z_{13} R_{21}(u),$$

$$Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12}$$

etc.
One of the solutions:

$$Z = \sum_{a,b} e^{i\theta(\hat{a}\cdot\hat{b})} e_{a,a} \otimes e_{b,b}$$

with Anyon Grading (bosonic/anyonic) $\hat{a} = 0, 1$ and and Anyon parameter θ.
We use simplest 4×4-matrix solution using gradings $\hat{1} = 0, \ \hat{2} = 1$:

$$Z = diag(1, 1, 1, e^{i\theta})$$

for $\theta = 0 \rightarrow Z = I$: BYBE $\rightarrow$ YBE :
$R(u-v)L_{1j}(u)L_{2j}(v) = L_{2j}(v)L_{1j}(u)R(u-v)$,
and BR $\rightarrow$ bosonic commutativity
$[L_{2k}(v), L_{1j}(u)] = 0$ (ultralocality), at sites $k \neq j$
Integrable model construction

Define transfer matrix (global object) :

$$\tau(u) = trace_a(L_{a1}(u) \dots L_{aN}(u))$$

generating conserved operators

$$\log \tau(u) = \sum_n C_n u^n$$

BYBE guarantees

$$[\tau(u), \tau(v)] = 0 \longrightarrow \ [C_n, C_m] = 0$$

Hence, (*Quantum Integrability !*)
Hamiltonian of the model = $H = C_n$, n=1, 2, 3,...

A. Class of **quantum integrable Anyon models** with

• known *rational R(u)-matrix*:
BYBE and BR fix through Anyonic Lax operator $L_j(u)$

$$L^b_{a(l)}(\lambda) = \lambda \delta_{ab} \ p^{0(l)}_b + \alpha p^{(l)}_{ba}$$

and different realizations of operators $p_{ba}^{(l)},\ p_b^{0(l)}, a, b = 1, 2$

1) Anyonic CR

2) Intregrable model Hamiltonian

I. *Lattice hard-core anyon model* (Batchelor et al '08)

We construct through above scheme nearest-neighbor interacting Anyon model:

$$C_1 = H^{(1a)} = \sum_{k=1}^{N} 2n_k n_{k+1} + + a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1}, \ \ n_k \equiv a_k^\dagger a_k$$

with Anyonic CR at space-separated points $k > l$:

$$a_k a_l^\dagger = e^{i\theta} a_l^\dagger a_k, \ \ a_k a_l = e^{-i\theta} a_l a_k$$

But fermionic CR at coinciding points: (not Anyonic!= existing drawback)

$$[a_k, a_k^\dagger]_+ = 1,$$

with additional hard-core (Fermionic) constraint $a^2 = 0$, , responsible for regularity condition of Lax operator $L(0) = P$ and hence NN-interacting Hamiltonian!

II. Novel **Anyon lattice model**

At another realization we costruct a new *quantum integrable* lattice Anyon model with:

i) *Next-nearest-neighbor* + higher order nonlinear interactions

$$C_3 = \mathrm{H}^{(2a)} = \sum_k (\psi^\dagger_{k+1}\psi_{k-1} - (n_k + n_{k+1})\psi^\dagger_{k+1}\psi_k + \frac{1}{3\Delta^2} n_k^3,$$

with $n_k = p_k + \Delta^2 \psi^\dagger_k \psi_k$,

Advantage: We get finally the needed *Anyon CR at all points*

i) At coinciding points:

$$\psi_k \psi^\dagger_k - e^{-i\theta} \psi^\dagger_k \psi_k = p_k \frac{1}{\Delta}$$

ii) at seperated points $k > j$.

$$\psi_k \psi_j^\dagger = e^{i\theta} \psi_j^\dagger \psi_k,$$

etc . (Thus we resolve an existing problem with Anyon models)

III. **Quantum Integrable Anyon NLS**

At continuum (field) limit of the above lattice model: (Lattice const. $\Delta \to 0$) $k \to x, \ \psi_k \to A(x)$
we derive *Anyon quantum field NLS model*

$$\hat{H}^{(3a)} = \int dx (A_x^\dagger A_x + c(A^\dagger A)^2)$$

with field operator $A(x)$ satisfying *Anyon CR* at all points
(follow from the BYBE and BR !):
I) At $x = y$:

$$A(x)A^\dagger(y) - e^{i\theta} A^\dagger(y)A(x) = \delta(x-y)$$

II) At $x > y$:

$$A(x)A^\dagger(y) = e^{i\theta} A^\dagger(y)A(x),$$

$$A(x)A(y) = e^{-i\theta} A(y)A(x),$$

interpolating: i) Boson ($\theta = 0$), ii) Fermion ($\theta = \pi$))

N-particle sector

$$|N >= \int d^N x \sum_{\{x_l\}} \Phi(x_1, x_2, \ldots x_N)$$

$$A^\dagger(x_1)A^\dagger(x_2) \cdots A^\dagger(x_n))|0 >$$

of the NLS Anyon quantum field model $\longrightarrow$ δ - Anyon Gas!

$$H_N = -\sum_k \partial_k^2 + c \sum_{k \neq j} \delta(x_k - x_l)$$

• Establishing thus the *missing link* between the well known *anyon gas* (δ-function) and a *Anyon quantum*

field model (NLS)!
Now we construct
B. another class of **quantum integrable Anyon models** using
known *trigonometric* $R_t(u)$*-matrix*
but with same Z-matrix

IV. **q-Anyon model**

From BYBE (following similar construction) we get
Novel *Anyonic q-oscillator*: (with two deformation parameters: $q = e^{i\alpha}$ and *anyon parameter*: $s = e^{i\theta}$)
i) At coinciding points k

$$\phi_k \phi_k^\dagger - e^{i\theta} \ \phi_k^\dagger \phi_k = e^{i\theta N_k} \cos 2\alpha N_k$$

ii) at separated points $k > j$:

$$\phi_k \phi_j^\dagger = e^{i\theta} \phi_j^\dagger \phi_k$$

Not giving details of this model, switch over to its QFT limit

At field limit $\Delta \to 0$: $\phi_k \to D(x)$ we obtain
V. Quantum Integrable **derivative- NLS Anyon field model**

$$\hat{H}^{(4a)} = \int dx (D_x^\dagger D_x + 2i\kappa (D^\dagger)^2 D D_x)$$

with *Anyon field CR*:
I) At $x = y$:

$$D(x)D^\dagger(y) - e^{i\theta} D^\dagger(y)D(x) = \kappa\delta(x-y)$$

II) At $x > y$:

$$D(x)D^\dagger(y) = e^{i\theta} D^\dagger(y)D(x),$$

etc.
N**-particle sector** $|N>$
of this Anyon DNLS field $\longrightarrow$ known δ'- Anyon gas !

$$H_N^d = -\sum_k \partial_k^2 + i\kappa \sum_{k\neq j} \delta(x_k - x_l)(\partial_{x_k} + \partial_{x_l})$$

• Establishing thus the final missing link between δ'-*Anyon gas and Anyon DNLS model* (a QFT model)
Note: All *Anyon models* constructed here (*Anyon lattice and q-oscillator model, NLS and DNLS Anyon field models*) are
i) **Quantum Integrable** and
ii) **Exactly solvable** by *algebraic Bethe Ansatz*.

VI. **Anyon Quantum Group**:
From BYBE and BR with same Z and Trig. R_t-matrix we discover Novel *Anyon Quantum Group* $A_\theta s_q u(2)$ (with two deformation parameters: $q, \quad s = e^{i\theta}$):

$$S^+S^- - sS^+S^+ = [2S^3]_q s^{-S^3},$$

$$q^{S^3}S^{\pm} = q^{\pm 1}S^{\pm}q^{S^3}, \quad s^{S^3}S^{\pm} = s^{\pm 1}S^{\pm}s^{S^3}$$

denoting $[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin \alpha x}{\sin \alpha}$.
Note: Hopf algebra structure :

(*relared to Braided Hopf algebra* of S.Majid)
i.) Unusual braided multiplication:

$$(I \otimes S^{\pm})(S^{\pm} \otimes I) = s^{-1}(S^{\pm} \otimes S^{\pm}),$$

$$(I \otimes S^{\mp})(S^{\pm} \otimes I) = s(S^{\pm} \otimes S^{\mp})$$

ii.) Two-parameter deformed coproduct:

$$\Delta(S^{+}) = q^{-S^3} \otimes S^{+} + S^{+} \otimes q^{S^3} s^{-S^3},$$

$$\Delta(S^{-}) = q^{-S^3} s^{-S^3} \otimes S^{-} + S^{-} \otimes q^{S^3}, \;\; \Delta(S^3) = S^3 \otimes I + I \otimes S^3$$

Remark : *check algebra by direct insertion* !
At $s = 1 \longrightarrow$ standard $su_q(2)$ quantum algebra

$$[S^{-}, S^{+}]_{-} = [2S^3]_q, \quad [S^3, S^{\pm}] = \pm S^3,$$

with usual commutative multiplication

$$(I \otimes S^{\pm})(S^{\pm} \otimes I) = (S^{\pm} \otimes S^{\pm}), ,$$

At $q \to 1$ and arbitrary s

VI. Novel **purely Anyonic-deformed** $A_\theta su(2)$ **algebra**

$$S^+S^- - sS^+S^+ = 2S^3 s^{-S^3}, s^{S^3}S^\pm = s^{\pm 1}S^\pm s^{S^3}$$

with braided multiplication

$$(I \otimes S^\pm)(S^\pm \otimes I) = s^{-1}(S^\pm \otimes S^\pm),$$

etc. and a *non-cocommutative coproduct*!:

$$\Delta(S^+) = I \otimes S^+ + S^+ \otimes s^{-S^3},\ \Delta(S^-) = s^{-S^3} \otimes S^- + S^- \otimes I,$$

etc.

What we have achieved

i) Systematic *construction of 1d Anyon models through Braided YBE*

1a) All models generated are *quantum integrable and*

solvable by algebraic Bethe Ansatz
ii) We have constructed Lattice anyon models:
Known hard-core Anyon and New Lattice Anyon next-NN model + q-oscillator model
iii) Novel Anyon field models: NLS and DNLS, their N-particle sectors gaving known δ and $\delta^{'}$ Anyon gases
iv) We obtain Novel Anyon Quantum Group with unusual Hopf algebra structure

v) **Future problems**:

- Discover Non-Abelian Anyon models by choosing more general Z-matrix!
- Construct Anyonic sine-Gordon model realizing Anyon quantum group!

Thank You!