Correlation functions in the 2D Ising model and Painlevé VI

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Outline

The 2D Ising model

The 2D symmetric Ising model on a square lattice in the ferromagnetic regime is defined by the interaction energy

$$E = -J \sum_{i,j} (\sigma_{i,j}\sigma_{i,j+1} + \sigma_{i,j}\sigma_{i+1,j}), \quad J > 0, \quad \sigma_i = \pm 1$$

$$t = s^4, \quad s = \begin{cases} \sinh(2J/k_B T) \quad , & T > T_c, \\ \sinh(2J/k_B T)^{-1}, & T < T_c. \end{cases}, \quad \sinh(2J/k_B T_c) = 1$$

$$C(M,N) = \langle \sigma_{0,0}\, \sigma_{M,N} \rangle \; - \; \text{pair correlation functions}$$

(Kaufman, Onsager, Montroll, Potts, Ward, McCoy, Wu, Kadanoff, Cheng, Jimbo, Miwa, Baxter, Perk, Ghosh, Shrock, Martinez, ...)

- Determinant and form factor representations for $C(M,N)$.
- Toeplitz determinants for $C(N,N)$ and $C(0,N)$
- Painlevé VI equation for $C(N,N)$

Form factor representation of correlations

The λ-extended (generalized) pair correlation functions

$$C^{+}(M,N;\lambda) = s^{-1}(1-t)^{1/4}\sum_{n=0}^{\infty}\lambda^{2n+1}\widehat{C}^{(2n+1)}(M,N), \quad T > T_c,$$

$$C^{-}(M,N;\lambda) = (1-t)^{1/4}\Big(1+\sum_{n=1}^{\infty}\lambda^{2n}\widehat{C}^{(2n)}(M,N)\Big), \quad T < T_c,$$

$$\widehat{C}^{(n)}(M,N) = \frac{1}{n!}\int_{-\pi}^{\pi}\frac{d\omega_1}{2\pi}\cdots\int_{-\pi}^{\pi}\frac{d\omega_n}{2\pi}\left[\prod_{i=1}^{n}\frac{x_i^M}{\sinh\gamma_i}\right]\left[\prod_{1\le i<j\le n}h_{ij}\right]^2\cos\Big(N\sum_{i=1}^{n}\omega_i\Big),$$

$$\sinh\gamma_i = (y_i^2-1)^{1/2}, \quad x_i = y_i-(y_i^2-1)^{1/2}, \quad y_i = s+s^{-1}-\cos\omega_i,$$

$$h_{ij} = \frac{2(x_ix_j)^{1/2}\sin((\omega_i-\omega_j)/2)}{1-x_ix_j}.$$

$$C^{\pm}(M,N) = C^{\pm}(M,N;1)$$

Jimbo-Miwa approach

In 1980 Jimbo and Miwa introduced the function

$$\sigma_N(t) = \begin{cases} t(t-1)\dfrac{d}{dt}\log C^+(N,N) - \dfrac{1}{4}, & T > T_c, \\ t(t-1)\dfrac{d}{dt}\log C^-(N,N) - \dfrac{t}{4}, & T < T_c. \end{cases}$$

and showed that it satisfies the "sigma" form of Painlevé VI (for a particular choice of parameters)

$$\left(t(t-1)\frac{d^2\sigma}{dt^2}\right)^2 + 4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}\right)\left(t\frac{d\sigma}{dt} - \sigma\right) = N^2\left((t-1)\frac{d\sigma}{dt} - \sigma\right)$$

Jimbo and Miwa also considered an isomonodromic λ-extension of $C^{\pm}(N,N)$ which also satisfies this equation with λ playing the role of initial condition.
In 2007 Bookraa et al. observed that the Jimbo-Miwa λ-extension is the same as the form-factor expansions for $C^{\pm}(N,N;\lambda)$.
A general solution of the above equation can be found for any integer N.

A short review of the Okamoto theory of the PVI equation

For each solution of the $\mathbf{P}_{VI}$ equation, one can construct a Hamiltonian function $H(t) \equiv H(t; \mathbf{b})$ depending on the parameters

$$\mathbf{b} = (b_1, b_2, b_3, b_4)$$

A related tau-function $\tau(t) \equiv \tau(t; \mathbf{b})$ defined by $H(t; \mathbf{b}) = \frac{d}{dt} \log \tau(t; \mathbf{b})$. The auxiliary hamiltonian

$$h(t) = t(t-1)H(t) + e_2(b_1, b_3, b_4)\, t - \frac{1}{2} e_2(b_1, b_2, b_3, b_4)$$

solves the $\mathbf{E}_{VI}[\mathbf{b}]$ equation

$$h'(t)\Big[t(1-t)h''(t)\Big]^2 + \Big[h'(t)[2h(t)-(2t-1)h'(t)] + b_1 b_2 b_3 b_4\Big]^2 = \prod_{k=1}^{4}\Big(h'(t) + b_k^2\Big)$$

The group G of Backlund transformations of $\mathbf{P}_{VI}$ is isomorphic to the affine Weyl group of type F_4: $W_a(F_4)$.

It contains the following transformations of parameters

$$w_1 : b_1 \leftrightarrow b_2, \quad w_2 : b_2 \leftrightarrow b_3, \quad w_3 : b_3 \leftrightarrow b_4, \quad w_4 : b_3 \rightarrow -b_3, b_4 \rightarrow -b_4$$

$$x^1 : \kappa_0 \leftrightarrow \kappa_1, \quad x^2 : \kappa_0 \leftrightarrow \kappa_\infty, \quad x^3 : \kappa_0 \leftrightarrow \theta$$

$$\kappa_0 = b_1 + b_2, \quad \kappa_1 = b_1 - b_2, \quad \kappa_\infty = b_3 - b_4, \quad \theta = b_3 + b_4 + 1$$

and the parallel transformation

$$l_3 : \mathbf{b} \equiv (b_1, b_2, b_3, b_4) \rightarrow \mathbf{b}^+ \equiv (b_1, b_2, b_3 + 1, b_4).$$

For any $s \in W_a(F_4)$

$$s : \mathbf{b} \mapsto \mathbf{b_s}$$

one can construct another $h(t; \mathbf{b_s})$ satisfying $\mathbf{E}_{VI}[\mathbf{b_s}]$ and $h(t; \mathbf{b_s})$ is a rational function of t and

$$\left\{ h(t; \mathbf{b}),\ \frac{d}{dt} h(t; \mathbf{b}),\ \frac{d^2}{dt^2} h(t; \mathbf{b}) \right\}$$

Consider a sequence of parameters

$$\mathbf{b}_N \equiv (b_1 + N/2, b_2 - N/2, b_3 + N/2, b_4 + N/2)$$

and associated sequences of Hamiltonians and tau-functions

$$h_N(t) = h(t, \mathbf{b}_N), \quad H_N(t) = H(t, \mathbf{b}_N), \quad \tau_N(t) = \tau(t, \mathbf{b}_N)$$

Following the Okamoto derivation of Toda relations one can show that

$$t \frac{d^2}{dt^2} \log \tau_N(t) + \frac{d}{dt} \log \tau_N(t) = B_N \frac{\tau_{N+1}(t)\, \tau_{N-1}(t)}{\tau^2(t)}.$$

Choosing

$$b_1 = -1/2, \quad b_2 = -1/2, \quad b_3 = 0, \quad b_4 = 0$$

and comparing the equations for $h_N(t)$ and $\sigma_N(t)$ we obtain

$$\tau_N(t) = \begin{cases} C_N^+(t)(1-t)^{-N^2} t^{\frac{N}{2}}, & T > T_c \\ C_N^-(t)(1-t)^{-N^2} t^{\frac{N}{2} - \frac{1}{4}}, & T < T_c. \end{cases}$$

Toda equations for diagonal correlation functions

The resulting Toda equation for $C^{\pm}(t)$

$$t\,\frac{d^2}{dt^2}\log C_N^{\pm}(t)+\frac{d}{dt}\log C_N^{\pm}(t)+\frac{N^2}{(1-t)^2}=\frac{\left(N^2-\frac{1}{4}\right)}{(1-t)^2}\,\frac{C_{N+1}^{\pm}(t)\,C_{N-1}^{\pm}(t)}{(C_N^{\pm}(t))^2}$$

For $\lambda=1$ case initial conditions are

$$C_0^{\pm}(t)=1,\quad C_1^{-}(t)=E(t),\quad C_1^{+}(t)=t^{-1/2}\left[(t-1)K(t)+E(t)\right]$$

For generic λ and the case $N=0$, the choice $\mathbf{b}=(-1/2,-1/2,0,0)$ corresponds to the Picard case of PVI where the general solution is known in terms of the Weierstrass $\mathcal{P}$-function.

The corresponding tau-function (depending on two initial conditions x and y) has been recently calculated (Brezhnev (2009) and VM (2010)). It gives

$$C_0^{\pm}(t)=c_0(x,y)\frac{(1-t)^{1/4}}{t^{1/4}}\,q^{y^2/\pi^2}\frac{\theta_1(x+\tau y|\tau)}{\theta_4(0|\tau)},\qquad t=k^2(\tau)$$

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For $\lambda = 1$ case initial conditions are

$$C_0^{\pm}(t) = 1, \quad C_1^{-}(t) = E(t), \quad C_1^{+}(t) = t^{-1/2}\left[(t-1)K(t) + E(t)\right]$$

For generic λ and the case $N = 0$, the choice $\mathbf{b} = (-1/2, -1/2, 0, 0)$ corresponds to the Picard case of PVI where the general solution is known in terms of the Weierstrass $\mathcal{P}$-function.

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We need to find relations connecting initial conditions:

$$\text{for}\, T > T_c: \quad c_0(x,y) = 1,\ \lambda = \sin(x),\ y = 0,$$

$$\text{for}\, T < T_c: \quad c_0(x,y) = -i\exp(ix),\ \lambda = \sin(x),\ y = \pi/2,$$

$$C_0^+(t) = \frac{(1-t)^{1/4}}{t^{1/4}}\frac{\theta_1(\arcsin(\lambda)|\tau)}{\theta_4(0|\tau)}, \qquad C_0^-(t) = \frac{\theta_4(\arcsin(\lambda)|\tau)}{\theta_3(0|\tau)}$$

The result for $C_0^-(t)$ was first conjectured by Booukraa et al. in (2007) based on long series numerical expansions.

For generic λ we have a symmetry $N \leftrightarrow -N$ which allows to calculate $C_1^\pm(t)$:

$$C_1^+(t) = \frac{1}{\cos(x)}\frac{\theta_4(x|\tau)}{\theta_3(0|\tau)}\frac{X}{t^{1/2}}, \quad C_1^-(t) = \frac{1}{\cos(x)}\frac{\theta_4(x|\tau)}{\theta_3(0|\tau)}(\mathrm{cn}(z,t)\mathrm{dn}(z,t) + \mathrm{sn}(z,t)X)$$

$$x = \arcsin(\lambda), \quad z = xK = \frac{2x\mathrm{K}(t)}{\pi}, \quad X = \frac{1}{K}\left[\log\theta_4(x|\tau)\right]'_x$$

Toeplitz determinant representation

Toeplitz determinant representation of $C_N^{\pm}(t)$ for $\lambda = 1$.

$$C_N^+(t) = \det |a_{i-j}(t)|_{i,j=1,\ldots,N}, \quad C_N^-(t) = (-1)^N \det |a_{i-j-1}(t)|_{i,j=1,\ldots,N},$$

where $a_n(t)$ are given explicitly in terms of the hypergeometric function and polynomials in $K(t), E(t)$ and t:

$$a_n(t) = K(t)p_n(t) + E(t)q_n(t)$$

Let M be the $(N+1) \times (N+1)$ matrix $M = (M_{ij})$, $D = Det(M)$, $D = D[i_1, i_2, \ldots; j_1, j_2, \ldots]$ is the minor determinant with $i_k's$ rows and $j_k's$ columns removed. The Plucker relation

$$D\, D[N, N+1; N, N+1] = D[N+1; N+1]\, D[N; N] - D[N; N+1]\, D[N+1; N]$$

Identify

$$D = C_{N+1}^+(t),\ D[N+1, N+1] = C_N^+(t),\ D[N, N+1; N, N+1] = C_{N-1}^+(t)$$

$$D\, D[N, N+1; N, N+1] = D[N+1; N+1]\, D[N; N] - D[N; N+1]\, D[N+1; N]$$

Then for $D[N+1; N+1] = C_N^+(t)$ we have

$$D[N; N] = \mathcal{D}_N^F(t) C_N^+(t),$$

$$D[N; N+1] = \mathcal{D}_N^A(t) C_N^+(t),$$

$$D[N+1; N] = \mathcal{D}_N^B(t) C_N^+(t),$$

where

$$\mathcal{D}_N^A(t) = \frac{1}{2N+1}\left[N\, t^{-1/2} + 2t^{1/2}(t-1)\partial_t\right]$$

$$\mathcal{D}_N^B(t) = \frac{1}{2N-1}\left[-N\, t^{-1/2} + 2t^{1/2}(t-1)\partial_t\right]$$

$$\mathcal{D}_N^F(t) = \frac{1}{N^2 - 1/4}\left[(1-(4t)^{-1})N^2 + (t-1)^2(t\partial_t^2 + \partial_t)\right]$$

Substituting back to the Plucker relation we immediately obtain Toda equation for diagonal correlations.