Symmetries of spin systems and Birman-Wenzl-Murakami algebra

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Abstract

We consider integrable open spin chains related to the quantum affine algebras $\mathcal{U}_q(\widehat{o(3)})$ and $\mathcal{U}_q(A_2^{(2)})$. We discuss the symmetry algebras of these chains with the local $\mathbb{C}^3$ space related to the Birman-Wenzl-Murakami algebra. The symmetry algebra and the Birman-Wenzl-Murakami algebra centralize each other in the representation space $\mathcal{H} = \otimes_1^N \mathbb{C}^3$ of the system, and this determines the structure of the spin system spectra. Consequently, the corresponding multiplet structure of the energy spectra is obtained.

In collaboration with Petr P. Kulish and Zoltán Nagy.

1. Introduction

In the case of the isotropic Heisenberg chain of spin $1/2$ the symmetry algebra is sl_2, the Hamiltonian is an element of the group algebra $\mathbb{C}[\mathfrak{S}_N]$ of the symmetric group $\mathfrak{S}_N$. The RLL-relations define an infinite dimensional quantum algebra – the Yangian $\mathcal{Y}(sl_2)$. The actions of sl_2 and $\mathfrak{S}_N$ on the state of space $\mathcal{H} = \otimes_1^N \mathbb{C}^2$ are mutually commuting (the Schur-Weyl duality). Here we consider a generalization to the case of the Birman-Wenzl-Murakami (BMW) algebra [1] and its specific representations in $\mathbb{C}^3 \otimes \mathbb{C}^3$ given by the spectral parameter dependent R-matrices. These R-matrices correspond to different quantum affine algebras $\mathcal{U}_q(\widehat{o(3)})$, $\mathcal{U}_q(A_2^{(2)})$, $\mathcal{U}_q(\widehat{osp(1|2)})$ and $\mathcal{U}_q(sl(1|2)^{(2)})$. Although corresponding spin systems were analyzed in a variety of papers, we stress the connection of the open spin chains with the BMW algebra as a centralizer of the symmetry algebra and thus unveil the multiplet structure of the energy spectra of the corresponding Hamiltonians.

For the XXZ-model of spin 1 the appropriate dynamical symmetry algebra is $\mathcal{U}_q(\widehat{o(3)})$ and its symmetry algebra is $\mathcal{U}_q(o(3))$ [2]. The corresponding R-matrix was found in [3], see also [4].

The R-matrix of $\mathcal{U}_q(A_2^{(2)})$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ was found in [5] and the corresponding periodic spin chain was solved by algebraic Bethe ansatz in [6].

These two spectral parameter dependent R-matrices are the two versions of the Yang-Baxterization procedure for a given representation of the BMW algebra $W_2(q, \nu = q^{-2})$ in $\mathbb{C}^3 \otimes \mathbb{C}^3$ [7, 8].

The two additional R-matrices related to the quantum affine super-algebras can be obtained by considering the BMW algebra $W_2(-q, -q^{-2})$. In this case the BMW algebra is the centralizer of the $\mathcal{U}_q(osp(1|2))$ action in the tensor product of its fundamental representation.

2. R-matrix of XXZ spin-1chain

The 9×9 R-matrix $R(\lambda, \eta)$ of the XXZ-chain of spin one satisfies the Yang-Baxter equation (YBE) in the space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and has the following properties:

- the $U(1)$ symmetry $[h_1 + h_2, R_{12}(\lambda, \eta)] = 0$
- the regularity condition at $\lambda = 0$ $R(0, \eta) = \sinh(\eta)\sinh(2\eta)\mathcal{P}$
- the unitarity $R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)\mathbb{1}$
- the PT-symmetry $R_{12}^t(\lambda) = R_{21}(\lambda)$
- the crossing symmetry $R(\lambda) = (Q \otimes \mathbb{1})R^{t_2}(-\lambda - \eta)(Q \otimes \mathbb{1})$,

  where t_2 denotes the transpose in the second space and the matrix Q is given by $Q = \begin{pmatrix} 0 & 0 & -e^{-\eta} \\ 0 & 1 & 0 \\ -e^{\eta} & 0 & 0 \end{pmatrix}$.

In the braid group form $\check{R}(\lambda, \eta) = \mathcal{P}R(\lambda, \eta)$, can be expressed as

$$\check{R}(\lambda, \eta) = \frac{e^{\eta}}{4}\left(e^{2\lambda} - 1\right)\check{R}(\eta) + (\sinh\eta\sinh 2\eta)\,\mathbb{1} + \frac{e^{-\eta}}{4}\left(e^{-2\lambda} - 1\right)\check{R}^{-1}(\eta).$$

The constant R-matrix $\check{R}^{\pm 1}(\eta) = \lim_{\lambda\to\pm\infty}\left(4\exp(\mp(2\lambda + \eta))\check{R}(\lambda, \eta)\right)$ is a solution of the YBE in the braid group form $\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$, and has the spectral decomposition $(q = e^{2\eta})$ $\check{R}(\eta) = qP_5(\eta) - \frac{1}{q}P_3(\eta) + \frac{1}{q^2}P_1(\eta)$. It is related to the quantum group $\mathcal{U}_q(o(3))$.

To establishing a relation with the BMW algebra, the projector $P_1(\eta)$ is related to the rank one matrix $\mathcal{E}(\eta) = \mu P_1(\eta)$, with $\mu = q + 1 + 1/q$. Then

$$\begin{aligned} \mathcal{E}^2(\eta) &= \mu\mathcal{E}(\eta), \\ \check{R}(\eta)\mathcal{E}(\eta) &= \mathcal{E}(\eta)\check{R}(\eta) = \frac{1}{q^2}\mathcal{E}(\eta), \\ \check{R}(\eta) - \check{R}^{-1}(\eta) &= \omega(q)\left(\mathbb{1} - \mathcal{E}(\eta)\right), \end{aligned}$$

where $\omega(q) = q - 1/q$. Thus $\check{R}, \check{R}^{-1}$ and $\mathcal{E}$ provide a realization of the BMW algebra $W_N(q, 1/q^2)$ in the space $\mathcal{H} = \otimes_1^N \mathbb{C}^3$.

3. Izergin-Korepin R-matrix

The Izergin-Korepin R-matrix $R(\lambda, \eta)$ satisfies the YBE and has the following properties:

- the $U(1)$ symmetry $[h_1 + h_2, R_{12}(\lambda, \eta)] = 0$
- the regularity condition at $\lambda = 0$ $R(0, \eta) = -2\cosh(3\eta)\sinh(2\eta)\mathcal{P}$
- the unitarity $R_{12}(\lambda)R_{21}(-\lambda) = \widetilde{\rho}(\lambda)\mathbb{1}$
- the PT-symmetry $R_{12}^t(\lambda) = R_{21}(\lambda)$
- the crossing symmetry $R(\lambda) = (Q \otimes \mathbb{1})R^{t_2}(-\lambda + 6\eta + \imath\pi)(Q \otimes \mathbb{1})$,
  the matrix Q is given by above.

In the braid group form this R-matrix can be expressed as

$$\check{R}(\lambda, \eta) = \frac{e^{3\eta}}{2}\left(1 - e^{-\lambda}\right)\check{R}(\eta) - 2(\cosh 3\eta\sinh 2\eta)\mathbb{1} - \frac{e^{-3\eta}}{2}\left(1 - e^{\lambda}\right)\check{R}^{-1}(\eta),$$

where the constant R-matrix is defined ptrviously.

At the degeneration point $\lambda = 4\eta$ this R-matrix is proportional to the rank 3 projector $P_3(\eta)$ which is a q-analogue of the antisymmetrizer in $\mathbb{C}^3 \otimes \mathbb{C}^3$. One can further obtain the rank one antisymmetrizer in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ as $\mathcal{A}_3 \simeq \check{R}_{12}(4\eta, \eta)\check{R}_{23}(8\eta, \eta)\check{R}_{12}(4\eta, \eta)$. It can be used to define a quantum determinant q-det$L(\lambda)$ of operator valued L-matrix $L(\lambda)$ satisfying the RLL-relation

$$\check{R}_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_1(\mu)L_2(\lambda)\check{R}_{12}(\lambda - \mu).$$

In this case, the quantum determinant is given by

$$\text{q-det}L(\lambda) \simeq \check{R}_{12}(4\eta, \eta)\check{R}_{23}(8\eta, \eta)\check{R}_{12}(4\eta, \eta)L_1(\lambda)L_2(\lambda - 4\eta)L_3(\lambda - 8\eta).$$

Finally, we stress that the XXZ_1 and $A_2^{(2)}$ R-matrices have different scaling limits. The $A_2^{(2)}$ R-matrix in the limit $\lambda \to \epsilon\lambda$, $\eta \to \epsilon\eta$ and $\epsilon \to 0$ yields the $sl(3)$-Yang R-matrix $R(\lambda, \eta) = \lambda\mathbb{1} - 4\eta\mathcal{P}$, while in the XXZ_1 case the limit yields $R(\lambda, \eta) = \lambda(\lambda + \eta)\mathbb{1} + 2\eta(\lambda + \eta)\mathcal{P} + 2\lambda\eta\mathcal{K}$, where $\mathcal{K}$ is a rank 1 matrix, invariant with respect to the group $O(3)$.

Also, in the quasi-classical limit $\eta \to 0$ these two trigonometric R-matrices also yield different classical r-matrices.

4. Birman-Wenzl-Murakami algebra $W_N(q, \nu)$

The defining relations of the BMW algebra $W_N(q, \nu)$, for the generators 1, σ_i , σ_i^{-1} and e_i, $i = 1, \ldots, N - 1$, are

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \ \text{for } |i - j| > 1, \\ e_i\sigma_i &= \sigma_i e_i = \nu e_i, \quad e_i\sigma_{i-1}^{\pm 1}e_i = \nu^{\mp 1}e_i, \\ \sigma_i - \sigma_i^{-1} &= \omega(q)(1 - e_i), \quad \text{where } \omega(q) = q - 1/q. \end{aligned}$$

It can be shown that its dimension is $(2N - 1)!!$. Many useful relations follow from the definition above, for example $e_i^2 = \mu e_i$, with $\mu = \frac{(q - \nu)(\nu + 1/q)}{\nu\omega}$, and also a cubic relation $(\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0$.

The Yang-Baxterization procedure yields two spectral parameter dependent elements

$$\sigma_i^{(\pm)}(u) = \frac{1}{\omega}\left(u^{-1}\sigma_i - u\,\sigma_i^{-1}\right) + \frac{\nu \pm q^{\pm 1}}{u\nu \pm q^{\pm 1}u^{-1}}e_i.$$

They satisfy the YBE in the braid group form

$$\sigma_i^{(\pm)}(u)\sigma_{i+1}^{(\pm)}(uv)\sigma_i^{(\pm)}(v) = \sigma_{i+1}^{(\pm)}(v)\sigma_i^{(\pm)}(uv)\sigma_{i+1}^{(\pm)}(u).$$

Their unitarity relation is $\sigma_i^{(\pm)}(u)\sigma_i^{(\pm)}(u^{-1}) = \left(1 - \omega^{-2}(u - u^{-1})^2\right)$. These elements are normalized so that $\sigma_i^{(\pm)}(\pm 1) = \pm 1$. We set $\nu = 1/q^2$ and find that $\sigma_i^{(-)}(e^{-\lambda}) \simeq \check{R}_{i,i+1}(\lambda, \eta)$ of XXZ_1 and $\sigma_i^{(+)}(e^{\lambda/2}) \simeq \check{R}_{i,i+1}(\lambda, \eta)$ of Izergin-Korepin.

The irreducible representations of the BMW algebra $W_N(q, \nu)$ are more complicated than the irreducible representations of the symmetric group $\mathcal{S}_N$ or the Hecke algebra $\mathrm{H}_N(q)$, although they can be parameterized by the Young diagrams.

The simplest, one-dimensional irreducible representations are defined by the symmetrizer and antisymmetrizer, respectively. The symmetrizer is given by

$$\mathcal{S}_N = \frac{1}{[N]_q!}\sigma_1^{(-)}(q^{-1})\sigma_2^{(-)}(q^{-2})\cdots\sigma_{N-1}^{(-)}(q^{-(N-1)})\mathcal{S}_{N-1},$$

with $\mathcal{S}_1 = 1$ and $\mathcal{S}_2 = \frac{1}{[2]_q}\sigma_1^{(-)}(q^{-1})$. We use the standard notation for the q-factorial $[n]_q! = [n]_q[n - 1]_q\cdots[2]_q[1]_q$ and the q-numbers $[n]_q = (q^n - q^{-n})/(q - q^{-1})$.

The elements $\mathcal{S}_n$, $n = 1, \ldots, N$ are idempotents, i.e. $\mathcal{S}_n^2 = \mathcal{S}_n$. In addition, the symmetrizer $\mathcal{S}_N$ is also central.

In the realization of the BMW algebra $W_2(q, q^{-2})$ on $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\sigma_1 = \check{R}(\eta) = qP_5 - q^{-1}P_3 + q^{-2}P_1, \quad \text{and} \quad e_1 = \mu P_1 = (q + 1 + q^{-1})P_1.$$

Thus $\mathcal{S}_2 \propto \sigma_1^{(-)}(q^{-1}) = (q + q^{-1})P_5$, and $\sigma_1^{\pm 1}P_5 = q^{\pm 1}P_5$, $e_1 P_5 = 0$.

Similarly, the antisymmetrizer of the $W_N(q, \nu)$ is given by

$$\mathcal{A}_N = \frac{1}{[N]_q!}\sigma_1^{(+)}(q)\sigma_2^{(+)}(q^2)\cdots\sigma_{N-1}^{(+)}(q^{N-1})\mathcal{A}_{N-1},$$

with $\mathcal{A}_1 = 1$ and $\mathcal{A}_2 = \frac{1}{[2]_q}\sigma_1^{(+)}(q)$. The elements $\mathcal{A}_n$, $n = 1, \ldots, N$ are idempotents and the antisymmetrizer $\mathcal{A}_N$ is also central. Also

$$\mathcal{A}_3 \simeq \sigma_1^{(+)}(q)\sigma_2^{(+)}(q^2)\sigma_1^{(+)}(q) = \sigma_2^{(+)}(q)\sigma_1^{(+)}(q^2)\sigma_2^{(+)}(q).$$

In the realization on $\mathbb{C}^3 \otimes \mathbb{C}^3$

$$\sigma_1^{(+)}(q) = [2]_q P_3, \quad \text{and} \quad \sigma_1^{\pm 1}P_3 = -q^{\mp 1}P_3, \quad e_1 P_3 = 0.$$

In the Izergin-Korepin realization (with $\sigma_i^{(+)}(e^{\lambda/2}) \simeq \check{R}_{i,i+1}(\lambda, \eta)$) the antisymmetrizer $\mathcal{A}_3$ has rank one, as it was already noticed.

A straightforward calculation yields $\mathcal{A}_4 = 0$. Consequently all the higher antisymmetrizers vanish identically, $\mathcal{A}_n \equiv 0$, for $n > 4$.

5. Open Spin Chain

The R-matrix $R(u, q)$ can be used to construct an L-operator for an integrable spin system$L_{0j}(u) = R_{0j}(u, q)$, in the case of XXZ_1 $u = \exp(-\lambda)$. Then the monodromy matrix of a spin chain with N sites is

$$T(u) = L_{0\,N}(u)L_{0\,N-1}(u)\cdots L_{0\,1}(u).$$

For integrable spin chains with non-periodic boundary condition one has to use the Sklyanin [9] formalism. The monodromy matrix $\mathcal{T}(u)$ consists of the two matrices $T(u)$ and a reflection matrix $K^-(u) \in \mathrm{End}(V)$

$$\mathcal{T}(u) = T(u)K^-(u)T^{-1}(u^{-1}).$$

The generating function $\tau(u)$ of the integrals of motion is the trace of $\mathcal{T}(u)$ over the auxiliary space with an extra reflection matrix $K^+(u)$

$$\tau(u) = \mathrm{tr}_0\left(K_0^+(u)\mathcal{T}(u)\right).$$

The reflection matrices $K^{\pm}(u)$ are solutions of the reflection equation with a property $K^-(1) = \mathbb{1} \in \mathrm{End}(V)$ and $\tau(1) \simeq \mathbb{1}$.

The Hamiltonian of the open chain is given by $H = \frac{1}{2}\frac{d}{du}\ln\tau(u)|_{u=1}$,

$$H = \sum_{i=1}^{N-1}\check{R}'_{i,i+1}(1) + \frac{\mathrm{tr}_0 K_0^+(1)\check{R}'_{N\,0}(1)}{\mathrm{tr}_0 K_0^+(1)} + \frac{1}{2}\left(\frac{dK_1^-(1)}{du} + \frac{1}{\mathrm{tr}_0 K_0^+(1)}\frac{d\,\mathrm{tr}_0 K_0^+(1)}{du}\right).$$

The Hamiltonian density $h_{i,i+1} = \frac{d}{du}\check{R}_{i,i+1}(u)|_{u=1}$ is a function of the generators of $W_N(q, q^{-2})$ on the space $\mathcal{H} = \otimes_1^N \mathbb{C}^3$. The two extra boundary terms are contributions from the two reflection matrices $K^{\pm}(u)$ at the sites 1 and N. We can take the constant K-matrices $K^-(u) = 1$ and $K^+(u) = Q^t Q$. It is easy to check that a non-zero contribution at the site N is proportional to the identity, hence it does not influence the structure of the spectrum.

In the space $\mathcal{H}$ algebras $\mathcal{U}_q(o(3))$ and $W_N(q, q^{-2})$ are mutual centralizers. This induces the decomposition of the representation space $\mathcal{H}$ into direct sum of irreducible representations of both algebras, as a generalisation of the Schur-Weyl duality,

$$\mathcal{H} = \bigotimes_1^N \mathbb{C}^3 = \sum_{s=0}^{N} \mathbb{C}^{2s+1} \otimes U_s,$$

where $\mathbb{C}^{2s+1}$ is an irrep. of $\mathcal{U}_q(o(3))$ while U_s is some irrep. of $W_N(q, q^{-2})$. The dimension of an irreducible representation of $W_N(q, q^{-2})$ is equal to the multiplicity of the corresponding irreducible representation of centralizer algebra $\mathcal{U}_q(o(3))$, and vice versa

$$m(\mathbb{C}^{2s+1}) = \dim U_s, \quad m(U_s) = \dim \mathbb{C}^{2s+1}.$$

The above decomposition permits to determine the structure of the multiplets of the Hamiltonian

$$H = \sum_{i=1}^{N-1} h_{i,i+1}, \quad h_{i,i+1} = \frac{d}{d\lambda}\check{R}(\lambda, \eta)|_{\lambda=0} = f(\check{R}_i) \in W_N(q, q^{-2}).$$

The R-matrices define the local Hamiltonian density for two sites of the corresponding spin chains. For the XXZ_1-model one gets

$$h_{XXZ} = \frac{d}{d\lambda}\check{R}(\lambda, \eta)|_{\lambda=0} \simeq q\check{R}(\eta) - \check{R}^{-1}(\eta).$$

In the $A_2^{(2)}$-case one gets

$$h_A = \frac{d}{d\lambda}\check{R}(\lambda, \eta)|_{\lambda=0} \simeq q\check{R}(\eta) + \frac{1}{q^2}\check{R}^{-1}(\eta).$$

Let us consider the case of $N = 3$ when the algebra $W_3(q, q^{-2})$ is realized on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and the corresponding Hamiltonians ($H = h_{12} + h_{23}$) are:

$$H_{XXZ}\mathcal{S}_3 = 2(q + 1 + \frac{1}{q})\mathcal{S}_3, \quad H_{XXZ}\mathcal{A}_3 = 2\mathcal{A}_3$$

$$H_A\mathcal{S}_3 = 2(q^2 + \frac{1}{q^3})\mathcal{S}_3, \quad H_A\mathcal{A}_3 = -2(1 + \frac{1}{q})\mathcal{A}_3.$$

In this case there are four irreps. of W_3: two one-dimensional irreps. generated by $\mathcal{S}_3$ and $\mathcal{A}_3$, respectively, the three-dim. irrep. d_3 (corresponding to the one-box Young diagram) and the two-dim. irrep. d_2 (corresponding to the three-box Young diagram with two rows).

Thus for $N = 3$ the above Hamiltonians can have up to seven distinct eigenvalues. Their multiplicities are obtained from the correspondence between the irreps of W_3 and $\mathcal{U}_q(o(3))$: $U(\mathcal{S}_3) \sim \mathbb{C}^7$, $U(\mathcal{A}_3) \sim \mathbb{C}^1$ $U(d_3) \sim \mathbb{C}^3$ $U(d_2) \sim \mathbb{C}^5$. The degeneracies of corresponding eigenvalues are $m(\epsilon(\mathcal{S}_3)) = 7$, $m(\epsilon(\mathcal{A}_3)) = 1$, $m(\epsilon_j(d_3)) = 3$, $m(\epsilon_k(d_2)) = 5$, $j = 1, 2, 3$; $k = 1, 2$. The eigenvalues of the XXZ_1 Hamiltonian are

$$\begin{aligned} \epsilon(\mathcal{S}_3) &= 2(q + 1 + \frac{1}{q}), \quad \epsilon(\mathcal{A}_3) = 2, \\ \epsilon_1(d_3) &= 1, \quad \epsilon_{2,3}(d_3) = \left(\frac{1}{2} \pm \sqrt{\frac{1}{2} + 2(q + 3 + \frac{1}{q})}\right), \\ \epsilon_1(d_2) &= (q + 1 + \frac{1}{q}), \quad \epsilon_2(d_2) = (q + 3 + \frac{1}{q}). \end{aligned}$$

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Recent Advances in Quantum Integrable Systems–RAQIS'10, 15-18 June 2010 LAPTH, Annecy-le-Vieux, France