Correlation functions of integrable spin chains with boundaries

Chihiro Matsui

Department of Physics, The University of Tokyo

RAQIS'10, Annecy
June 18, 2010

Outline

Correlation functions in multi-integral forms

- The spin-$\frac{1}{2}$ XXZ model
  - VO: Jimbo, Miki, Miwa, Nakayashiki (1992)
  - qKZ equations: Jimbo, Miwa (1996)
  - ABA: Kitanine, Maillet, Terras (1999)

Correlation functions in multi-integral forms

- The spin-$\frac{1}{2}$ XXZ model
  - VO: Jimbo, Miki, Miwa, Nakayashiki (1992)
  - qKZ equations: Jimbo, Miwa (1996)
  - ABA: Kitanine, Maillet, Terras (1999)
- The spin-$\frac{1}{2}$ XXZ model with boundary
  - VO: Jimbo, Kedem, Kojima, Konno, Miwa (1995)
  - ABA: Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras (2007)

Correlation functions in multi-integral forms

- The spin-$\frac{1}{2}$ XXZ model
  - VO: Jimbo, Miki, Miwa, Nakayashiki (1992)
  - qKZ equations: Jimbo, Miwa (1996)
  - ABA: Kitanine, Maillet, Terras (1999)
- The spin-$\frac{1}{2}$ XXZ model with boundary
  - VO: Jimbo, Kedem, Kojima, Konno, Miwa (1995)
  - ABA: Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras (2007)
- The XXZ model at finite temperature
  - ABA: Göhmann, Klümper, Seel (2004)

Correlation functions in multi-integral forms

- The spin-$\frac{1}{2}$ XXZ model
  - VO: Jimbo, Miki, Miwa, Nakayashiki (1992)
  - qKZ equations: Jimbo, Miwa (1996)
  - ABA: Kitanine, Maillet, Terras (1999)
- The spin-$\frac{1}{2}$ XXZ model with boundary
  - VO: Jimbo, Kedem, Kojima, Konno, Miwa (1995)
  - ABA: Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras (2007)
- The XXZ model at finite temperature
  - ABA: Göhmann, Klümper, Seel (2004)
- The spin-1 XXZ (XXX) model
  - VO: Idzumi (1994)
  - ABA: Kitanine (2001)
- The spin-s XXZ model
  - ABA: Deguchi, C. M. (2009)

Correlation functions in multi-integral forms

- The spin-$\frac{1}{2}$ XXZ model
  - VO: Jimbo, Miki, Miwa, Nakayashiki (1992)
  - qKZ equations: Jimbo, Miwa (1996)
  - ABA: Kitanine, Maillet, Terras (1999)
- The spin-$\frac{1}{2}$ XXZ model with boundary
  - VO: Jimbo, Kedem, Kojima, Konno, Miwa (1995)
  - ABA: Kitanine, Kozlowski, Maillet, Niccoli, Slavnov, Terras (2007)
- The XXZ model at finite temperature
  - ABA: Göhmann, Klümper, Seel (2004)
- The spin-1 XXZ (XXX) model
  - VO: Idzumi (1994)
  - ABA: Kitanine (2001)
- The spin-s XXZ model
  - ABA: Deguchi, C. M. (2009)

Algebraic Bethe ansatz

The R-matrix to the spin-$\frac{1}{2}$ representation of $\mathcal{U}_q(sl_2)$

$$R_{0j}(\lambda;\xi_j) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{0j} & c_{0j} & 0 \\ 0 & c_{0j} & b_{0j} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{[0j]} \qquad \begin{aligned} b_{0j} &:= \frac{\sinh(\lambda-\xi_j)}{\sinh(\lambda-\xi_j+\eta)} \\ c_{0j} &:= \frac{\sinh\eta}{\sinh(\lambda-\xi_j+\eta)}\,, \end{aligned}$$

Algebraic Bethe ansatz

The R-matrix to the spin-$\frac{1}{2}$ representation of $\mathcal{U}_q(sl_2)$

$$R_{0j}(\lambda;\xi_j) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{0j} & c_{0j} & 0 \\ 0 & c_{0j} & b_{0j} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{[0j]} \qquad \begin{aligned} b_{0j} &:= \frac{\sinh(\lambda-\xi_j)}{\sinh(\lambda-\xi_j+\eta)} \\ c_{0j} &:= \frac{\sinh\eta}{\sinh(\lambda-\xi_j+\eta)} \end{aligned},$$

satisfies the Yang-Baxter equation

$$\begin{aligned} & R_{12}(\lambda_{12})R_{13}(\lambda_{13})R_{23}(\lambda_{23}) \\ &= R_{13}(\lambda_{13})R_{23}(\lambda_{13})R_{12}(\lambda_{12}) \end{aligned} \qquad \begin{aligned} \lambda_{ij} &:= \lambda_i - \lambda_j\,, \\ \bar{\lambda}_{ij} &:= \lambda_i + \lambda_j\,. \end{aligned}$$

Algebraic Bethe ansatz

The boundary K-matrix

$$K(\lambda;\xi) = \begin{bmatrix} \sinh(\lambda+\xi) & 0 \\ 0 & \sinh(\xi-\lambda) \end{bmatrix}$$

$$K_\pm(\lambda;\xi_\pm) := K(\lambda \pm \tfrac{\eta}{2};\xi_\pm)\,,$$

Algebraic Bethe ansatz

The boundary K-matrix

$$K(\lambda;\xi) = \begin{bmatrix} \sinh(\lambda+\xi) & 0 \\ 0 & \sinh(\xi-\lambda) \end{bmatrix}$$
$$K_\pm(\lambda;\xi_\pm) := K(\lambda \pm \tfrac{\eta}{2};\xi_\pm)\,,$$

satisfies the reflection relation

$$R_{12}(\lambda_{12})K_1(\lambda_1)R_{12}^{t_1t_2}(\bar\lambda_{12})K_2(\lambda_2)$$
$$= K_2(\lambda_2)R_{12}(\bar\lambda_{12})K_1(\lambda_1)R_{12}^{t_1t_2}(\lambda_{12})$$

The monodromy matrices

$$T_0(\lambda) := R_{0N}(\lambda - \xi_N) \cdots R_{01}(\lambda - \xi_1) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}_{[0]}$$

$$\hat{T}_0(\lambda) := R_{10}(\lambda + \xi_1 - \eta) \cdots R_{N0}(\lambda - \xi_N - \eta)$$

$$= (-1)^N \begin{bmatrix} D(-\lambda) & -B(-\lambda) \\ -C(-\lambda) & A(-\lambda) \end{bmatrix}_{[0]}$$

$$X \in \mathrm{End}(\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{\mathrm{N}})$$
$$(X = A, B, C, D)$$

Algebraic Bethe ansatz

The monodromy matrices

$$T_0(\lambda) := R_{0N}(\lambda - \xi_N) \cdots R_{01}(\lambda - \xi_1) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}_{[0]}$$

$$\hat{T}_0(\lambda) := R_{10}(\lambda + \xi_1 - \eta) \cdots R_{N0}(\lambda - \xi_N - \eta)$$

$$= (-1)^N \begin{bmatrix} D(-\lambda) & -B(-\lambda) \\ -C(-\lambda) & A(-\lambda) \end{bmatrix}_{[0]}$$

$$X \in \mathrm{End}(\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{\mathrm{N}})$$
$$(X = A, B, C, D)$$

satisfy the following relation

$$R_{12}(\lambda_{12}) T_1(\lambda_1) T_2(\lambda_2)$$
$$= T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_{12})$$

Algebraic Bethe ansatz

The monodromy matrices

$$T_0(\lambda) := R_{0N}(\lambda - \xi_N) \cdots R_{01}(\lambda - \xi_1) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}_{[0]}$$

$$\hat{T}_0(\lambda) := R_{10}(\lambda + \xi_1 - \eta) \cdots R_{N0}(\lambda - \xi_N - \eta)$$

$$= (-1)^N \begin{bmatrix} D(-\lambda) & -B(-\lambda) \\ -C(-\lambda) & A(-\lambda) \end{bmatrix}_{[0]}$$

$$X \in \mathrm{End}(\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{\mathrm{N}}) \quad (X = A, B, C, D)$$

satisfy the following relation

$$R_{12}(\lambda_{12}) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_{12}) \quad \Rightarrow$$

- $[t(\lambda),\ t(\mu)] = 0 \quad (t(\lambda) := \mathrm{tr}_j T_j(\lambda))$

Algebraic Bethe ansatz

The monodromy matrices

$$T_0(\lambda) := R_{0N}(\lambda - \xi_N) \cdots R_{01}(\lambda - \xi_1) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix}_{[0]}$$

$$\hat{T}_0(\lambda) := R_{10}(\lambda + \xi_1 - \eta) \cdots R_{N0}(\lambda - \xi_N - \eta)$$

$$= (-1)^N \begin{bmatrix} D(-\lambda) & -B(-\lambda) \\ -C(-\lambda) & A(-\lambda) \end{bmatrix}_{[0]}$$

$$X \in \mathrm{End}(\overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{\mathrm{N}})$$
$$(X = A, B, C, D)$$

satisfy the following relation

$$R_{12}(\lambda_{12}) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_{12}) \quad \Rightarrow$$

- $[t(\lambda),\ t(\mu)] = 0 \quad (t(\lambda) := \mathrm{tr}_j T_j(\lambda))$
- comm. rel. among A, B, C, D

Algebraic Bethe ansatz

The double-row monodromy matrices

$$\mathcal{U}_-(\lambda) := T(\lambda)K_-(\lambda)\hat{T}(\lambda) := \begin{bmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{bmatrix}$$

$$\mathcal{U}_+(\lambda) := \hat{T}(\lambda)K_+(\lambda)T(\lambda) := \begin{bmatrix} \mathcal{A}_+(\lambda) & \mathcal{B}_+(\lambda) \\ \mathcal{C}_+(\lambda) & \mathcal{D}_+(\lambda) \end{bmatrix}$$

Algebraic Bethe ansatz

The double-row monodromy matrices

$$\mathcal{U}_-(\lambda) := T(\lambda)K_-(\lambda)\hat{T}(\lambda) := \begin{bmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{bmatrix}$$

$$\mathcal{U}_+(\lambda) := \hat{T}(\lambda)K_+(\lambda)T(\lambda) := \begin{bmatrix} \mathcal{A}_+(\lambda) & \mathcal{B}_+(\lambda) \\ \mathcal{C}_+(\lambda) & \mathcal{D}_+(\lambda) \end{bmatrix}$$

satisfy the following relations

$$\begin{aligned} & R_{12}(\lambda_{12})(\mathcal{U}_-)_1(\lambda_1)R_{12}^{t_1t_2}(\bar{\lambda}_{12}-\eta)(\mathcal{U}_-)_2(\lambda_2) \\ & = (\mathcal{U}_-)_2(\lambda_2)R_{12}(\bar{\lambda}_{12}-\eta)(\mathcal{U}_-)_1(\lambda_1)R_{12}^{t_1t_2}(\lambda_{12}) \\ & R_{12}(-\lambda_{12})(\mathcal{U}_+)_1^{t_1}(\lambda_1)R_{12}^{t_1t_2}(-\bar{\lambda}_{12}-\eta)(\mathcal{U}_+)_2^{t_2}(\lambda_2) \\ & = (\mathcal{U}_+)_2^{t_2}(\lambda_2)R_{12}(-\bar{\lambda}_{12}-\eta)(\mathcal{U}_+)_1^{t_1}(\lambda_1)R_{12}^{t_1t_2}(-\lambda_{12}) \,. \end{aligned}$$

Integrable higher spin XXZ models

The higher spin XXZ model is described by the following Hamiltonian

$$\mathcal{H}^{(s)} := \frac{d}{d\lambda}\mathcal{T}^{(s,s)}(\lambda)\Big|_{\lambda=s\eta} + \mathsf{const.}$$

$$\mathcal{T}^{(s,s)}(\lambda) := \mathrm{tr}_0[K_+^{(s)}(\lambda)T^{(s,s)}(\lambda)K_-^{(s)}(\lambda)\hat{T}^{(s,s)}(\lambda)]\,.$$

Integrable higher spin XXZ models

The higher spin XXZ model is described by the following Hamiltonian

$$\mathcal{H}^{(s)} := \frac{d}{d\lambda}\mathcal{T}^{(s,s)}(\lambda)\Big|_{\lambda=s\eta} + \text{const.}$$

$$\mathcal{T}^{(s,s)}(\lambda) := \mathrm{tr}_0[K_+^{(s)}(\lambda)T^{(s,s)}(\lambda)K_-^{(s)}(\lambda)\hat{T}^{(s,s)}(\lambda)]\,.$$

$K^{(s)} \in \mathrm{End}(\mathbb{C}^{2s+1})$ and $T^{(s,s)} \in \mathrm{End}(\mathbb{C}^{2s+1} \overset{N}{\otimes} \mathbb{C}^{2s+1})$ are constructed by the fusion procedure

$$K^{(1)}(\lambda) = P_{12}^+ K_1(\lambda+\eta) R_{12}^{t_1 t_2}(2\lambda+\eta) K_2(\lambda) P_{12}^+$$

$$T^{(s,s)}(\lambda) := R_{0N}^{(s,s)}(\lambda-\xi_N)\cdots R_{01}^{(s,s)}(\lambda-\xi_1)$$

$$R^{(\frac{1}{2},1)}(\lambda) = P_{12}^+ R_{02}(\lambda+\eta) R_{01}(\lambda) P_{12}^+$$

$$R^{(1,\frac{1}{2})}(\lambda) = P_{12}^+ R_{10}(\lambda+\eta) R_{20}(\lambda) P_{12}^+ \qquad (P_{12}^+ = R_{12}(\eta))$$

Integrable higher spin XXZ models

Due to the following commutativity

$$[\mathcal{T}^{(j,s)}(\lambda),\ \mathcal{T}^{(k,s)}(\mu)] = 0\,,$$

Integrable higher spin XXZ models

Due to the following commutativity

$$[\mathcal{T}^{(j,s)}(\lambda),\ \mathcal{T}^{(k,s)}(\mu)] = 0\,,$$

the eigenstates of $\mathcal{T}^{(s,s)}(\lambda)$ are constructed by

$$|\Psi_g^{\pm}\rangle := \prod_{j=1}^{n} \mathcal{B}_{\pm}^{(s)}(\lambda_j)|0\rangle$$

$$|0\rangle := \overset{N}{\otimes}|s,s\rangle \qquad (\mathcal{B}_{\pm}^{(s)} \in \mathrm{End}(\overset{N}{\otimes}\mathbb{C}^{2s+1}))$$

Integrable higher spin XXZ models

Due to the following commutativity

$$[\mathcal{T}^{(j,s)}(\lambda),\ \mathcal{T}^{(k,s)}(\mu)] = 0\,,$$

the eigenstates of $\mathcal{T}^{(s,s)}(\lambda)$ are constructed by

$$|\Psi_g^{\pm}\rangle := \prod_{j=1}^{n} \mathcal{B}_{\pm}^{(s)}(\lambda_j)|0\rangle$$

$$|0\rangle := \overset{N}{\otimes}|s,s\rangle \qquad (\mathcal{B}_{\pm}^{(s)} \in \mathrm{End}(\overset{N}{\otimes}\mathbb{C}^{2s+1}))$$

with $\{\lambda\}$ as the solutions of the Bethe equations

$$\left[\frac{\sinh(\lambda_j + s\eta)\sinh(-\lambda_j - s\eta)}{\sinh(-\lambda_j + s\eta)\sinh(\lambda_j - s\eta)}\right]^N$$

$$= -\frac{\sinh(-\lambda_j + \xi_+ - \frac{\eta}{2})\sinh(-\lambda_j + \xi_- - \frac{\eta}{2})}{\sinh(\lambda_j + \xi_+ - \frac{\eta}{2})\sinh(\lambda_j + \xi_- - \frac{\eta}{2})}$$

$$\times \prod_{k=1}^{n} \frac{\sinh(-\bar{\lambda}_{jk} - \eta)\sinh(-\lambda_{jk} - \eta)}{\sinh(\bar{\lambda}_{jk} - \eta)\sinh(\lambda_{jk} - \eta)} \cdot \frac{\sinh(2\lambda_j - \eta)}{\sinh(-2\lambda_j - \eta)}$$

Integrable higher spin XXZ models

Due to the following commutativity

$$[\mathcal{T}^{(j,s)}(\lambda),\ \mathcal{T}^{(k,s)}(\mu)] = 0\,,$$

the eigenstates of $\mathcal{T}^{(s,s)}(\lambda)$ are constructed by

$$|\Psi_g^{\pm}\rangle := \prod_{j=1}^{n} \mathcal{B}_{\pm}^{(s)}(\lambda_j)|0\rangle$$

$$|0\rangle := \overset{N}{\otimes}|s,s\rangle \qquad (\mathcal{B}_{\pm}^{(s)} \in \mathrm{End}(\overset{N}{\otimes}\mathbb{C}^{2s+1}))$$

with $\{\lambda\}$ as the solutions of the Bethe equations

$$\left[\frac{\sinh(\lambda_j + s\eta)\sinh(-\lambda_j - s\eta)}{\sinh(-\lambda_j + s\eta)\sinh(\lambda_j - s\eta)}\right]^N$$

$$= -\frac{\sinh(-\lambda_j + \xi_+ - \frac{\eta}{2})\sinh(-\lambda_j + \xi_- - \frac{\eta}{2})}{\sinh(\lambda_j + \xi_+ - \frac{\eta}{2})\sinh(\lambda_j + \xi_- - \frac{\eta}{2})}$$

$$\times \prod_{k=1}^{n} \frac{\sinh(-\bar{\lambda}_{jk} - \eta)\sinh(-\lambda_{jk} - \eta)}{\sinh(\bar{\lambda}_{jk} - \eta)\sinh(\lambda_{jk} - \eta)} \cdot \frac{\sinh(2\lambda_j - \eta)}{\sinh(-2\lambda_j - \eta)}$$

Integrable higher spin XXZ models

There exist the boundary string solutions

$$\lambda^{\mathrm{B}} = \cdots\,,\, -\xi_- - \frac{\eta}{2}\,, \quad -\xi_- + \frac{\eta}{2}\,, \quad -\xi_- + \frac{3\eta}{2}\,, \cdots$$

in the regime $0 < \tilde{\xi}_- < \frac{\zeta}{2}$ and $\zeta < \frac{\pi}{2s}$, where

$$\begin{cases} \zeta := i\eta > 0\,, \quad \tilde{\xi}_- := i\xi_- > 0 & |\cosh\eta| \le 1 \\ \zeta := -\eta > 0\,, \quad \tilde{\xi}_- := -\xi_- > 0 & \\ \qquad\qquad\qquad \tilde{\xi}_- := -\xi_- + \frac{\pi i}{2} > 0 & |\cosh\eta| > 1\,. \end{cases}$$

Integrable higher spin XXZ models

There exist the boundary string solutions

$$\lambda^{\mathsf{B}} = \cdots , -\xi_- - \frac{\eta}{2} , \quad -\xi_- + \frac{\eta}{2} , \quad -\xi_- + \frac{3\eta}{2} , \cdots$$

in the regime $0 < \tilde{\xi}_- < \frac{\zeta}{2}$ and $\zeta < \frac{\pi}{2s}$, where

$$\begin{cases} \zeta := i\eta > 0 , \quad \tilde{\xi}_- := i\xi_- > 0 & |\cosh \eta| \leq 1 \\ \zeta := -\eta > 0 , \quad \tilde{\xi}_- := -\xi_- > 0 & \\ \qquad\qquad\quad\ \ \tilde{\xi}_- := -\xi_- + \frac{\pi i}{2} > 0 & |\cosh \eta| > 1 . \end{cases}$$

- The boundary bound states reduce the free energy.

Integrable higher spin XXZ models

There exist the boundary string solutions

$$\lambda^{\mathrm{B}} = \cdots, -\xi_- - \frac{\eta}{2}, \quad -\xi_- + \frac{\eta}{2}, \quad -\xi_- + \frac{3\eta}{2}, \cdots$$

in the regime $0 < \tilde{\xi}_- < \frac{\zeta}{2}$ and $\zeta < \frac{\pi}{2s}$, where

$$\begin{cases} \zeta := i\eta > 0, \quad \tilde{\xi}_- := i\xi_- > 0 & |\cosh\eta| \leq 1 \\ \zeta := -\eta > 0, \quad \tilde{\xi}_- := -\xi_- > 0 & \\ \qquad\qquad\qquad \tilde{\xi}_- := -\xi_- + \frac{\pi i}{2} > 0 & |\cosh\eta| > 1\,. \end{cases}$$

- The boundary bound states reduce the free energy.
- Strings longer than $2s$ do not contribute to the free energy.

Integrable higher spin XXZ models

There exist the boundary string solutions

$$\lambda^{\mathrm{B}} = \cdots, -\xi_- - \frac{\eta}{2}, \quad -\xi_- + \frac{\eta}{2}, \quad -\xi_- + \frac{3\eta}{2}, \cdots$$

in the regime $0 < \tilde{\xi}_- < \frac{\zeta}{2}$ and $\zeta < \frac{\pi}{2s}$, where

$$\begin{cases} \zeta := i\eta > 0, \quad \tilde{\xi}_- := i\xi_- > 0 & |\cosh\eta| \le 1 \\ \zeta := -\eta > 0, \quad \tilde{\xi}_- := -\xi_- > 0 & \\ \qquad\qquad\qquad \tilde{\xi}_- := -\xi_- + \frac{\pi i}{2} > 0 & |\cosh\eta| > 1. \end{cases}$$

- The boundary bound states reduce the free energy.
- Strings longer than $2s$ do not contribute to the free energy.

$\Rightarrow$ Assume the boundary bound $2s$-string solutions

$$\begin{cases} \lambda_r^{\mathrm{B}} = -\xi_- + (-s - \frac{1}{2} + r)\eta & \text{for } s \in \mathbb{Z}_{\ge 0} \\ \lambda_r^{\mathrm{B}} = -\xi_- + (-s + r)\eta & \text{for } s \in \mathbb{Z}^{\frac{1}{2}}_{\ge 0} \quad (r = 1, \ldots, 2s) \end{cases}$$

Integrable higher spin XXZ models

Assume the $2s$-string solutions for the ground state

$$\lambda_{2s(j-1)+r} = \mu_j + (-s - \tfrac{1}{2} + r)\eta \qquad (r = 1, \dots, 2s)\,.$$

Integrable higher spin XXZ models

Assume the $2s$-string solutions for the ground state

$$\lambda_{2s(j-1)+r} = \mu_j + (-s - \tfrac{1}{2} + r)\eta \qquad (r = 1, \ldots, 2s)\,.$$

- Taking the logarithmic derivative of BE

Integrable higher spin XXZ models

Assume the $2s$-string solutions for the ground state

$$\lambda_{2s(j-1)+r} = \mu_j + (-s - \tfrac{1}{2} + r)\eta \qquad (r = 1, \ldots, 2s)\,.$$

- Taking the logarithmic derivative of BE
- In the thermodynamic limit $N \to \infty$ by fixing $\frac{n}{N}$

Integrable higher spin XXZ models

Assume the $2s$-string solutions for the ground state

$$\lambda_{2s(j-1)+r} = \mu_j + (-s - \tfrac{1}{2} + r)\eta \qquad (r = 1, \ldots, 2s)\,.$$

- Taking the logarithmic derivative of BE
- In the thermodynamic limit $N \to \infty$ by fixing $\frac{n}{N}$

$$\Rightarrow \;\; 2\sum_{r=1}^{2s} K_{(2r-1)\eta}(\mu) \qquad\qquad \left(\rho(\lambda_j) := \lim_{N\to\infty} \frac{1}{N(\lambda_{j+1}-\lambda_j)}\right)$$

$$= \rho(\mu) + \int_{-\Lambda}^{\Lambda} \left[K_{4s\eta}(\mu - \mu') + 2\sum_{r=1}^{2s-1} K_{2r\eta}(\mu - \mu')\rho(\mu')d\mu'\right]$$

$$\Lambda := \infty \quad (|\cosh\eta| \le 1)\,, \qquad \Lambda := \frac{i\pi}{2} \quad (|\cosh\eta| > 1)$$

$$\Rightarrow \quad \rho(\lambda) = \begin{cases} \dfrac{1}{\eta\cosh\frac{\pi\lambda}{i\eta}} & |\cosh\eta| \le 1 \\ \frac{1}{\pi}\prod_{n=1}^{\infty}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)^2 \frac{\theta_3(i\lambda, e^{\eta})}{\theta_4(i\lambda, e^{\eta})} & |\cosh\eta| > 1\,. \end{cases}$$

Quantum inverse scattering method

Quantum inverse scattering method

For the spin-$\frac{1}{2}$ case [Kitanine et al. (07)]

$$E_n^{\varepsilon'_n,\varepsilon_n} = \left[\prod_{\alpha=1}^{n-1} \mathrm{tr}_0 T_0(\xi_\alpha)\right] \mathrm{tr}_0[T_0(\xi_n) E_0^{\varepsilon'_n,\varepsilon_n}] \left[\prod_{\alpha=1}^{n} \mathrm{tr}_0 T_0(\xi_\alpha)\right]^{-1} .$$

Quantum inverse scattering method

For the spin-$\frac{1}{2}$ case [Kitanine et al. (07)]

$$E_n^{\varepsilon'_n,\varepsilon_n} = \left[\prod_{\alpha=1}^{n-1} \mathrm{tr}_0 T_0(\xi_\alpha)\right] \mathrm{tr}_0[T_0(\xi_n) E_0^{\varepsilon'_n,\varepsilon_n}] \left[\prod_{\alpha=1}^{n} \mathrm{tr}_0 T_0(\xi_\alpha)\right]^{-1} .$$

- $E^{\varepsilon'^s,\varepsilon^s}$ is expressed as a tensor product of two vectors

$$[0 \cdots \underset{\varepsilon'^s}{1} \cdots 0]^t \otimes [0 \cdots \underset{\varepsilon^s}{1} \cdots 0]$$

For the spin-$\frac{1}{2}$ case [Kitanine et al. (07)]

$$E_n^{\varepsilon'_n,\varepsilon_n} = \left[\prod_{\alpha=1}^{n-1} \mathrm{tr}_0 T_0(\xi_\alpha)\right] \mathrm{tr}_0[T_0(\xi_n) E_0^{\varepsilon'_n,\varepsilon_n}] \left[\prod_{\alpha=1}^{n} \mathrm{tr}_0 T_0(\xi_\alpha)\right]^{-1}.$$

- $E^{\varepsilon'^s,\varepsilon^s}$ is expressed as a tensor product of two vectors

$$[0 \cdots \underset{\varepsilon'^s}{1} \cdots 0]^t \otimes [0 \cdots \underset{\varepsilon^s}{1} \cdots 0]$$

- A spin-s column (row) vector is mapped to a tensor product of spin-$\frac{1}{2}$ vectors

$$F^{(s)}[1\ 0 \cdots 0]^t \mapsto \sum_{j=1}^{2s} \left(\overset{j-1}{\otimes} K \otimes F \overset{2s-j}{\otimes} K^{-1} \right) \overbrace{[1\ 0]^t \otimes \cdots \otimes [1\ 0]^t}^{2s}$$

$$[1\ 0 \cdots 0]\, E^{(s)} \mapsto \overbrace{[1\ 0] \otimes \cdots \otimes [1\ 0]}^{2s} \sum_{j=1}^{2s} \left(\overset{j-1}{\otimes} K \otimes E \overset{2s-j}{\otimes} K^{-1} \right)$$

Quantum inverse scattering method

The elementary matrix of spin-s representation is mapped to a tensor product of $2s$ elementary matrices of spin-$\frac{1}{2}$ representations

$$E^{\varepsilon'^s,\varepsilon^s}_{1_{1\dots 2s}} \mapsto \begin{bmatrix} 2s \\ \varepsilon'^s \end{bmatrix}_q^{-\frac{1}{2}} \begin{bmatrix} 2s \\ \varepsilon^s \end{bmatrix}_q^{-\frac{1}{2}} P^s \prod_{j=1}^{\varepsilon^s} E_j^{2,2} \prod_{j=\varepsilon^s+1}^{\varepsilon'^s} E_j^{2,1} \prod_{j=\varepsilon'^s+1}^{2s} E_j^{1,1} P^s \qquad (\varepsilon'^s > \varepsilon^s)$$

$$E^{\varepsilon'^s,\varepsilon^s}_{1_{1\dots 2s}} \mapsto \begin{bmatrix} 2s \\ \varepsilon'^s \end{bmatrix}_q^{-\frac{1}{2}} \begin{bmatrix} 2s \\ \varepsilon^s \end{bmatrix}_q^{-\frac{1}{2}} P^s \prod_{j=1}^{\varepsilon'^s} E_j^{2,2} \prod_{j=\varepsilon'^s+1}^{\varepsilon^s} E_j^{1,2} \prod_{j=\varepsilon^s+1}^{2s} E_j^{1,1} P^s \qquad (\varepsilon'^s < \varepsilon^s)$$

$$E^{\varepsilon'^s,\varepsilon^s}_{1_{1\dots 2s}} \mapsto \begin{bmatrix} 2s \\ \varepsilon^s \end{bmatrix}_q P^s \prod_{j=1}^{\varepsilon^s} E_j^{2,2} \prod_{j=\varepsilon^s+1}^{2s} E_j^{1,1} P^s \qquad (\varepsilon'^s = \varepsilon^s)$$

Quantum inverse scattering method

The quantum inverse scattering for the elementary matrices of spin-s representations

$$E_i^{\varepsilon_i'^s,\varepsilon_i^s} \mapsto \begin{bmatrix} 2s \\ \varepsilon'^s \end{bmatrix}_q^{-\frac{1}{2}} \begin{bmatrix} 2s \\ \varepsilon^s \end{bmatrix}_q^{-\frac{1}{2}} P^s \prod_{j=1}^{2s(i-1)} (A+D)(w_j) \prod_{j=2s(i-1)+1}^{2s(i-1)+\varepsilon_i} D(w_j)$$

$$\times \prod_{j=2s(i-1)+\varepsilon_i+1}^{2s(i-1)+\varepsilon_i'} C(w_j) \prod_{j=2s(i-1)+\varepsilon_i'+1}^{2si} A(w_j)$$

$$\times \left[\prod_{j=1}^{i}(A+D)(w_j)\right]^{-1} P^s \qquad (\varepsilon_i'^s > \varepsilon_i^s)$$

$$w_{2s(j-1)+r} := \xi_j + (-s - \tfrac{1}{2} + r)\eta \qquad (r = 1, \ldots, 2s)$$

Correlation functions

The ground state of the open system

$$|\Psi_g^+\rangle := \prod_{j=1}^{n} \mathcal{B}_+^{(s)}(\lambda_j)|0\rangle$$

Correlation functions

The ground state of the open system

$$|\Psi_g^+\rangle := \prod_{j=1}^n \mathcal{B}_+^{(s)}(\lambda_j)|0\rangle$$

is expressed in terms of the ground state of the closed system as

$$|\Psi_g^+\rangle = \sum_{\sigma_1,\ldots,\sigma_n=\pm} H^{\mathcal{B}_+}_{(\sigma_1,\ldots,\sigma_n)}(\lambda_1,\ldots,\lambda_n;\xi_-)\prod_{j=1}^n B^{(s)}(\lambda_j^\sigma)|0\rangle \quad (\lambda_j^\sigma := \sigma_j\lambda_j)$$

$$H^{\mathcal{B}_+^{(s)}}_{(\sigma_1,\ldots,\sigma_n)}(\lambda_1,\ldots,\lambda_n;\xi_+) := \prod_{j=1}^n \Bigg[(-1)^N\sigma_j\prod_{k=1}^N \sinh(-\lambda_j^\sigma-\xi_k-s\eta) \times \frac{\sinh(2\lambda_j+\eta)}{\sinh(2\lambda_j)}\sinh(\lambda_j^\sigma+\xi_+-\tfrac{\eta}{2})\Bigg] \times \prod_{1\le r<s\le n}\frac{\sinh(\bar\lambda_{rs}^\sigma-\eta)}{\sinh\bar\lambda_{rs}^\sigma}. \quad (\bar\lambda_{rs}^\sigma := \sigma_r\lambda_r+\sigma_s\lambda_s)$$

Correlation functions

The m-point correlation function

$$F_m^{(s)} := \frac{\langle \Psi_g^+ | \prod_{j=1}^m E_j^{\varepsilon'^s_j, \varepsilon^s_j} | \Psi_g^+ \rangle}{\langle \Psi_g^+ | \Psi_g^+ \rangle}$$

Correlation functions

The m-point correlation function

$$F_m^{(s)} := \frac{\langle \Psi_g^+ | \prod_{j=1}^m E_j^{\varepsilon_j'^s, \varepsilon_j^s} | \Psi_g^+ \rangle}{\langle \Psi_g^+ | \Psi_g^+ \rangle}$$

is obtained as a multi-integral form in the thermodynamic limit

$$F_m^{(s)} = \prod_{j=1}^{m} \begin{bmatrix} 2s \\ \varepsilon_j'^s \end{bmatrix}_q^{-\frac{1}{2}} \begin{bmatrix} 2s \\ \varepsilon_j^s \end{bmatrix}_q^{-\frac{1}{2}} \mathcal{G}(\{\xi\})$$

$$\times \sum_{\sigma \in \mathfrak{S}_{2sm}} \mathrm{sgn}(\sigma) \prod_{\mathrm{j}=1}^{\mathrm{m}} \left(\prod_{\mathrm{r}=1}^{\varepsilon_{\mathrm{j}}^{\mathrm{s}}-1} \int_{\mathcal{C}_{\mathrm{r}}} \mathrm{d}\lambda_{\sigma(2\mathrm{s}(\mathrm{j}-1)+\mathrm{r})} \prod_{\mathrm{r}=\varepsilon_{\mathrm{j}}'^{\mathrm{s}}}^{2\mathrm{s}} \int_{\bar{\mathcal{C}}_{\mathrm{r}}} \mathrm{d}\lambda_{\sigma(2\mathrm{s}(\mathrm{j}-1)+\mathrm{r})} \right)$$

$$\times H_{2sm}(\{\lambda\}, \{\xi\}) \det_{2sm} \Phi(\{\lambda\}, \{\xi\}) \qquad (c := \textstyle\sum_{k=1}^m (\varepsilon_k^s - 1))$$

Correlation functions

The contours are set as

$$\begin{cases} \mathcal{C}_r = (-\Lambda - (s + \frac{1}{2} - r)\eta, \Lambda - (s + \frac{1}{2} - r)\eta) \cup \Gamma(\{\lambda^{\mathsf{B}}\}) \\ \bar{\mathcal{C}}_r = \mathcal{C}_r \cup \Gamma(\{\xi\})\,. \end{cases}$$

Correlation functions

The contours are set as

$$\begin{cases} \mathcal{C}_r = (-\Lambda - (s + \frac{1}{2} - r)\eta, \Lambda - (s + \frac{1}{2} - r)\eta) \cup \Gamma(\{\lambda^{\mathsf{B}}\}) \\ \bar{\mathcal{C}}_r = \mathcal{C}_r \cup \Gamma(\{\xi\}) \, . \end{cases}$$

The functions $\mathcal{G}$, H_{2sm}, $\det_{2sm} \Phi$ are defined as follows

$$\mathcal{G}(\{\xi\}) = \frac{(-1)^{2sm-c}}{\prod_{j<i} \prod_{p,q=1}^{2s} \sinh(\xi_{ij} + (p-q)\eta) \prod_{i \le j} \sinh(\bar{\xi}_{ij} + (2s-p-q)\eta)}$$
$$\times \frac{1}{\prod_{r=1}^{2s-1} \sinh^m((2s-r)\eta) \prod_{k=1}^{m} \prod_{1 \le j \le r \le 2s} \sinh(2\xi_k + (2s-r-j)\eta)}$$

Correlation functions

$$H_{2sm}(\{\lambda\},\{\xi\}) = \frac{\prod_{j=1}^{2sm}\prod_{k=1}^{m}\prod_{r=1}^{2s}\sinh(\lambda_{\sigma(j)}+\xi_k+(s-\frac{1}{2}-r)\eta)}{\prod_{1\le i<j\le 2sm}\sinh(\lambda_{\sigma(i)\sigma(j)}+\eta+\epsilon_{ij})\sinh(\bar{\lambda}_{\sigma(i)\sigma(j)}-\eta+\bar{\epsilon}_{ij})}$$

$$\times\frac{\prod_{k=1}^{m}\prod_{r=1}^{2s}\sinh(\xi_k+\xi_- -(s+\frac{1}{2}-r)\eta)}{\prod_{k=1}^{2sm}\sinh(\lambda_{\sigma(k)}+\xi_- -\frac{\eta}{2})}$$

$$\times\prod_{j=1}^{2sm}\prod_{k=1}^{m}\prod_{r=1}^{2s-1}\sinh(\lambda_{\sigma(j)}-\xi_k+(-s+r)\eta)$$

$$\times\prod_{j=1}^{m}\prod_{r=1}^{\varepsilon_j^s-1}\left(\prod_{k=1}^{j-1}\sinh(\lambda_{\sigma(i_{p(j,r)})}-\xi_k-s\eta)\prod_{k=j+1}^{m}\sinh(\lambda_{\sigma(i_{p(j,r)})}-\xi_k+s\eta)\right)$$

$$\times\prod_{j=1}^{m}\prod_{r=\varepsilon_j'^s}^{2s}\left(\prod_{k=1}^{j-1}\sinh(\lambda_{\sigma(i_{p(j,r)})}-\xi_k+s\eta)\prod_{k=j+1}^{m}\sinh(\lambda_{\sigma(i_{p(j,r)})}-\xi_k-s\eta)\right)$$

Correlation functions

$$\det_{2sm} \Phi(\{\lambda\},\{\xi\})$$

$$= \begin{cases} \left(\frac{1}{\eta}\right)^{2sm} \prod_{r=1}^{2s} \left[\frac{\prod_{1\le i<j\le m} \sinh(\frac{\pi}{\zeta}\lambda_{\sigma(2s(i-1)+r)\sigma(2s(j-1)+r)})}{\prod_{i=1}^{m}\prod_{j=1}^{m} \cosh(\frac{\pi}{\zeta}(\lambda_{\sigma(2s(i-1)+r)} - \xi_j))} \right. \\ \left. \times \frac{\sinh(\frac{\pi}{\zeta}\bar{\lambda}_{\sigma(2s(i-1)+r)\sigma(2s(j-1)+r)})}{\cosh(\frac{\pi}{\zeta}(\lambda_{\sigma(2s(i-1)+r)} + \xi_j))} \sinh(\tfrac{\pi}{\zeta}\xi_{ij}) \sinh(\tfrac{\pi}{\zeta}\bar{\xi}_{ij}) \right] \\ \qquad (|\cosh\eta| \le 1) \\ \left(-\frac{1}{\pi}\right)^{2sm} \prod_{r=1}^{2s}\prod_{j=1}^{m} \theta_1(i\lambda_{\sigma(2s(j-1)+r)})\theta_2(i\lambda_{\sigma(2s(j-1)+r)})\theta_3(i\xi_j)\theta_4(i\xi_j) \\ \times \frac{\prod_{1\le j<k\le m} \theta_1(i\lambda_{\sigma(2s(j-1)+r)\sigma(2s(k-1)+r)})}{\prod_{j,k=1}^{m} \theta_1(i(\lambda_{\sigma(2s(j-1)+r)} - \xi_k))} \\ \times \frac{\theta_1(i\bar{\lambda}_{\sigma(2s(j-1)+r)\sigma(2s(k-1)+r)})}{\theta_1(i(\lambda_{\sigma(2s(j-1)+r)} + \xi_k))} \theta_1(i\xi_{kj})\theta_1(i\bar{\xi}_{kj}) \\ \qquad (|\cosh\eta| > 1) \end{cases}$$

Concluding remarks

- We obtained the correlation functions of the integrable XXZ spin-s spin chains with boundaries in the multi-integral forms.

Concluding remarks

- We obtained the correlation functions of the integrable XXZ spin-s spin chains with boundaries in the multi-integral forms.
- The two point function $\langle \sigma_1^z \sigma_m^z \rangle$.

Concluding remarks

- We obtained the correlation functions of the integrable XXZ spin-s spin chains with boundaries in the multi-integral forms.
- The two point function $\langle \sigma_1^z \sigma_m^z \rangle$.
- The asymptotic behavior of correlation functions.

Concluding remarks

- We obtained the correlation functions of the integrable XXZ spin-s spin chains with boundaries in the multi-integral forms.
- The two point function $\langle \sigma_1^z \sigma_m^z \rangle$.
- The asymptotic behavior of correlation functions.
- The solutions of the boundary qKZ equations in the multi-integral forms.

Thank you for your attention!

Correlation functions

For the massless regime ($|\cosh\eta| \leq 1$)

$$\epsilon_{ij} = \begin{cases} i\epsilon & \mathcal{I}m(\lambda_{ij}) > 0 \\ -i\epsilon & \mathcal{I}m(\lambda_{ij}) < 0 \end{cases} \qquad \bar{\epsilon}_{ij} = \begin{cases} i\epsilon & \mathcal{I}m(\bar{\lambda}_{ij}) > 0 \\ -i\epsilon & \mathcal{I}m(\bar{\lambda}_{ij}) < 0 \end{cases}$$

and for the massive regime ($|\cosh\eta| > 1$)

$$\epsilon_{ij} = \begin{cases} \epsilon & \mathcal{R}e(\lambda_{ij}) > 0 \\ -\epsilon & \mathcal{R}e(\lambda_{ij}) < 0 \end{cases} \qquad \bar{\epsilon}_{ij} = \begin{cases} \epsilon & \mathcal{R}e(\bar{\lambda}_{ij}) > 0 \\ -\epsilon & \mathcal{R}e(\bar{\lambda}_{ij}) < 0 \end{cases}$$

Correlation functions

The indices $i_{p(j,r)}$ are defined as

$$\{i_{p(j,r)};\ 2s-\varepsilon_j^s+1 \le p(j,r) \le 2s\} \quad i_{p(j,r)} > i_{p(j',r')}$$

$$1 \le i_{p(j,r)} < i_{p(j',r')} \le c$$

$$p(j,r) < p(j',r') \quad 1 \le j < j' \le m$$

$$p(j,r) < p(j,r') \quad 1 \le r < r' \le 2s$$

$$\{i_{p(j,r)};\ 1 \le p(j,r) \le \varepsilon_j'^s - 1\} \quad i_{p(j,r)} < i_{p(j',r')}$$

$$c+1 \le i_{p(j,r)} < i_{p(j',r')} \le 2sm$$

$$p(j,r) < p(j',r') \quad 1 \le j < j' \le m$$

$$p(j,r) < p(j,r') \quad 1 \le r < r' \le 2s$$