THE MANY REPRESENTATIONS OF ISING FORM FACTORS

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THEOREM

YOU CANNOT UNDERSTAND A PAPER UNTIL YOU HAVE GENERALIZED IT

COROLLARY

NO ONE UNDERSTANDS HIS/HER MOST RECENT PAPER

THIS TALK WILL ILLUSTRATE THE COROLLARY

1. INTRODUCTION

2. THE FORM FACTOR $f^{(2)}_{N,N}$

3. THE FORM FACTOR $f^{(3)}_{N,N}$

4. THETA FUNCTION $q - k$ DUALITY

5. FUTURE DIRECTIONS

1.INTRODUCTION

ISING MODEL INTERACTION ENERGY

$$\mathcal{E} = -\sum_{j,k}\{E^v\sigma_{j,k}\sigma_{j+1,k} + E^h\sigma_{j,k}\sigma_{j,k+1}\}$$

DIAGONAL CORRELATION FUNCTION FORM FACTOR REPRESENTATION

FOR $T < T_c$

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = (1-t)^{1/4}\{1 + \sum_{n=1}^{\infty} f_{N,N}^{(2n)}(t)\}$$

WITH

$$t = (\sinh 2E^v/k_BT \sinh 2E^h/k_BT)^{-2}$$

FOR $T > T_c$

$$C(N,N) = (1-t)^{1/4}\sum_{n=0}^{\infty} f_{N,N}^{(2n+1)}(t)$$

WITH

$$t = (\sinh 2E^v/k_BT \sinh 2E^h/k_BT)^{2}$$

AND $f_{N,N}^{(n)}$ **IS AN** n **FOLD INTEGRAL.**

$$f_{N,N}^{(2n+1)}(t) = \frac{t^{((2n+1)N/2+n(n+1))}}{\pi^{2n+1}n!(n+1)!}$$

$$\int_0^1 \prod_{k=1}^{2n+1} x_k^N dx_k \prod_{j=1}^{n} ((1-x_{2j})(1-tx_{2j})x_{2j})^{1/2}$$

$$\times \prod_{j=1}^{n+1} ((1-x_{2j-1})(1-tx_{2j-1})x_{2j-1})^{-1/2}$$

$$\prod_{1\leq j\leq n+1} \prod_{1\leq k\leq n} (1-tx_{2j-1}x_{2k})^{-2}$$

$$\prod_{1\leq j<k\leq n+1} (x_{2j-1}-x_{2k-1})^2 \prod_{1\leq j<k\leq n} (x_{2j}-x_{2k})^2$$

$$f_{N,N}^{(1)}(t) = \frac{t^{N/2}}{\pi} \int_0^1 x^{N-1/2}[(1-x)(1-tx)]^{-1/2}$$

$$= \frac{t^{N/2}(1/2)_N}{N!} F_N$$

WHERE

$$F_N = F(1/2, N+1/2; \mathrm{N+1}; \mathrm{t}).$$

AND

$$(a)_0 = 1, \quad (a)_N = a(a+1)\cdots(a+N-1)$$

AS $t \to 0$

$$f_{N,N}^{(3)}(t) \to t^{3N/2+2} \frac{(1/2)_{N+1}^3 (N+2)}{2(1+2N)[(N+2)!]^3}$$

$$f_{N,N}^{(2n)} = \frac{t^{n(N+n)}}{\pi^{2n} n!^2} \int_0^1 \prod_{k=1}^{2n} x_k^N dx_k$$

$$\times \prod_{j=1}^{n} \left(\frac{x_{2j-1}(1-x_{2j})(1-tx_{2j})}{x_{2j}(1-x_{2j-1})(1-tx_{2j-1})} \right)^{1/2}$$

$$\times \prod_{1 \le j \le n} \prod_{1 \le k \le n} (1 - tx_{2j-1}x_{2k})^{-2}$$

$$\times \prod_{1 \le j < k \le n} (x_{2j-1} - x_{2k-1})^2 (x_{2j} - x_{2k})^2$$

IN "HOLONOMY OF THE ISING MODEL FORM FACTORS"

BOUKRAA, HASSANI, MAILLARD, MCCOY, ORRICK AND ZENINE
J. PHYS A40 (2OO7) 75-112

EXPLICIT EVALUATIONS BY MAPLE

$$2f_0^{(2)} = (K-E)K$$
$$2f_1^{(2)} = 1 - 3KE - (t-2)K^2$$
$$6tf_2^{(2)} = 6t - (6t^2 - 11t + 2)K^2 - (15t-4)KE - 2(t+1)E^2$$

$$6f_0^{(3)} = K - (t-2)K^3 - 3K^2E$$
$$6t^{1/2}f_1^{(3)} = 4(K-E) - 6K^2E - (2t-3)K^3 + 3KE^2$$
$$\begin{aligned} 18f_3^{(3)} &= 7(t+2)K - 14(t+1)E \\ &\quad + 24E^3 + 3(2t^2 - 11t + 2)EK^2 \\ &\quad + 36(t-1)KE^2 - 3(t^2-2)K^3 \end{aligned}$$

$$24f_0^{(4)} = 4(K-E)K - (2t-3)K^4 - 6K^3E + 3K^2E^2$$
$$\begin{aligned} 24f_1^{(4)} &= 9 - 30KE - 10(t-2)K^2 \\ &\quad + (t^2 - 6t + 6)K^4 + 15K^2E^2 + 10(t-2)K^3E \end{aligned}$$

WHERE

$K = F(1/2, 1/2; 1; t) \qquad E = F(1/2, -1/2; 1; t)$

THESE RESULTS WERE FOUND SEPARATELY FOR EACH N AND n BY

1. EXPANDING THE INTEGRALS IN A SERIES IN t;

2 HAVING MAPLE FIND AN ODE IN t;

3.PROCESSING THE ODE ON MAPLE.

OUTSTANDING OPEN QUESTIONS

1. FIND AN ANALYTIC PROOF FOR THESE COMPUTER RESULTS

2. FOR EACH $F^{(n)}_{N,N}$ FIND A RESULT VALID FOR ALL N

2. THE FORM FACTOR $f^{(2)}_{N,N}$

$$f^{(2)}_{N,N}(t) =$$

$$\int_0^1 dx \int_0^1 dy \frac{t^{N+1}x^{N+1/2}y^{N-1/2}}{\pi^2(1-txy)^2} \frac{(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}}$$

$$= \oint_C \frac{dxdy}{(2\pi i)^2} \frac{x^{N+1/2}y^{N-1/2}}{(1-xy)^2} \frac{(y-t^{1/2})^{1/2}(1-t^{1/2}y)^{1/2}}{(x-t^{1/2})^{1/2}(1-t^{1/2}x)^{1/2}}$$

BY USING CONTIGUOUS RELATIONS ON HYPERGEOMETRIC FUNCTIONS WE REWRITE THE EXPANSION IN TERMS OF E AND K IN TERMS OF F_N AND F_{N+1}

$$f^{(2)}_{N,N} =$$
$$C + C_0(N;t)F_N^2 + C_1(N;t)F_N F_{N+1} + C_2(N;t)F_{N+1}^2$$

with $C = N/2$

FOR $N = 0, \quad C_0 = C_2 = 0, C_1 = t/4$

FOR $N = 1$

$$C_0(1;t) = -\frac{1}{4}(t+1)\left(2t^2+t+2\right)$$
$$C_1(1;t) = \frac{3^2}{2^5}t\left(4t^2+5t+4\right)$$
$$C_2(1;t) = -\frac{3^4}{2^7}t^2(t+1)$$

FOR N=2

$$C_0(2;t) = -\frac{1}{2^6}(t+1)\left(64t^4+16t^3+99t^2+16t+64\right)$$
$$C_1(2;t) = \frac{5^2}{2^8\cdot 3}t\left(64t^4+88t^3+105t^2+88t+64\right)$$
$$C_2(2;t) = -\frac{5^4}{2^7\cdot 3^2}t^2(t+1)\left(2t^2+t+2\right)$$

THE POLYNOMIALS ARE PALINDROMIC!

Result

$$C_0(N;t) = -\frac{N}{2}\sum_{n=0}^{2N+1} c_n(N+1)t^n$$

$$= \frac{N(N+1)(N+2)^2}{(N+3/2)^4}t^{-2}C_2(N+1;t)$$

$$C_1(N;t) = \frac{(N+1/2)^2}{(N+1)}t\sum_{n=0}^{2N} d_n(N)t^n$$

$$C_2(N;t) = -\frac{(N+1/2)^4}{2N(N+1)^2}t^2\sum_{n=0}^{2N-1} c_n(N)t^n$$

where for $0 \le n \le N-1$

$$c_{2N-1-n}(N) = c_n(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N)$$

$$d_{2N-n}(N) = d_n(N) = \sum_{k=0}^{n} a_k(N)a_{n-k}(N+1)$$

and

$$d_N(N) = \frac{a_N(N+1)}{2N+1} + \sum_{k=0}^{N-1} a_k(N)a_{N-1-k}(N)$$

where for $0 \le k \le N-1$

$$a_k(N) = \frac{(1/2)_k(1/2-N)_k}{(1-N)_k k!}$$

Sketch of proof

1. INTEGRATE BY PARTS

2. DIFFERENTIATE USING

$$\frac{d}{dt}\left[\frac{(y-t^{1/2})(1-t^{1/2}y)}{x-t^{1/2})(1-t^{1/2}x)}\right]^{1/2} =$$

$$\frac{1}{t^{1/2}}\left[\frac{(y-t^{1/2})(1-t^{1/2}y)}{x-t^{1/2})(1-t^{1/2}x)}\right]^{1/2}$$

$$\times\frac{(xy-1)(x-y)(t-1)}{(y-t^{1/2})(1-t^{1/2}y)(x-t^{1/2})(1-t^{1/2}x)}$$

to find

$$\frac{df^{(2)}_{N,N}(t)}{dt} = \frac{(1-t)t^N}{4\pi}\left[\frac{(2N+1)\Gamma(\mathrm{N}+3/2)^2}{\Gamma(N+1)\Gamma(\mathrm{N}+2)}F_{N+1}G_N \right.$$
$$\left. -\frac{(2N-1)\Gamma(\mathrm{N}+1/2)\Gamma(\mathrm{N}+5/2)}{\Gamma(N+1)\Gamma(\mathrm{N}+2)}F_N G_{N+1}\right]$$

where

$$G_N = F(3/2, N+3/2; N+1; t)$$

3. EQUATE THE RESULT OF STEP 2 WITH THE DERIVATIVE OF THE ASSUMED FORM TO OBTAIN 3 COUPLED FIRST ORDER EQUATIONS.

4. CONVERT THE COUPLED SYSTEM TO 3 THIRD ORDER LINEAR INHOMOGENEOUS EQUATIONS

5. SOLVE THE HOMOGENEOUS ODE

FOR $C_2(N;t)$ THE SOLUTION OF THE HOMOGENEOUS PART IS THE SYMMETRIC SQUARE OF

$$O_2 \equiv Dt^2 - \frac{1+N-Nt}{t(1-t)}Dt + \frac{4+4N-t-2Nt}{4t^2(1-t)}$$

WHICH HAS SOLUTIONS

$$u_1 = t^{N+1}F_N$$

$$u_2 = tF(1/2, 1/2-N; 1; 1-t) =$$

$$t\sum_{n=0}^{N-1} \frac{(1/2)_n(1/2-N)_n}{(1-N)_n n!}t^n + t^{N+1}\frac{(-1)^N\Gamma(1/2+N)}{\Gamma(N)\Gamma(1/2-N)}$$

$$\times \sum_{n=0}^{\infty} \frac{(1/2)_N(1/2+N)_n}{n!(n+N)}[k_n - \ln t]t^n$$

$$k_n = \psi(n+1) + \psi(n+1+N) - \psi(\frac{1}{2}+n) - \psi(\frac{1}{2}+n+N)$$

6. SOLVE THE INHOMOGENEOUS ODE USING THE HOMOGENEOUS SOLUTION

EXPONENTS OF THE HOMOGENEOUS $C_2(N;t)$ EQUATION AT $t = 0$ ARE $2, \; N + 2, 2N + 2$

THE INHOMOGENEOUS TERM IS $A_2(N)t^{N+2}(1 - t)$

THE INHOMOGENEOUS EQUATION IS INVARIANT UNDER

$$C_2(N;t) \to t^{2N+3}C_2(N;1/t)$$

THE SOLUTION IS BUILT OUT OF THE SQUARE OF THE POLYNOMIAL PART OF u_2

3. THE FORM FACTOR $f^{(3)}_{N,N}$ THE K-E RESULTS IN THE $F_N - F_{N+1}$ BASIS ARE

$$f^{(3)}_{0,0} = \frac{1}{6}F_0 - \frac{1}{6}\left(1+t\right){F_0}^3 + \frac{1}{4}t{F_0}^2F_1$$

$$\frac{f^{(3)}_{1,1}}{t^{1/2}} = \frac{1}{3}F_1 - \frac{1}{2^3\cdot 3}\left(1+t\right)\left(8t^2+13\,t+8\right){F_1}^3$$
$$+\frac{3^2}{2^6}t\left(8t^2+15\,t+8\right){F_1}^2F_2 - \frac{3^4}{2^6}t^2(t+1)F_1F_2^2 + \frac{3^5}{2^9}t^3{F_2}^3$$

$$\frac{f^{(3)}_{2,2}}{t} = \frac{7}{2^4}F_2$$
$$-\frac{1}{2^{10}\cdot 3}\left(1+t\right)\left(2^6\cdot 3\cdot 7\,t^4+1136\,t^3+3229\,t^2+1136\,t+1344\right){F_2}^3$$
$$+\frac{5^2}{2^{11}\cdot 3}\,t\left(2^5\cdot 3^2t^4+596\,t^3+859\,t^2+596\,t+2^5\cdot 3^2\right){F_2}^2F_3$$
$$-\frac{5^5}{2^{10}\cdot 3^2}(t+1)(3t^2+4t+3)t^2F_2F_3^2 + \frac{5^6}{2^{11}\cdot 3^4}t^3\left(3t^2+8\,t+3\right){F_3}^3$$

$$\frac{f^{(3)}_{3,3}}{t^{3/2}} = \frac{5^2}{2^4\cdot 3}F_3$$
$$-\frac{1}{2^{11}\cdot 3^4}(t+1)(2^7\cdot 3^3\cdot 5^2t^6+49680\,t^5+153306\,t^4+160427\,t^3$$
$$+153306\,t^2+49680\,t+2^7\cdot 3^3\cdot 5^2){F_3}^3$$
$$+\frac{7^2}{2^{16}\cdot 3^3}t(2^{10}\cdot 3^2\cdot 5t^6+79200\,t^5+128104\,t^4+168593\,t^3$$
$$+128104\,t^2+79200\,t+2^{10}\cdot 3^2\cdot 5)F_3^2F_4$$
$$-\frac{7^4}{2^{16}\cdot 3^3}(t+1)t^2(2^4\cdot 3^2\cdot 5t^4+670\,t^3+1763\,t^2+670\,t+2^4\cdot 3^2\cdot 5)F_3F_4^2$$
$$+\frac{7^6}{2^{21}\cdot 3^4}t^3(2^6\cdot 5t^4+740\,t^3+1407\,t^2+740\,t+2^6\cdot 5){F_4}^3$$

1. THE FORM OF THE RESULT IS

$$\frac{f_{N,N}^{(3)}}{t^{N/2}} = C(N)F_N + \sum_{k=0}^{3} C_k(N;t)F_N^{3-k}F_{N+1}^k$$

2. OUR STARTING POINT IS NOT THE INTEGRAL BUT INSTEAD THE OPERATOR WHICH ANNIHILATES $f_{N,N}^{(3)}$

$$L_4(N)\cdot L_2(N)\cdot f_{N,N}^{(3)} = 0$$

WHERE

$$L_2(N) = Dt^2 + \frac{2t-1}{(t(t-1)}Dt - \frac{1}{4t} + \frac{1}{4(t-1)} - \frac{N^2}{4t^2}$$

$$L_4(N)\cdot A(N) = B(N)\cdot SYM^3(L_2(N))$$

$$A(N) = (t-1)tDt^3 + \frac{7}{2}(2t-1)Dt^2$$

$$+\frac{41t^2-41t+6}{4t(t-1)}Dt + \frac{9(2t-1)}{8(t-1)t}$$

$$+\frac{9(t-1)N^2}{4t}Dt - \frac{9(2t-1)N^2}{8t^2}$$

3. THE PROCEDURE USED FOR $f^{(2)}_{N,N}$ GIVES 4 COUPLED SECOND ORDER EQUATIONS THAT LEADS TO 8TH ORDER INHOMOGENEOUS ODE'S.

4.$C_0(N;t)$ **ODE HAS EXPONENTS AT** $t=0$
$-N,\ 0,\ 1,\ N+1,\ N+2,\ 2N+2,\ 2N+3,\ 3N+3$

THE SOLUTION IS THE DIRECT SUM OF

$Y_3 = t^{-2}\mathrm{SYM}^2(O_2(N+1))$
$Y_5 =$
$t^{-N-5}[(t-1)tdt - 2(2N+3)t + 4]\mathrm{SYM}^4(O_2(N+1))$

THE PALINDROMIC PROPERTY HOLDS
$C_0(t) = t^{2N+1}C_0(1/t)$

THE INHOMOGENEOUS TERM BEGINS WITH t^N

THE RELEVANT SOLUTION OF Y_5 **IS CONSTRUCTED OUT OF** $u_1(N+1;t)$ **AND** $u_2(N+1;t)$ **AS**

$$y_5 = t^{-N-5}[(t-1)tDt - 2(2N+3)t + 4] \times u_1(N+1;t)u_2(N+1;t)^3$$

5. THE POLYNOMIAL SOLUTIONS ARE

$$y_5 = \sum_{n=0}^{2N+1} d_n t^n =$$

$$- \sum_{n=0}^{N} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{k=0}^{l} (N+n+1) a_k a_{l-1} a_{m-l} b_{n-m} t^n$$

$$- \sum_{n=0}^{N-1} \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{k=0}^{l} (3N+1-n) a_k a_{l-1} a_{m-l} b_{n-m} t^{n+1}$$

$$+ \sum_{n=N+1}^{2N+1} d_n t^n$$

WITH

$$d_{2N+1-n} = d_n$$

$$b_k = \frac{(1/2)_k (N+3/2)_k}{N+2)_k k!}$$

AND

$$y_3 = \sum_{n=0}^{2N+1} c) t^n$$

$$c_{2N+1-n} = c_n = \sum_{k=0}^{n} a_k a_{n-k}$$

WHERE

$$a_k = \frac{1/2)_k (-N-1/2)_k}{(-N)_k k!}$$

6. SOME FINAL RESULTS

USING THE NORMALIZING CONDITION ON $f^{(3)}_{N,N}$ WE FIND

$$C = \frac{(3N+1)(1/2)_N}{6N!}$$

$$C_0(N;t) = -\frac{N(1/2)_N}{2N!}y_3 + \frac{(1/2)_N}{6(N+1)N!}y_5$$

4. THETA FUNCTION $q-k$ DUALITY

IN 2001 ORRICK, NICKEL, GUTTMANN AND PERK PRESENTED CONJECTURES FOR $f_{0,0}^{(n)}$ and $f_{1,1}^{(n)}$ FOR ALL n IN TERMS OF

$$\Phi_0(\sum_{n=0}^{\infty} c_n z^n) = \sum_{n=0}^{\infty} c_n q^{n^2/4}$$

$$2^{-n}(1-k^2)^{1/4} f_{0,0}^{(n)} = \frac{(1,k^{-1/2})}{\theta_3}\Phi_0(\frac{z^n(1-z^2)}{(1+z^2)^{n+1}})$$

and

$$2^{-n}(1-k^2)^{1/4} f_{1,1}^{(n)} = \frac{2(n+1)(1,k^{-1/2})}{\theta_2\theta_3^2}\Phi_0(\frac{z^{n+1}(1-z^2)}{(1+z^2)^{n+2}})$$

where $k = t^{1/2}$

$$(1,k^{-1/2}) = \begin{cases} 1 & \text{for} \quad T < T_c \quad (n \text{ even}) \\ k^{-1/2} & \text{for} \quad T > T_c \quad (n \text{ odd}) \end{cases}$$

WE HAVE DEFINED

$$\theta_1(u,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin[(2n+1)u]$$
$$\theta_2(u,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos[(2n+1)u]$$
$$\theta_3(u,q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nu$$
$$\theta_4(u,q) = 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nu$$

FOR $u=0$ **WE USE**

$$\theta_2 = \theta_2(0,q), \qquad \theta_3 = \theta_3(0,q), \qquad \theta_4 = \theta_4(0,q)$$

$q = e^{i\pi\tau}$ **where** $\tau = iK(k')/K(k)$

$$k = 4q^{1/2}\prod_{n=1}^{\infty}\left[\frac{1+q^{2n}}{1+q^{2n-1}}\right]^4 = \frac{\theta_2^2}{\theta_3^2}$$
$$k' = (1-k^2)^{1/2} = \frac{\theta_4^2}{\theta_3^2}$$

$$\frac{2}{\pi}K = \theta_3^2, \qquad \textbf{and:} \qquad \frac{dq}{dk} = \frac{\pi^2}{2}\frac{q}{kk'^2K^2}$$

$$q\frac{d}{dq} = \frac{2}{\pi^2}k\,k'^2K^2\frac{d}{dk}$$

We will also use

$$\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2}, \qquad \frac{dE}{dk} = \frac{E-K}{k}$$

EXAMPLE OF $f_{0,0}^{(2n)}$

$$f_{0,0}^{(2n)} = \frac{1}{\theta_4} \cdot \Phi_0\left(\frac{2^{2n} z^{2n}(1-z^2)}{(1+z^2)^{2n+1}}\right)$$

USING

$$\frac{2^{2n} z^{2n}(1-z^2)}{(1+z^2)^{2n+1}}$$

$$= 2\,\frac{(-1)^n}{(2n)!} \sum_{j=0}^{\infty} (-1)^j\, z^{2j} \prod_{m=0}^{n-1} 4\,[j^2 - m^2]$$

WE FIND

$$f_{0,0}^{(2n)} = 2\,\frac{(-1)^n\, 4^n}{\theta_4 (2n)!} \sum_{j=0}^{\infty} (-1)^j\, q^{j^2} \prod_{m=0}^{n-1} [j^2 - m^2]$$

$$= 2\frac{(-1)^n\, 4^n}{\theta_4 (2n)!} \sum_{j=0}^{\infty} (-1)^j \prod_{m=0}^{n-1} [q\,\frac{d}{dq} - m^2] q^{j^2}$$

$$= \frac{(-1)^n\, 4^n}{\theta_4 (2n)!} \prod_{m=1}^{n-1} [q\,\frac{d}{dq} - m^2] q \frac{d}{dq} \theta_4$$

CONVERSION FROM q TO k

$$\theta_4^2 = \frac{2}{\pi}k' K$$

AND THUS

$$q\frac{d}{dq}\theta_4^2 = \frac{2}{\pi^2}k\,k'^2 K^2 \frac{d}{dk}\left(\frac{2}{\pi}k' K\right)$$

WHICH REDUCES TO

$$q\frac{d}{dq}\theta_4^2 = \frac{2}{\pi^2}\cdot k' K^2 \frac{2}{\pi}\{E-K\}$$

AND

$$2\theta_4 q\frac{d}{dq}\theta_4 = \frac{2}{\pi^2}\theta_4^2 K\{E-K\}$$

SO

$$\frac{1}{\theta_4}q\,\frac{d}{dq}\theta_4 = \frac{1}{\pi^2}K\{E-K\}$$

TO EVALUATE $f_{0,0}^{(2)}$ SET $n=1$ TO OBTAIN

$$f_{0,0}^{(2)} = \frac{2}{\pi^2}K\{K-E\}$$

FOR $f_{0,0}^{(2n)}$ EACH $q\frac{d}{dq}$
GIVES TWO FACTORS OF K AND E

EXAMPLE $f_{0,0}^{(2n+1)}$

$$f_{0,0}^{(2n+1)} = \frac{\theta_3}{\theta_2\theta_4}\Phi_0(\frac{2^{2n+1}z^{2n+1}(1-z^2)}{(1+z^2)^{2n+2}})$$

USING

$$\frac{2^{2n+1}z^{2n+1}(1-z^2)}{(1+z^2)^{2n+2}} = \frac{(-1)^n}{(2n+1)!}\sum_{j=0}^{\infty}(2j+1)(-1)^j z^{2j+1}$$

$$\times \prod_{m=0}^{n-1}[(2j+1)^2-(2m+1)^2]$$

WE FIND

$$f_{0,0}^{(2n+1)} = \frac{\theta_3}{\theta_2\theta_4}\frac{2(-1)^n}{(2n+1)!}\sum_{j=0}^{\infty}(2j+1)(-1)^j q^{(2j+1)^2/4}$$

$$\prod_{m=0}^{n-1}[(2j+1)^2-(2m+1)^2]$$

$$= \frac{\theta_3}{\theta_2\theta_4}\frac{2\ (-1)^n}{(2n+1)!}\prod_{m=0}^{n-1}[4\,q\frac{d}{dq}-(2m+1)^2]$$

$$\times \sum_{j=0}^{\infty}(2j+1)(-1)^j q^{(2j+1)^2/4}$$

THUS USING

$$2\sum_{j=0}^{\infty}(2j+1)(-1)^j q^{(2j+1)^2/4} = \frac{\partial}{\partial u}\theta_1(u,q)|_{u=0} = \theta_2\theta_3\theta_4$$

WE FIND THE RESULT

$$f_{0,0}^{(2n+1)} = \frac{\theta_3}{\theta_2\theta_4}\frac{(-1)^n}{(2n+1)!}\prod_{m=0}^{n-1}[4q\frac{d}{dq}-(2m+1)^2]\theta_2\theta_3\theta_4$$

REDUCTION FROM q TO k.

FOR $n = 0$

$$f_{0,0}^{(1)} = \theta_3^2 = \frac{2}{\pi} \cdot K$$

FOR $n \geq 3$ WE USE

$$\theta_2^2\theta_3^2\theta_4^2 = k\,k'(2\,K/\pi)^3$$

AND THUS

$$\begin{aligned} q\frac{d}{dq}\theta_2^2\theta_3^3\theta_4^2 &= 2\theta_2\theta_3\theta_4 q\frac{d}{dq}\theta_2\theta_3\theta_4 \\ &= \frac{2}{\pi^2}k\,k'^2\,K^2\frac{d}{dk}\{k\,k'(2\,K/\pi)^3\} \\ &= \frac{2}{\pi^2}k\,k'(2K/\pi)^3K\{(k^2-2)\,K + 3E\} \\ &= \frac{2}{\pi^2}\cdot\theta_2^2\theta_3^2\theta_4^2K\{(k^2-2)K+3E\} \end{aligned}$$

THEREFORE

$$q\frac{d}{dq}\theta_2\theta_3\theta_4 = \frac{1}{\pi^2}\theta_2\theta_3\theta_4\cdot K\,\{(k^2-2)\,K + 3E\} \quad (1)$$

AND THUS

$$f_{0,0}^{(3)} = \frac{1}{3!}\{(2/\pi)K - (2/\pi)^3K^2[(k^2-2)K+3E\}$$

WE FINALLY NOTE THAT FOR ALL N

$$f^{(1)}_{N,N} = \frac{t^{N/2}(1/2)_N}{N!} F(1/2, N+1/2; N+1; t)$$
$$= (-1)^N \prod_{j=0}^{N-1} \left[\frac{2}{(2j+1)\theta_2^2\theta_3^2} q\frac{d}{dq} - \frac{\theta_2^2}{\theta_3^2} - \frac{j\theta_4^4}{(2j+1)\theta^2\theta_3^2} \right] \theta_3^2$$

FUTURE DIRECTIONS

1. EXTEND THE RESULTS TO ALL N AND ALL n.

2. UNDERSTAND THE RESULTS IN TERMS OF MODULAR PROPERTIES OF ELLIPTIC FUNCTIONS

3. FIND AN ANALYTIC DERIVATION OF ALL RESULTS WHICH STARTS FROM THE INTEGRAL.

IN THIS RESPECT THERE IS AN ANALOGUE BETWEEN KOREPIN'S FACTORIZATION OF XXX CORRELATIONS STARTING FROM THE INTEGRALS AND THE JIMBO, MIWA, SMIRNOV DERIVATION STARTING FROM THE FUNCTIONAL EQUATIONS FOR THE INHOMOGENEOUS PROBLEM.