The Relativistic Avatars of Giant Magnons

The symmetric space sine-Gordon theories

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The emergence of integrability is a remarkable property of the AdS/CFT correspondence

- Existence of an infinite tower of hidden conserved charges on both sides of the correspondence.
- Implies exact spectrum, S-matrix, etc.
- Enables the quantitative investigation of the conjectured duality.
- Manifested by the appearance of special solvable models and equations in explicit calculations.

Outline

1. The symmetric space sine-Gordon theories
2. SSSG theories and the AdS/CFT correspondence
3. Giant magnons and their solitonic avatars
4. The $\mathbb{C}P^{n+1}$ SSSG theories
5. Open problems

The SSSG theories

Symmetric space sine-Gordon (SSSG) theories

- Two-dimensional Integrable relativistic theories.
- Obtained from sigma models via the Pohlmeyer reduction.
- Relevant for the investigation of the AdS/CFT correspondence.
- Admit soliton solutions $\longrightarrow$ Giant Magnons

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- Relevant for the investigation of the AdS/CFT correspondence.
- Admit soliton solutions $\longrightarrow$ Giant Magnons
- Quantum S-matrices not known in general!

Pohlmeyer reduction of symmetric space sigma models

1. Take a symmetric space F/G (compact groups)
   - Involution: $\sigma^2 = 1, \quad \sigma(G) = G$
   - Lie algebra decomposition: $\quad \mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}, \quad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$
2. Define a sigma model with target F/G:

$$\boxed{\mathscr{L} = \text{Tr}\left(\partial_\mu \mathcal{F} \partial^\mu \mathcal{F}\right)}$$

with $\mathcal{F} \in F$ and $\sigma(\mathcal{F}) = \mathcal{F}^{-1}$.

3. Impose the constraints (breaking conformal and relativistic invariance)

$$\boxed{T_{++} = T_{--} = \mu^2} \quad \Rightarrow \quad \boxed{\partial_\pm \mathcal{F} \mathcal{F}^{-1} = f_\pm \Lambda f_\pm^{-1}}$$

where $\sigma(\Lambda) = -\Lambda$, and $f_\pm \in F$.

The Reduced Model

- The constrained model can be re-formulated in terms of

$$\boxed{\gamma = f_-^{-1} f_+ \in G}$$

- There is a $H_L \times H_R$ gauge symmetry arising from $f_\pm \to f_\pm h_\pm$ giving

$$\gamma \to h_-^{-1} \gamma h_+, \quad \text{for} \quad h_\pm \in H \subset G \quad \text{such that} \quad h_\pm \Lambda h_\pm^{-1} = \Lambda$$

- The equations of the reduced model are zero-curvature conditions

$$\boxed{\left[\partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1} A_+^{(L)} \gamma - \Lambda\,,\ \partial_- + A_-^{(R)} - \gamma^{-1}\Lambda\gamma\right] = 0}$$

$\Rightarrow$ Classical integrability.

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$\Rightarrow$ Classical integrability.

★ These are relativistic equations!

The Reduced Model

- With particular gauge fixing conditions, the SSSG equations become the non-abelian affine Toda equations

Pohlmeyer, Eichenherr, Forger, D'Auria, Regge, ...'79-81

$$\boxed{\partial_-(\gamma^{-1}\partial_+\gamma) = [\Lambda, \gamma^{-1}\Lambda\gamma]}$$

Leznov-Saveliev'83
Ferreira-JLM-SanchezGuillen'97
Nirov-Razumov'07
....

associated to the affine Lie algebra

$$\bigoplus_{n\in\mathbb{Z}} \left(\lambda^{2n} \otimes \mathfrak{g} + \lambda^{2n+1} \otimes \mathfrak{p}\right)$$

Lagrangian formalism

Bakas-Park-Shin'95
Grigoriev-Tseytlin'08
JLM'08

★ Choosing partial gauge fixing conditions

$$H_L \times H_R \;\longrightarrow\; H_{\rm vec}$$

the SSSG equations can be derived from a relativistic Lagrangian

$$\boxed{\mathscr{L} = \mathscr{L}_{\rm gWZW}(G/H) + {\rm Tr}(\Lambda\gamma^{-1}\Lambda\gamma)}$$

with gauge group $\gamma \rightarrow h^{-1}\gamma h$, for $h \in H$.

Some features

- Degenerate vacuum $\gamma_0 \in H \;\Rightarrow\;$ non-abelian global symmetries.
- Solitons carry a topological charge $\gamma_0(t,+\infty)\gamma_0^{-1}(t,-\infty)$.
- Perturbative expansion often becomes problematic.
- Coupling constant is the level of WZW.
- Natural interpretation as perturbed CFTs.

Examples

Pohlmeyer'76

$F/G = SO(3)/SO(2) \simeq S^2, \;\; H = \emptyset$

$\longrightarrow$ sine-Gordon theory

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \cos\phi$$

$F/G = SO(4)/SO(3) \simeq S^3, \;\; H = SO(2)$

$\longrightarrow$ complex sine-Gordon theory

$$\mathscr{L} = \partial_\mu\phi\partial^\mu\phi + \cot^2\phi\,\partial_\mu\theta\partial^\mu\theta + \cos 2\phi$$

Zamolodchikov and Zamolodchikov'79
Dorey-Hollowood'95

- Both SSSG theories are exactly solved in terms of solitons: Exact spectrum and S-matrix.

Strings on curved space-times are described by worldsheet sigma models

In string theory, gauge fixing leads naturally to the Pohlmeyer constraints

Tseytlin'03

$$\boxed{\text{Virasoro constraints on } \mathbb{R}_t \times \mathfrak{M}} \xrightarrow{X^0=\mu t} \boxed{T^{\mathfrak{M}}_{\pm\pm} = \mu^2}$$

Examples of compact symmetric spaces

$$S^n = SO(n+1)/SO(n) \longrightarrow \boxed{\mathbb{R}_t \times S^n \subset \mathrm{AdS}_5 \times S^5}$$

$$\mathbb{C}P^n = SU(n+1)/U(n) \longrightarrow \boxed{\mathbb{R}_t \times \mathbb{C}P^n \subset \mathrm{AdS}_4 \times \mathbb{C}P^3}$$

Examples of non-compact symmetric spaces $\longrightarrow$ different types of Pohlmeyer reductions

$$\mathrm{AdS}_n = SO(2, n-1)/SO(1, n-1)$$

$$\mu^2 > 0 \;\rightarrow\; \mathrm{AdS}_n \times \mathbb{R}_t$$

$$\mu^2 < 0 \;\rightarrow\; \boxed{\mathrm{AdS}_n \times \mathbb{S}^1_\theta \subset \mathrm{AdS}_n \times S^5 \quad \text{or} \quad \mathrm{AdS}_n \times \mathbb{C}P^3}$$

Alday-Maldacena'09

$$\mu^2 = 0 \;\rightarrow\; \boxed{\mathrm{AdS}_n} \quad \longrightarrow \text{ gluon scattering amplitudes}$$

all relevant for the AdS/CFT correspondence!

Giant magnons

Minahan-Zarembo'04

- On the CFT side, integrability is manifested by the appearance of integrable spin chains whose Hamiltonians provide the spectrum of exact scaling/conformal dimensions Δ.

Hofman-Maldacena'06

- In the limit where Δ and a conserved charge J become infinite, with the difference $\Delta - J$ and the 't Hooft coupling held fixed, the string dual of the fundamental magnon excitations are lump-like solutions known as Giant magnons, which propagate in an infinite long string.

- Giant magnons describe the classical motion of (bosonic) strings on curved space-times of the form $R_t \times \mathfrak{M}$, with $\mathfrak{M} = F/G$ a symmetric space $\longrightarrow$ SSSG theories

Staudacher'04
Beisert'05
Arutyunov-Frolov-Zamaklar'06
Ahn-Nepomechie'08
. . . .

- For $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_4 \times \mathbb{C}P^3$, the spectrum and S-matrix of giant magnons is already known.
- The S-matrix is complicated by the fact that the worldsheet theory is non-relativistic.
- The non-relativistic giant magnons map to a relativistic soliton avatar in the SSSG theory via the (complicated) Pohlmeyer map.

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★ In principle, the equivalence between the gauged fixed worldsheet theory and the SSSG theory is at the classical level.

★ Quantum equivalence may hold in the full (conformal invariant) theory with all the fermions included!

Generalized Pohlmeyer reduction for $AdS_5 \times S^5$

Grigoriev-Tseytlin'08
Mikhailov-SchaferNakemi'08

Virasoro constraints

$$T_{\pm\pm} = T_{\pm\pm}^{AdS_5} + T_{\pm\pm}^{S^5} = 0 \rightsquigarrow \boxed{\begin{array}{l} T_{\pm\pm}^{S^5} = +\mu^2 \\ T_{\pm\pm}^{AdS_5} = -\mu^2 \leftarrow \end{array}}$$

$\longrightarrow$ Lorentz invariant Lagrangian action for $AdS_5 \times S^5$ superstring theory

$$\boxed{\mathscr{L} = \mathscr{L}_{gWZW}\left[\frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}\right] + \text{potential} + \textbf{fermions}}$$

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Outstanding problem:

Find the exact relativistic S-matrix of the SSSG theories

An approach to quantization of the generic SSSG theories

- Focus not so much on the Lagrangian but rather on the solitons themselves.
- Experience with the SG and CSG cases is that the perturbative fields re-appear anyway when implementing the bootstrap programme for the solitons.
- The route to the spectrum and S-matrix is to calculate the time-delay of classical soliton scattering

$$\lim_{\lambda\to 0} S \sim \exp\left[\frac{i}{2}\int dE\,\Delta t(E)\right]$$

- Impose all the axioms of S-matrix theory and solve the bootstrap (account for all the bound state poles).

Giant magnons and solitons

Giant magnons and solitons are related via the complicated Pohlmeyer "map" and we need a method that constructs both at the same time without the need to perform the map:

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Reformulate equations of sigma model as auxiliary linear problem

$$\left(\partial_\pm - \frac{\partial_\pm \mathcal{F}\mathcal{F}^{-1}}{1\pm\lambda}\right)\Psi(\lambda) = 0$$

equations of motion

$$\left[\partial_+ - \frac{\partial_+ \mathcal{F}\mathcal{F}^{-1}}{1+\lambda}, \partial_- - \frac{\partial_- \mathcal{F}\mathcal{F}^{-1}}{1-\lambda}\right] = 0$$

with $\mathcal{F} = \Psi(0)$.

Dressing transformation

Zakharov-Mikhailov'78
Harnad-SaintAubin-Shnider'84

$$\boxed{\Psi(\lambda) = \chi(\lambda)\Psi_0(\lambda)}$$

$$\chi(\lambda) = 1 + \frac{H_k \Gamma_{kj}^{-1} F_j^{\dagger}}{\lambda - \xi_j}, \qquad \chi^{-1}(\lambda) = 1 - \frac{H_j \Gamma_{jk}^{-1} F_k^{\dagger}}{\lambda - \mu_j},$$

$$F_j = (\Psi_0(\xi_j)^{\dagger})^{-1}\varpi_j, \qquad H_j = \Psi_0(\mu_j)\pi_j, \qquad \Gamma_{jk} = F_j^{\dagger} F_k/(\xi_j - \mu_k)$$

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Dressing data

- ξ_j and μ_j determine rapidity and angular velocity.
- The vectors ϖ_j and π_j are collective coordinates.

Giant magnons and their soliton avatars

The key fact

Hollowood-JLM'09

If the "vacuum" satisfies the Pohlmeyer constraints

$$\Psi_0(\lambda) = \exp\left(\frac{x_+}{1+\lambda} + \frac{x_-}{1-\lambda}\right)\Lambda$$

then the dressed solution also satisfies Pohlmeyer constraints

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The dressing determines both the giant magnon and the soliton

$$\mathcal{F}_{\text{magnon}} = \chi(0)e^{2t\Lambda}$$

$$\gamma_{\text{soliton}} = e^{-t\Lambda}\chi(1)^{-1}\chi(-1)e^{t\Lambda}$$

Hence the magnon and soliton have the same collective coordinates

$\mathbb{C}P^{n+1}$ giant magnons and their solitonic avatars

The $\mathbb{C}P^{n+1}$ symmetric space

$$\boxed{\mathbb{C}P^{n+1} = F/G = SU(n+2)/U(n+1)} \qquad \boxed{H = U(n)}$$

$$\Lambda = \left(\begin{array}{cc|c} 0 & -1 & \mathbf{0} \\ 1 & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right) \quad G = \left(\begin{array}{c|cc} e^{i\alpha} & 0 & \mathbf{0} \\ \hline 0 & * & * \\ \mathbf{0} & * & * \end{array}\right) \quad H = \left(\begin{array}{cc|c} e^{i\alpha} & 0 & \mathbf{0} \\ 0 & e^{i\alpha} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & * \end{array}\right)$$

Simplest case: $\mathbb{C}P^2$ SSSG

- $\gamma \in U(2)$ and we fix the $H = U(1)$ gauge by taking the slice

$$\gamma = \begin{pmatrix} e^{i\psi/2} & 0 & 0 \\ 0 & \cos\theta e^{i\varphi+i\psi/2} & e^{-i\psi/2}\sin\theta \\ 0 & -e^{i\psi/2}\sin\theta & \cos\theta e^{-i\varphi-i\psi} \end{pmatrix}$$

- The Lagrangian is Eichenherr-Honerkamp'81

$$\mathscr{L} = \frac{1}{\lambda^2}\Big[\partial_\mu\theta\partial^\mu\theta + \tfrac{1}{4}\partial_\mu\psi\partial^\mu\psi + \cot^2\theta\partial_\mu(\psi+\varphi)\partial^\mu(\psi+\varphi) + 2\mu^2\cos\theta\cos\varphi\Big]$$

with vacua at $\theta = 0$, $\phi = 0$, and $0 \leq \psi < 4\pi$.

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with vacua at $\theta = 0$, $\phi = 0$, and $0 \leq \psi < 4\pi$.

General $\mathbb{C}P^{n+1}$ case

Integrable perturbation of the $U(n+1)_k/U(n)_k$ gauged WZW model.

The elementary (non relativistic) $\mathbb{C}P^{n+1}$ giant magnons

Hollowood-JLM'09
Abbott-Aniceto-Sax'09

- Constructed using the dressing method with two poles:

$$\xi_1 = r\, e^{ip/2}, \qquad \xi_2 = 1/\xi_1, \qquad 0 \leq p \leq 2\pi$$

- Take values in a $\mathbb{C}P^2 \subset \mathbb{C}P^{n+2}$ subspace specified by a collective coordinate $\mathbf{\Omega} \in \mathbb{C}P^{n-1}$
- Noether $SU(n+2)$ charges:

$$\boxed{\Delta\mathcal{Q} = J_\Lambda \Lambda + J_H h_{\mathbf{\Omega}}}$$

$$J_\Lambda = -\frac{1+r^2}{r}\left|\sin\frac{p}{2}\right|, \quad J_H = -\frac{1-r^2}{r}\left|\sin\frac{p}{2}\right|, \quad h_{\mathbf{\Omega}} = i\left(\begin{array}{c|c} \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & -2\mathbf{\Omega}\mathbf{\Omega}^\dagger \end{array}\right)$$

Non-relativistic dispersion relation

$$\Delta - \frac{1}{2}J = -\sqrt{\frac{\lambda}{2}}\, J_\Lambda, \qquad \frac{1}{2}Q = \sqrt{\frac{\lambda}{2}} J_H, \qquad \lambda = \text{'t Hooft coupling}$$

$$\Rightarrow \boxed{\Delta - \frac{1}{2}J = \sqrt{\frac{1}{4}Q^2 + 2\lambda \sin^2\frac{p}{2}}}$$

Beisert'05
Chen-Dorey-Okamura'06

- Consequence of centrally extended $SU(2|2)$ symmetry.
- Bound state of Q elementary giant magnons of charge $Q = 1$.

The $\mathbb{C}P^{n+1}$ (relativistic) solitonic avatars

Hollowood-JLM'09

- Classical vacuum solutions: $\gamma_v \in H$ modulo gauge transformations $\gamma_v \sim U\gamma_v U^{-1} \;\Rightarrow\; \gamma_v \in$ Cartan torus of $H = U(n)$.
- $\boxed{(\xi = re^{ip/2}, \boldsymbol{\Omega})} \longrightarrow \tan q = \frac{2r}{1-r^2} \sin \frac{p}{2}$

$$\rightarrow \begin{cases} \text{Mass:} & M = \frac{4k}{\pi} \big| \sin q \big| \\ \text{Topological charge:} & \gamma^{-1}(-\infty)\gamma(+\infty) = \exp\left(-2qh_{\boldsymbol{\Omega}}\right) \\ \text{Rapidity:} & \tanh \vartheta = \frac{2r}{1+r^2} \cos \frac{p}{2} \end{cases}$$

- Classical time-delays for $(\xi_1, \mathbf{\Omega}_1) \times (\xi_2, \mathbf{\Omega}_2)$

$$\Delta t = \frac{\Delta F}{\mu |\sin Q_2| \sinh \vartheta_2}$$

Hatsuda-Tanaka'09
Kalousios-Papathanasiou'10
Hollowood-JLM'10

$$\Delta F = \log\left[\left| \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2^*} \right|^4 \left| \frac{1 - \xi_1 \xi_2}{1 - \xi_1 \xi_2^*} \right|^2 \cos^2 \Theta + \left| \frac{\xi_1 - \xi_2}{\xi_1 - \xi_2^*} \right|^2 \sin^2 \Theta \right]$$

$$= \log\left[\left| \frac{\sinh\left(\frac{\vartheta}{2} - i\frac{q_1 - q_2}{2}\right)}{\sinh\left(\frac{\vartheta}{2} - i\frac{q_1 + q_2}{2}\right)} \right|^4 \left| \frac{\cosh\left(\frac{\vartheta}{2} - i\frac{q_1 - q_2}{2}\right)}{\cosh\left(\frac{\vartheta}{2} - i\frac{q_1 + q_2}{2}\right)} \right|^2 \cos^2 \Theta \right.$$

$$\left. + \left| \frac{\sinh\left(\frac{\vartheta}{2} - i\frac{q_1 - q_2}{2}\right)}{\sinh\left(\frac{\vartheta}{2} - i\frac{q_1 + q_2}{2}\right)} \right|^2 \sin^2 \Theta \right] .$$

where $\boxed{\vartheta = \vartheta_2 - \vartheta_1}$ and $|\mathbf{\Omega}^{(2)*} \cdot \mathbf{\Omega}^{(1)}| = \cos \Theta$.

The Soliton S-matrix conjecture: $\mathbb{C}P^2$

- The classical $\mathbb{C}P^2$ SSSG theory has global (abelian) $U(1)$ symmetry.
- Bohr-Sommerfeld quantization of the $U(1)$ charge q:

$$q = \frac{\pi a}{N}, \qquad a = 1, 2, \ldots$$

- $\Rightarrow \; M_a = M \sin\left(\pi a/N\right)$, spectrum of a_{N-1} minimal S-matrix theory.
- Perturbative excitations are the solitons with smallest charge $q = 1$ (just as in CSG, or breathers in SG).

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★ Discretization of the classical moduli space of vacuum solutions:

$$\boxed{U(1) \rightarrow \mathbb{Z}_N}$$

- S-matrix with $\mathbb{Z}_N$ symmetry:

$$\boxed{S_{11}(\vartheta) = \frac{\sinh\left(\frac{\vartheta}{2} + \frac{i\pi}{N}\right)}{\sinh\left(\frac{\vartheta}{2} - \frac{i\pi}{N}\right)} \frac{\cosh\left(\frac{\vartheta}{2} + \frac{i\pi}{2N}\right)}{\cosh\left(\frac{\vartheta}{2} - \frac{i\pi}{2N}\right)}}$$

- Fusing rules:

$$a \circ b = \begin{cases} a+b, & a+b \leq N \\ a+b-N, & a+b > N \end{cases}$$

$$\Rightarrow \;\; S_{ab}(\vartheta) = S_{ab}^{(min)}(\vartheta)\, S_{ab}^{(CDD)}(\vartheta), \qquad a, b = 1, \ldots, N-1$$

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★ TBA calculation leads to $\boxed{N = 2k+1}$ to reproduce the central charge of $U(2)_k/U(1)$.

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- Global (non-abelian) $U(1) \times SU(n)$ symmetry
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Exact spectrum

- Particles $a = 1, \ldots, k$ in completely symmetric rank-a representations of $SU(n)$: $[a]$

$$q_a = \frac{\pi a}{2k+n}, \qquad M_a = M \sin(q_a)$$

- Antiparticles $a = 1, \ldots, k$ in conjugate representations: $[a^{n-1}]$

$$q_{\bar{a}} = -q_a, \qquad M_{\bar{a}} = M_a.$$

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★ Discretization of the classical moduli space of vacuum solutions

$$\boxed{\text{Cartan torus } \; U(1)^{n-1} \to \Lambda_S^*(k)}$$

The S-matrix

- Involves the trigonometric solution of the Yang-Baxter equations associated to $U_q\big(SU(n)\big)$ with $q = -\exp\big(i\pi/(k+n)\big)$

$$S_{[1][1]}(\vartheta) = \frac{\sinh(\frac{\lambda\vartheta}{2} + \frac{i\omega}{2})}{\sinh(\frac{\lambda\vartheta}{2} - \frac{i\omega}{2})} \frac{\cosh(\frac{\lambda\vartheta}{2} + \frac{i\omega}{4})}{\cosh(\frac{\lambda\vartheta}{2} - \frac{i\omega}{4})} \left(\mathbb{P}_{[2]} - \frac{\sinh(\lambda\vartheta - i\omega)}{\sinh(\lambda\vartheta + i\omega)} \mathbb{P}_{[1^2]}\right)$$

- Crossing: $S_{[a][b]}(i\pi - \vartheta) = S_{[b^{n-1}][a]}(\vartheta)$
- Fusing rules $\Rightarrow$ $\lambda = \frac{2k+n}{2k+2n}$ and $\omega = \frac{\pi}{k+n}$:

$$[a] \circ [b] = \begin{cases} [a+b] & a+b \leq k \\ 0 & a+b > k \end{cases}, \qquad [a] \circ [b^{n-1}] = \begin{cases} [a-b] & a > b \\ [(b-a)^{n-1}] & a < b. \end{cases}.$$

The S-matrix

- Involves the trigonometric solution of the Yang-Baxter equations associated to $U_q\big(SU(n)\big)$ with $q = -\exp\big(i\pi/(k+n)\big)$

$$S_{[1][1]}(\vartheta) = \frac{\sinh(\frac{\lambda\vartheta}{2} + \frac{i\omega}{2})}{\sinh(\frac{\lambda\vartheta}{2} - \frac{i\omega}{2})} \frac{\cosh(\frac{\lambda\vartheta}{2} + \frac{i\omega}{4})}{\cosh(\frac{\lambda\vartheta}{2} - \frac{i\omega}{4})} \left(\mathbb{P}_{[2]} - \frac{\sinh(\lambda\vartheta - i\omega)}{\sinh(\lambda\vartheta + i\omega)}\mathbb{P}_{[1^2]}\right)$$

- Crossing: $S_{[a][b]}(i\pi - \vartheta) = S_{[b^{n-1}][a]}(\vartheta)$
- Fusing rules $\Rightarrow$ $\lambda = \frac{2k+n}{2k+2n}$ and $\omega = \frac{\pi}{k+n}$:

$$[a]\circ[b] = \begin{cases} [a+b] & a+b \leq k \\ 0 & a+b > k \end{cases}, \qquad [a]\circ[b^{n-1}] = \begin{cases} [a-b] & a > b \\ [(b-a)^{n-1}] & a < b. \end{cases}.$$

Ahn-Bernard-LeClair'90
deVega-Fateev'91
Hollowood'93

★ The appearance of the symmetric representations is unusual. The known $SU(n)$ S-matrices give the anti-symmetric representations!

Semiclassical limit

- In the semiclassical limit

$$\boxed{k \to \infty, \qquad a \to \infty, \qquad a/k \quad \text{fixed}}$$

  The scattering amplitudes for the special (coherent) states $||\mathbf{\Omega}, a>> = \big(\Omega_i|\mathbf{e}_i>\big)^{\otimes a}$ match the classical time-delays.

- The symmetric representation of $SU(n)$ can be thought of as a fuzzy $\mathbb{C}P^{n-1}$. In the semiclassical limit, the fuzzy $\mathbb{C}P^{n-1}$ becomes a closer approximation of $\mathbb{C}P^{n-1}$ itself, which matches the fact that classical solitons exhibit a $\mathbb{C}P^{n-1}$ moduli space of solutions.

Open problems

Are magnons and solitons quantum equivalent?

- Magnons also come in symmetric representations, but their spectrum is different.
- But at the classical the charges are non-trivially related

$$\Delta \mathcal{Q}_{\text{magnon}} = F(q_{\text{soliton}}, \vartheta)$$

- Compare with the string theory results.
- Consider the SSSG theory corresponding to the whole $AdS_4 \times CP^3$.

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Other open problems

- Solve all the other classes of symmetric space sine-Gordon theories.
- Further checks of the conjectured S-matrix: TBA, etc.
- . . . .

Thank you!