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A generalization of Shapiro-Shapiro conjecture

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Theorem 0. *Let* $p(x) \in \mathbb{C}[x]$ *be a polynomial with complex coefficients. If all roots of* $p(x)$ *are real then there exists a non-zero constant* $a \in \mathbb{C}^*$ *so that the polynomial* $cp(x)$ *has real coefficients.*

Proof. By the fundamental theorem of algebra

$$p(x) = c\prod_{i=1}^{n}(x - z_i).$$

Take $a = 1/c$. $\square$

Theorem 1 (A. Eremenko, A. Gabrielov, 2002)**.** *Let $G(z)$ be a rational function with complex coefficients. Assume that all critical points of $G(z)$ are real. Then there exists a linear fractional transformation ϕ such that $\phi(G(z))$ is a rational function with real coefficients.*

Let $G(z) = p(z)/q(z)$, where $p(z), q(z) \in \mathbb{C}[z]$.

Then if all roots of the polynomial

$$p'(x)q(x) - p(x)q'(x)$$

are real, then there exist complex numbers a, b, c, d such that $ad - bc \neq 0$ and polynomials

$$ap(x) + bq(x), \qquad cp(x) + dq(x)$$

have real coefficients.

In the theorem $\phi(z) = (az + b)/(cz + d)$.

Theorem 2 (MTV, 2005). *Let $p_1(x), \ldots, p_N(x) \in \mathbb{C}[x]$ be polynomials with complex coefficients. Assume that all roots of the polynomial $\mathrm{Wr}(p_1, \ldots, p_N)$ are real. Then the complex vector space*

$$\mathrm{span}\{p_1(x), \ldots, p_N(x)\} \subset \mathbb{C}[x]$$

has a basis consisting of polynomials with real coefficients.

Here the Wronskian is

$$\mathrm{Wr}(p_1(x), \ldots, p_N(x)) = \det(p_i^{(j-1)})_{i,j=1,\ldots,N}.$$

Theorem 3 (MTV, 2007). *Let $p_1(x), \ldots, p_N(x) \in \mathbb{C}[x]$ be polynomials with complex coefficients. Let $\lambda_1, \ldots, \lambda_N$ be real numbers. Assume that all roots of the quasi-exponential function $\mathrm{Wr}(p_1 e^{\lambda_1 x}, \ldots, p_N e^{\lambda_N x})$ are real. Then the complex vector space*

$$\mathrm{span}\{p_1(x)e^{\lambda_1 x}, \ldots, p_N(x)e^{\lambda_N x}\}$$

has a basis consisting of quasi-exponentials with real coefficients.

Theorem 3 has the following curious reformulation.

Theorem 3'. *If the numbers $\lambda_1, \ldots, \lambda_N$ are real and distinct, and all eigenvalues of the matrix*

$$\begin{pmatrix} a_1 & \frac{1}{\lambda_2-\lambda_1} & \frac{1}{\lambda_3-\lambda_1} & \cdots & \frac{1}{\lambda_N-\lambda_1} \\ \frac{1}{\lambda_1-\lambda_2} & a_2 & \frac{1}{\lambda_3-\lambda_2} & \cdots & \frac{1}{\lambda_N-\lambda_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\lambda_1-\lambda_N} & \frac{1}{\lambda_2-\lambda_N} & \frac{1}{\lambda_3-\lambda_N} & \cdots & a_N \end{pmatrix}$$

are real then the numbers $a_1, \ldots, a_N$ are real.

Theorem 2 corresponds to the statement that the nilpotent matrices of that form has real diagonal elements.

This reformulation is related to properties of Calogero-Moser spaces. It also implies a criterion for the reality of irreducible representations of Cherednik algebras, (E. Horozov, M. Yakimov).

Theorem 4 (MTV, 2007)**.** *Let $p_1(x), \dots, p_N(x) \in \mathbb{C}[x]$ be polynomials with complex coefficients. Let $Q_1, \dots, Q_N$ and h be non-zero real numbers. Assume that all roots $z_1, \dots, z_n$ of the quasi-exponential function $\mathrm{Wr}_h^d(p_1 Q_1^x, \dots, p_N Q_N^x)$ are real and that $|z_i - z_j| \geqslant |2h|$ for all $i \neq j$. Then the complex vector space*

$$\mathrm{span}\{p_1(x)Q_1^x, \dots, p_N(x)Q_N^x\}$$

has a basis consisting of quasi-exponentials with real coefficients.

The discrete Wronskian $\mathrm{Wr}_h^d(f_1, \dots, f_N)$ of functions $f_1(x), \dots, f_N(x)$ (aka the Casorati determinant) is the determinant

$$\det \begin{pmatrix} f_1(x - h(N-1)) & f_1(x - h(N-3)) & \dots & f_1(x + h(N-1)) \\ f_2(x - h(N-1)) & f_2(x - h(N-3)) & \dots & f_2(x + h(N-1)) \\ \dots & \dots & \dots & \dots \\ f_N(x - h(N-1)) & f_N(x - h(N-3)) & \dots & f_N(x + h(N-1)) \end{pmatrix}.$$

The case $N = 2$ was first treated by A. Eremenko, A. Gabrielov, M. Shapiro, and A. Vainshtein (2004).

Theorem 4 also has a matrix reformulation.

Theorem 4'. Let Q_i be real disctinct numbers. Assume that the eigenvalues z_i of the matrix

$$\begin{pmatrix} a_1 & \dfrac{Q_1}{Q_2-Q_1} & \dfrac{Q_1}{Q_3-Q_1} & \dots & \dfrac{Q_1}{Q_N-Q_1} \\ \dfrac{Q_2}{Q_1-Q_2} & a_2 & \dfrac{Q_2}{Q_3-Q_2} & \dots & \dfrac{Q_2}{Q_N-Q_2} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{Q_N}{Q_1-Q_N} & \dfrac{Q_N}{Q_2-Q_N} & \dfrac{Q_N}{Q_3-Q_N} & \dots & a_N \end{pmatrix}.$$

are all real and differ at least by 1, $|z_i - z_j| \geqslant 1$ for all $i \neq j$. Then the diagonal entries a_i are all real.

The main result of this presentation is the following theorem.

Theorem 5. *Let* $p_1(x), \ldots, p_N(x) \in \mathbb{C}[x]$ *be polynomials with complex coefficients. Let* $\tilde{Q}_1, \ldots, \tilde{Q}_N$ *be non-zero real numbers. Let* h *be a purely imaginery number. Assume that the quasi-exponential function* $\mathrm{Wr}_h^d(p_1\tilde{Q}_1^x, \ldots, p_N\tilde{Q}_N^x)$ *has real coefficients and that all complex zeroes of this function have imaginery part at most* $|h|$*. Then the complex vector space*

$$\mathrm{span}\{p_1(x)\tilde{Q}_1^x, \ldots, p_N(x)\tilde{Q}_N^x\}$$

has a basis consisting of quasi-exponentials with real coefficients.

Taking the limit $h \to 0$, one can deduce differential Theorems 2 and 3 from either one of the difference Theorems 4 or 5.

The matrix reformulation of Theorem 5 has the following form.

Theorem 5'. *Let* $\lambda_1, \ldots, \lambda_N$ *be distinct real numbers. Assume that the characteristic polynomial of the matrix*

$$\begin{pmatrix} a_1 & \frac{1}{\sin(\lambda_1-\lambda_2)} & \frac{1}{\sin(\lambda_1-\lambda_3)} & \ldots & \frac{1}{\sin(\lambda_1-\lambda_N)} \\ \frac{1}{\sin(\lambda_2-\lambda_1)} & a_2 & \frac{1}{\sin(\lambda_2-\lambda_3)} & \ldots & \frac{1}{\sin(\lambda_2-\lambda_N)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{\sin(\lambda_N-\lambda_1)} & \frac{1}{\sin(\lambda_N-\lambda_2)} & \frac{1}{\sin(\lambda_N-\lambda_3)} & \ldots & a_N \end{pmatrix}.$$

has real coefficients. Assume that all eigenvalues have imaginery part at most 1. *Then the numbers* $a_1, \ldots, a_N$ *are real.*

THE PROOF via BETHE ANSATZ

Theorem 2	$\Longleftrightarrow$	$\mathfrak{sl}_N$ Gaudin model
Theorem 3	$\Longleftrightarrow$	quasi-periodic $\mathfrak{sl}_N$ Gaudin model
Theorems 4 and 5	$\Longleftrightarrow$	Non-homogeneous $\mathfrak{sl}_N$ XXX model

Spaces of quasi-exponentials	$\Longleftrightarrow$	Bethe vectors
Coeff. of diff. operators	$\Longleftrightarrow$	Eigenvalues of transfer matrices
Spaces are real	$\Longleftrightarrow$	Transfer matrices are Hermitian

Generic situation $\Longleftrightarrow$ Tensor products of vector representations

THE XXX MODEL

Yangian is the algebra generated by elements of $N \times N$ matrix $T(x)$ with relations

$$R_{(12)}(x-y)T_{(13)}(x)T_{(23)}(y) = T_{(23)}(y)T_{(13)}(x)R_{(12)}(x-y),$$

where $R(x) = x + P$.

Let $Q = \mathrm{diag}(Q_1, \dots, Q_N)$ be the diagonal matrix and set

$$\mathcal{D}_{\boldsymbol{Q}} = \mathrm{rdet}(1 - QT(x)e^{-\partial}).$$

Then

$$\mathcal{D}_{\boldsymbol{Q}} = 1 - B_{1,\boldsymbol{Q}}(x)e^{-\partial} + B_{2,\boldsymbol{Q}}(x)e^{-2\partial} - \dots + (-1)^N B_{N,\boldsymbol{Q}}(x)e^{-N\partial},$$

where $B_{i,\boldsymbol{Q}}(x)$ are series in x^{-1} with coefficients in $Y(\mathfrak{gl}_N)$. The series $B_i(x)$ are commuting series called transfer-matrices.

$T_{ij}(u)$ acts on $V(z) = \mathbb{C}^N = \langle v_i \rangle_{i=1}^N$ by series $\delta_{ij} + E_{ji}/(x-z)$.

XXX problem: diagonalize $B_{i,\boldsymbol{Q}}(x)$ on $V(z_1) \otimes \dots \otimes V(z_n)$.

The form on $V(z_1) \otimes \cdots \otimes V(z_n)$ is given by

$$\langle v_{a_1} \otimes \cdots \otimes v_{a_n}, v_{b_1} \otimes \cdots \otimes v_{b_n} \rangle = \prod_{i=1}^{n} \delta_{a_i b_i}.$$

It is a sesquilinear, positive-definite form.

Lemma 6. *For $j = 0, \ldots, N$, and $v, w \in \boldsymbol{W}(\boldsymbol{z})$,*

$$\langle B_{j,\boldsymbol{Q},\boldsymbol{z}}(x) v, w \rangle = b_{\boldsymbol{Q},\boldsymbol{z}}(x) \langle v, B_{N-j,\bar{\boldsymbol{Q}}^{-1},-\bar{\boldsymbol{z}}}(-\bar{x}-1) w \rangle,$$

where

$$b_{\boldsymbol{Q},\boldsymbol{z}}(x) = (-1)^N \prod_{j=1}^{N} Q_j \prod_{i=1}^{n} \frac{x - z_i + 1}{x - z_i}.$$

Here $B_{j,\boldsymbol{Q},\boldsymbol{z}}(x) \in \mathrm{End}(C^N)^{\otimes n}$ are the images of $B_{j,\boldsymbol{Q}}(x)$ acting in $V(z_1) \otimes \cdots \otimes V(z_n)$.

Corollary 7. *If* $Q_i\bar{Q}_i = 1$ *for* $i = 1, \ldots, N$, $-\bar{z}_{2j-1} = z_{2j}$ *for* $j = 1, \ldots, k$, *and* $-\bar{z}_j = z_j$ *for* $j > 2k$, *then for* $j = 0, \ldots, N$,

$$(B_{j,\boldsymbol{Q},\boldsymbol{z}}(x)(b_{\boldsymbol{Q},\boldsymbol{z}}(x))^{-1})^* = B_{N-j,\boldsymbol{Q},\boldsymbol{z}}(-\bar{x}-1)$$

with respect to the modified form $\langle\cdot,\cdot\rangle_k$ *given by*

$$\langle v, w\rangle_k = \langle v, \prod_{i=1}^{k} R^{\vee}_{(2i-1,2i)}(z_{2i-1} - z_{2i})w\rangle.$$

Here $R^{\vee}(x) = PR(x) = Px + 1$.

The modified form is positive-definite if $\operatorname{Re} z_i < 1/2$. Therefore the eigenvalues of $B_{j,\boldsymbol{Q},\boldsymbol{z}}(x)(b_{\boldsymbol{Q},\boldsymbol{z}}(x))^{-1}$ and $B_{N-j,\boldsymbol{Q},\boldsymbol{z}}(-\bar{x}-1)$ are complex conjugated to each other on each eigenvector.

The theorem follows.