Universal Bethe Ansatz and Scalar Products of Bethe Vectors

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Plan

- Nested Bethe ansatz (NBA) for quantum $\mathfrak{gl}_N$-symmetric models
- Tarasov-Varchenko construction of the universal Bethe vectors (UBV)
- UBV in terms of current generators of $U_q(\widehat{\mathfrak{gl}}_N)$ (method of projections)
- Universal Bethe ansatz (UBA) as the ordering problem of the current generators (via Cartan-Weyl basis)
- Integral formulas for the UBV and calculation of the scalar products of the UBV

Nested Bethe ansatz for quantum $\mathfrak{gl}_N$-symmetric models

NBA for quantum $\mathfrak{gl}_N$-symmetric models

P.Kulish, N.Reshetikhin *J.Phys. A: Math. Gen.* **16** (1983) L591

- $N \times N$ monodromy matrix $T(z)$
- Yang-Baxter relation for the monodromy matrix

$$\mathrm{R}(z,w)\cdot(T(z)\otimes\mathbf{1})\cdot(\mathbf{1}\otimes T(w)) = (\mathbf{1}\otimes T(w))\cdot(T(z)\otimes\mathbf{1})\cdot\mathrm{R}(z,w)$$

- Quantum integrals of motion: $\mathcal{T}(z) = \mathrm{tr}\ T(z)$ and higher integrals
- The problem is to find Ψ in terms of matrix elements of $T(z)$ such that $\mathcal{T}(z)\cdot\Psi = \tau(z)\cdot\Psi$
- Existing reference state Ω such that $T_{ij}(u)\Omega = 0$ for $i > j$
- NBA is an implicit procedure which relates the eigenvalue problem of the $\mathfrak{gl}_N$ model with analogous problem for $\mathfrak{gl}_{N-1}$ model
- Two types of embedding $T^{(N-1)}(z) \hookrightarrow T^{(N)}(z)$ $(i,j = 1,\ldots,N-1)$

$$T^{(N-1)}_{ij}(z) = T^{(N)}_{ij}(z) \quad \text{or} \quad T^{(N-1)}_{ij}(z) = T^{(N)}_{i+1,j+1}(z)$$

R-matrix formulation of the QAA $U_q(\widehat{\mathfrak{gl}}_N)$

R-matrix for $U_q(\widehat{\mathfrak{gl}}_N)$

$$
\begin{aligned}
\mathrm{R}(u,v) \;=\; & \sum_{1\le i\le N} \mathrm{E}_{ii}\otimes\mathrm{E}_{ii} + \\
& + \frac{u-v}{qu-q^{-1}v}\sum_{1\le i<j\le N}(\mathrm{E}_{ii}\otimes\mathrm{E}_{jj}+\mathrm{E}_{jj}\otimes\mathrm{E}_{ii}) \\
& + \frac{q-q^{-1}}{qu-q^{-1}v}\sum_{1\le i<j\le N}(u\mathrm{E}_{ij}\otimes\mathrm{E}_{ji}+v\mathrm{E}_{ji}\otimes\mathrm{E}_{ij})
\end{aligned}
$$

$$
\mathrm{R}_{12}(u_1,u_2)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{23}(u_2,u_3)=\mathrm{R}_{23}(u_2,u_3)\mathrm{R}_{13}(u_1,u_3)\mathrm{R}_{12}(u_1,u_2)
$$

L-operator formulation of $U_q(\widehat{\mathfrak{gl}}_N)$

Reshetikhin, N., Semenov-Tian-Shansky, M. *Lett. Math. Phys.* **19** (1990) 133

L-operators

$$\mathrm{L}^{\pm}(z) = \sum_{k=0}^{\infty} \sum_{i,j=1}^{N} \mathrm{E}_{ij} \otimes \mathrm{L}^{\pm}_{ij}[\pm k] z^{\mp k}$$

$$\mathrm{L}^{+}_{ji}[0] = \mathrm{L}^{-}_{ij}[0] = 0, \quad 1 \le i < j \le N, \quad \mathrm{L}^{+}_{kk}[0]\mathrm{L}^{-}_{kk}[0] = 1, \quad 1 \le k \le N$$

Commutation relations ($c = 0$, $\mu, \nu = \pm$)

$$\mathrm{R}(u,v) \cdot (\mathrm{L}^{\mu}(u) \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes \mathrm{L}^{\nu}(v)) = (\mathbf{1} \otimes \mathrm{L}^{\nu}(v)) \cdot (\mathrm{L}^{\mu}(u) \otimes \mathbf{1}) \cdot \mathrm{R}(u,v)$$

Borel subalgebras and coalgebraic structure

$$\mathrm{L}^{\pm}[n] \in U_q(\mathfrak{b}^{\pm}) \subset U_q(\widehat{\mathfrak{gl}}_N), \quad \Delta\left(\mathrm{L}^{\pm}_{ij}(u)\right) = \sum_{k=1}^{N} \mathrm{L}^{\pm}_{kj}(u) \otimes \mathrm{L}^{\pm}_{ik}(u)$$

Tarasov-Varchenko construction of the Universal Bethe vectors

Tarasov-Varchenko construction of the off-shell UBV

Tarasov, V., Varchenko, A. *Algebra and Analysis* **2** (1995) no.2, 275

$$\mathbb{T}(\bar{u}) = \mathrm{L}^{(1)}(u_1)\cdots \mathrm{L}^{(M)}(u_M)\cdot \overleftarrow{\prod_{1\le i<j\le M}} \mathrm{R}^{(ji)}(u_j,u_i) \in \left(\mathrm{End}(\mathbb{C}^N)\right)^{\otimes M} \otimes U_q(\mathfrak{b}^+)$$

Off-Shell UBV

$$\mathbb{B}_V(\bar{t}_{[\bar{n}]}) = \prod_{1\le a<b\le N-1} \prod_{1\le j\le n_b} \prod_{1\le i\le n_a} \frac{q t^b_j - q^{-1} t^a_i}{t^b_j - t^a_i}$$

$$\times\, \mathrm{tr}_{\mathbb{C}^{\otimes M}} \left(\mathbb{T}(t^1_1,\ldots,t^1_{n_1};\ \ldots\ ; t^{N-1}_1,\ldots,t^{N-1}_{n_{N-1}})\, \mathrm{E}_{21}^{\otimes n_1} \otimes\cdots\otimes \mathrm{E}_{N,N-1}^{\otimes n_{N-1}} \right) v$$

$$\bar{u} \to \bar{t}_{[\bar{n}]} = \left\{ t^1_1,\ldots,t^1_{n_1}; t^2_1,\ldots,t^2_{n_2};\ldots\ldots; t^{N-1}_1,\ldots,t^{N-1}_{n_{N-1}} \right\}, \quad M = \textstyle\sum_k n_k$$

$$\mathrm{L}^+_{ij}(t)v = 0, \quad i>j, \quad \mathrm{L}^+_{kk}(t)v = \Lambda_k(t)v$$

Examples

$$V = U_q(\mathfrak{b}^+)v, \quad \mathrm{L}^+_{ij}(t)v = 0, \quad i > j, \quad \mathrm{L}^+_{kk}(t)v = \Lambda_k(t)v$$

- $N = 2$

$$\mathbb{B}_V(\bar{t}_{[\bar{n}]}) = \mathrm{L}^+_{12}(t^1_1)\mathrm{L}^+_{12}(t^1_2)\cdots\mathrm{L}^+_{12}(t^1_{n_1})\cdot v$$

- $N = 3,\ n_1 = n_2 = 1$

$$\begin{aligned}\mathbb{B}_V(\bar{t}_{[\bar{n}]}) &= \left(\mathrm{L}^+_{23}(t^2_1)\mathrm{L}^+_{12}(t^1_1) + \frac{(q-q^{-1})t^1_1}{t^2_1 - t^1_1}\mathrm{L}^+_{13}(t^2_1)\mathrm{L}^+_{22}(t^1_1)\right)\cdot v \\ &= \left(\mathrm{L}^+_{12}(t^1_1)\mathrm{L}^+_{23}(t^2_1) + \frac{(q^{-1}-q)t^2_1}{t^1_1 - t^2_1}\mathrm{L}^+_{13}(t^1_1)\mathrm{L}^+_{22}(t^2_1)\right)\cdot v\end{aligned}$$

Futher development

- Although TV construction resolve the hierarchical relations of the NBA, it gives the expressions for the UBV in the implicit form
- Calculation of the traces in definition of the off-shell UBV $\mathbb{B}_V(\bar{t}_{[\bar{n}]})$ is a complicated problem
- In [Tarasov, V., Varchenko, A. `math.QA/0702277`] it was solved only on the level of the evaluation representations
- These formulas can be hardly used for the calculation of the scalar products of the UBV, where dual UBV $\mathbb{C}_V(\bar{\tau}_{[\bar{n}]})$ are defined similarly

Universal Bethe vectors in terms of the currents

$U_q(\widehat{\mathfrak{gl}}_N)$ in the current ('new') realization

Drinveld, V. *Soviet Math. Dokl.* **36** (1988), 212–216

Ding, J., Frenkel, I.B. *Comm. Math. Phys.* **156** (1993), 277–300

$$\mathrm{L}^{\pm}_{a,b}(t) = \mathrm{F}^{\pm}_{b,a}(t) k^{+}_{b}(t) + \sum_{b<m\leq N} \mathrm{F}^{\pm}_{m,a}(t) k^{+}_{m}(t) \mathrm{E}^{\pm}_{b,m}(t) \quad a<b$$

$$\mathrm{L}^{\pm}_{b,b}(t) = k^{\pm}_{b}(t) + \sum_{b<m\leq N} \mathrm{F}^{\pm}_{m,b}(t) k^{\pm}_{m}(t) \mathrm{E}^{\pm}_{b,m}(t)$$

$$\mathrm{L}^{\pm}_{a,b}(t) = k^{\pm}_{a}(t) \mathrm{E}^{\pm}_{b,a}(t) + \sum_{a<m\leq N} \mathrm{F}^{\pm}_{m,a}(t) k^{\pm}_{m}(t) \mathrm{E}^{\pm}_{b,m}(t) \quad a>b$$

$$L^{\pm}_{a,a}[0] = k^{\pm}_{a}[0] \qquad k^{+}_{a}[0]\, k^{-}_{a}[0] = 1$$

$$\mathrm{F}^{\pm}_{b,a}(t) = \sum_{\substack{n\geq 0\\ n<0}} \mathrm{F}^{\pm}_{b,a}[n] t^{-n} \qquad \mathrm{E}^{\pm}_{a,b}(t) = \sum_{\substack{n>0\\ n\leq 0}} \mathrm{E}^{\pm}_{a,b}[n] t^{-n}$$

Two isomorphic current realizations of $U_q(\widehat{\mathfrak{gl}}_N)$

Oskin, A., Pakuliak, S., Silantyev, A. *St. Petersburg Math. J.* **21** (2010), 651–680

$$\mathrm{L}^{\pm}_{a,b}(t) = \hat{\mathrm{F}}^{\pm}_{b,a}(t)\hat{k}^{+}_{a}(t) + \sum_{1\le m<a} \hat{\mathrm{F}}^{\pm}_{b,m}(t)\hat{k}^{+}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,a}(t) \quad a<b$$

$$\mathrm{L}^{\pm}_{a,a}(t) = \hat{k}^{\pm}_{a}(t) + \sum_{1\le m<a} \hat{\mathrm{F}}^{\pm}_{a,m}(t)\hat{k}^{\pm}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,a}(t)$$

$$\mathrm{L}^{\pm}_{a,b}(t) = \hat{k}^{\pm}_{b}(t)\hat{\mathrm{E}}^{\pm}_{b,a}(t) + \sum_{1\le m<b} \hat{\mathrm{F}}^{\pm}_{b,m}(t)\hat{k}^{\pm}_{m}(t)\hat{\mathrm{E}}^{\pm}_{m,i}(t) \quad a>b$$

These two different Gauss decompositions leads to different isomorphic current realizations of $U_q(\widehat{\mathfrak{gl}}_N)$ and correspond to two different embedding

$$U_q(\widehat{\mathfrak{gl}}_{N-1}) \hookrightarrow U_q(\widehat{\mathfrak{gl}}_N)$$

Current realization of $U_q(\widehat{\mathfrak{gl}}_N)$ [DF]

$$F_i(t) = \mathrm{F}^{+}_{i+1,i}(t) - \mathrm{F}^{-}_{i+1,i}(t) \qquad E_i(t) = \mathrm{E}^{+}_{i,i+1}(t) - \mathrm{E}^{-}_{i,i+1}(t)$$

Commutation relations

$$\ldots \quad \ldots$$

$$(qz - q^{-1}w)F_i(z)F_i(w) = F_i(w)F_i(z)(q^{-1}z - qw)$$

$$(q^{-1}z - qw)F_i(z)F_{i+1}(w) = F_{i+1}(w)F_i(z)(z - w)$$

$$k_i^{\pm}(z)F_i(w)\left(k_i^{\pm}(z)\right)^{-1} = \frac{q^{-1}z - qw}{z - w}F_i(w)$$

$$k_{i+1}^{\pm}(z)F_i(w)\left(k_{i+1}^{\pm}(z)\right)^{-1} = \frac{qz - q^{-1}w}{z - w}F_i(w)$$

$$[E_i(z), F_j(w)] = (q - q^{-1})\delta_{i,j}\delta(z/w)(k_i^{+}(z)/k_{i+1}^{+}(z) - k_i^{-}(w)/k_{i+1}^{-}(w))$$

$$\ldots \quad \ldots$$

Serre relations

Current realization of $U_q(\widehat{\mathfrak{gl}}_N)$ as quantum double

Current Borel subalgebras

$$U_F = \{F_i[n], k_j^+[m]; n \in \mathbb{Z},\ m \geq 0\} \subset U_q(\widehat{\mathfrak{gl}}_N)$$
$$U_E = \{E_i[n], k_j^-[m]; n \in \mathbb{Z},\ m \geq 0\} \subset U_q(\widehat{\mathfrak{gl}}_N)$$

Drinfeld coproduct

$$\Delta^{(D)} F_i(z) = 1 \otimes F_i(z) + F_i(z) \otimes k_i^+(z) k_{i+1}^+(z)^{-1}$$
$$\Delta^{(D)} k_i^\pm(z) = k_i^\pm(z) \otimes k_i^\pm(z)$$
$$\Delta^{(D)} E_i(z) = E_i(z) \otimes 1 + k_i^-(z) k_{i+1}^-(z)^{-1} \otimes E_i(z)$$

Intersections of the different type Borel subalgebras

Intersections

$$U_f^- = U_F \cap U_q(\mathfrak{b}^-) \qquad U_F^+ = U_F \cap U_q(\mathfrak{b}^+)$$
$$U_E^- = U_E \cap U_q(\mathfrak{b}^-) \qquad U_e^+ = U_E \cap U_q(\mathfrak{b}^+)$$

Coideal properties

$$\Delta^{(D)}(U_F^+) \subset U_F \otimes U_F^+ \qquad \Delta^{(D)}(U_f^-) \subset U_f^- \otimes U_F$$
$$\Delta^{(D)}(U_e^+) \subset U_E \otimes U_e^+ \qquad \Delta^{(D)}(U_E^-) \subset U_E^- \otimes U_E$$

Isomorphisms of vector spaces

$$m : U_f^- \otimes U_F^+ \to U_F \qquad m : U_e^+ \otimes U_E^- \to U_E$$

Method of projections

Projections

$$P^{+} : U_F \to U_F^{+} \qquad P^{-} : U_F \to U_f^{-}$$

$$P^{+}(f_{-} \cdot f_{+}) = \varepsilon(f_{-})\ f_{+} \qquad P^{-}(f_{-} \cdot f_{+}) = f_{-}\ \varepsilon(f_{+})$$

$$f_{-} \in U_f^{-} \qquad f_{+} \in U_F^{+}$$

$$P^{+*} : U_E \to U_e^{+} \qquad P^{-*} : U_E \to U_E^{-}$$

$$P^{-*}(e_{+} \cdot e_{-}) = \varepsilon(e_{+})\ e_{-} \qquad P^{+*}(e_{+} \cdot e_{-}) = e_{+}\ \varepsilon(e_{-})$$

$$e_{-} \in U_E^{-} \qquad e_{+} \in U_e^{+}$$

- Enriquez, B., Rubtsov, V. *Israel J. Math* **112** (1999) 61–108
- Enriquez, B., Khoroshkin, S., Pakuliak, S. *Comm. Math. Phys.* **276** (2007) 691–725

Let $\overline{U}_F$ be an extension of the algebra U_F formed by infinite sums of monomials which are ordered products $a_{i_1}[n_1]\cdots a_{i_k}[n_k]$ with $n_1\le\cdots\le n_k$, where $a_{i_l}[n_l]$ is either $F_{i_l}[n_l]$ or $k^+_{i_l}[n_l]$
Let $\overline{U}_E$ be an extension of the algebra U_E formed by infinite sums of monomials which are ordered products $a_{i_1}[n_1]\cdots a_{i_k}[n_k]$ with $n_1\ge\cdots\ge n_k$, where $a_{i_l}[n_l]$ is either $E_{i_l}[n_l]$ or $k^-_{i_l}[n_l]$

- the action of the projections $P^\pm$ can be extended to the algebra $\overline{U}_F$
- for any $f\in\overline{U}_F$ with $\Delta^{(D)}(f)=\sum_i f'_i\otimes f''_i$: $f=\sum_i P^-(f''_i)\cdot P^+(f'_i)$
- the action of the projections $P^{\pm *}$ can be extended to the algebra $\overline{U}_E$
- for any $e\in\overline{U}_E$ with $\Delta^{(D)}(e)=\sum_i e'_i\otimes e''_i$: $e=\sum_i P^{+*}(e'_i)\cdot P^{-*}(e''_i)$

Extension of Ding-Frenkel isomorphism

$$P^+\left(F_i(t)F_{i+1}(t)\cdots F_{j-2}(t)F_{j-1}(t)\right)=\mathrm{F}^+_{j,i}(t)$$
$$P^{+*}\left(E_{j-1}(t)E_{j-2}(t)\cdots E_{i+1}(t)E_i(t)\right)=\mathrm{E}^+_{i,j}(t)$$

Universal Bethe Vectors and Projection Method

$$\mathcal{W}(\bar{t}_{[\bar{n}]}) = P^+ \left(F_{N-1}(t^{N-1}_{n_{N-1}}) \cdots F_{N-1}(t^{N-1}_1) \quad \cdots \quad F_1(t^1_{n_1}) \cdots F_1(t^1_1) \right)$$

$$\mathcal{W}'(\bar{\tau}_{[\bar{m}]}) = P^{+*} \left(E_1(t^1_1) \cdots E_1(\tau^1_{m_1}) \quad \cdots \quad E_{N-1}(\tau^{N-1}_1) \cdots E_{N-1}(\tau^{N-1}_{m_{N-1}}) \right)$$

Weight singular vectors

Right $U_q(\mathfrak{b}^+)$-module V generated by v : $\begin{cases} E_i[n]v = 0, & n > 0; \\ k^+_j(t)v = \Lambda_j(t)v \end{cases}$

Left $U_q(\mathfrak{b}^+)$-module V' generated by v' : $\begin{cases} 0 = v'F_i[n], & n \geq 0; \\ v'k^+_j(t) = \Lambda'_j(t)v \end{cases}$

$$\mathbf{w}_V(\bar{t}_{[\bar{n}]}) = \prod_{j=2}^{N} \prod_{\ell=1}^{n_j} \Lambda_j(t^j_\ell)\, \mathcal{W}(\bar{t}_{[\bar{n}]})\, v \qquad \mathbf{w}'_{V'}(\bar{\tau}_{[\bar{m}]}) = v'\, \mathcal{W}'(\bar{\tau}_{[\bar{m}]}) \prod_{j=2}^{N} \prod_{\ell=1}^{m_j} \Lambda'_j(t^j_\ell)$$

Theorem [KP]

$$\mathbf{w}_V(\bar{t}_{[\bar{n}]}) = \mathbb{B}_V(\bar{t}_{[\bar{n}]}) \qquad \mathbf{w}'_{V'}(\bar{\tau}_{[\bar{m}]}) = \mathbb{C}_{V'}(\bar{\tau}_{[\bar{m}]})$$

UBV in terms of the matrix elements of the monodromy

$$\mathbf{w}_V(\bar{t}_{[\bar{n}]}) = \overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}} \sum_{[[\bar{s}]]} \Bigg((q-q^{-1})^{\sum_{b=1}^{N-1}(n_b - s_b^b)} \prod_{a\le b}^{N-1} \frac{1}{[s_a^b - s_{a-1}^b]_q!} \quad \times$$

$$\times \prod_{b=2}^{N-1}\prod_{a=1}^{b-1}\prod_{\ell=1}^{s_a^b} \frac{1}{1 - t^a_{\ell+\tilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell+\tilde{\mathbf{s}}_{a+1}^b}} \prod_{\ell'=1}^{\ell+\tilde{\mathbf{s}}_{a+1}^b-1} \frac{q-q^{-1}t^a_{\ell+\tilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell'}}{1-t^a_{\ell+\tilde{\mathbf{s}}_a^b}/t^{a+1}_{\ell'}}$$

$$\times \overleftarrow{\prod_{N-1\ge b\ge 1}} \left(\overrightarrow{\prod_{1\le a\le b}} \left(\prod_{\ell=s_{a-1}^b+1}^{s_a^b} \mathrm{L}^+_{a,b+1}(t_\ell^b) \right) \prod_{\ell=s_b^b+1}^{n_b} \mathrm{L}^+_{b+1,b+1}(t_\ell^b) \right) \Bigg) v$$

$$\overline{\mathrm{Sym}}_{\ \bar{t}_{[\bar{n}]}}\ G(\bar{t}_{[\bar{n}]}) = \beta(\bar{t}_{[\bar{n}]})^{-1} \mathrm{Sym}_{\ \bar{t}_{[\bar{n}]}} \left(\beta(\bar{t}_{[\bar{n}]}) \cdot G(\bar{t}_{[\bar{n}]}) \right) \qquad \beta(\bar{t}_{[\bar{n}]}) = \prod_{a=1}^{N-1}\prod_{\ell<\ell'}^{n_a} \frac{q^{-1}t_\ell^a - q t_{\ell'}^a}{t_\ell^a - t_{\ell'}^a}$$

$$0 = s_0^a \le s_1^a \le \cdots \le s_a^a\,, \quad n_a = \sum_{b=a}^{N-1} s_a^b\,, \quad \tilde{\mathbf{s}}_a^b = s_a^a + s_a^{a+1} + \cdots + s_a^{b-1}$$

Evaluation homomorphism of $U_q(\widehat{\mathfrak{gl}}_N)$ onto $U_q(\mathfrak{gl}_N)$

$$\mathcal{E}v_z\left(\mathrm{L}^+(u)\right) = \mathrm{L}^+ - \frac{z}{u}\,\mathrm{L}^- \qquad \mathcal{E}v_z\left(\mathrm{L}^-(u)\right) = \mathrm{L}^- - \frac{u}{z}\,\mathrm{L}^+$$

$$\mathrm{L}^+_{ab} = \begin{cases} (q-q^{-1})\mathsf{E}_{b,a}\mathsf{E}_{b,b} & a<b \\ \mathsf{E}_{a,a} & a=b \\ 0 & a>b \end{cases} \qquad \mathrm{L}^-_{ab} = \begin{cases} 0 & a<b \\ \mathsf{E}^{-1}_{a,a} & a=b \\ (q^{-1}-q)\mathsf{E}^{-1}_{a,a}\mathsf{E}_{b,a} & a>b \end{cases}$$

$$\mathsf{E}_{c,a} = \mathsf{E}_{c,b}\mathsf{E}_{b,a} - q^{-1}\mathsf{E}_{b,a}\mathsf{E}_{c,b}\,, \quad \mathsf{E}_{a,c} = \mathsf{E}_{a,b}\mathsf{E}_{b,c} - q\mathsf{E}_{b,c}\mathsf{E}_{a,b}\,, \quad a<b<c$$

Chevalley generators of $U_q(\mathfrak{gl}_N)$

$$\mathsf{E}_{a,a}\mathsf{E}_{b,c}\mathsf{E}^{-1}_{a,a} = q^{\delta_{ab}-\delta_{ac}}\mathsf{E}_{b,c}$$

$$[\mathsf{E}_{a,a+1},\mathsf{E}_{b+1,b}] = \delta_{ab}\frac{\mathsf{E}_{a,a}\mathsf{E}^{-1}_{a+1,a+1} - \mathsf{E}^{-1}_{a,a}\mathsf{E}_{a+1,a+1}}{q-q^{-1}}$$

$$\mathsf{E}^2_{a\pm1,a}\mathsf{E}_{a,a\mp1} - (q+q^{-1})\mathsf{E}_{a\pm1,a}\mathsf{E}_{a,a\mp1}\mathsf{E}_{a\pm1,a} + \mathsf{E}_{a,a-1}\mathsf{E}^2_{a\pm1,a} = 0$$

Off-shell UBV for the evaluation modules

Let $V_\Lambda(z)$ be an evaluation module generated by a singular vector v such that $\mathrm{E}_{a,a}v = q^{\Lambda_a}v$ [V.Tarasov, A.Varchenko. math.QA/0702277]

$$\mathbf{w}_{V_\Lambda(z)}(\bar t_{[\bar n]}) = (q-q^{-1})^{\sum_{a=1}^{N-1} n_a} \sum_{[[\bar s]]} \left(\left(\prod_{N-1\geq b>a\geq 1}^{\longleftarrow} \frac{q^{s^b_{a-1}(s^b_{a-1}-s^b_a)}}{[s^b_a - s^b_{a-1}]_q!}\, \check{\mathrm{E}}_{b+1,a}^{s^b_a - s^b_{a-1}} \right) v \right.$$

$$\left. \times\ \overline{\mathrm{Sym}}_{\ \bar t_{[\bar n]}} \left(\prod_{b=2}^{N-1} \prod_{a=1}^{b-1} \prod_{\ell=1}^{s^b_a} \frac{q^{\Lambda_{a+1}} t^a_{\ell+\tilde s^b_a} - q^{-\Lambda_{a+1}} z}{t^{a+1}_{\ell+\tilde s^b_{a+1}} - t^a_{\ell+\tilde s^b_a}} \prod_{\ell'=1}^{\ell+\tilde s^b_{a+1}-1} \frac{q t^{a+1}_{\ell'} - q^{-1} t^a_{\ell+\tilde s^b_a}}{t^{a+1}_{\ell'} - t^a_{\ell+\tilde s^b_a}} \right) \right)$$

$$\check{\mathrm{E}}_{b,a} = \mathrm{E}_{b,a}\mathrm{E}_{b,b}$$

UBA as the ordering problem of the current generators

Current generators and the Cartan-Weyl construction

It is known [Khoroshkin, S., Tolstoy, V. *J. Geom. and Phys.* **11** (1993) 101–108] that current generators coincide with the Cartan-Weyl basis

Root Ordering

If α, β, $\gamma \in \Sigma_+$ and $\gamma = \alpha + \beta$ then $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$

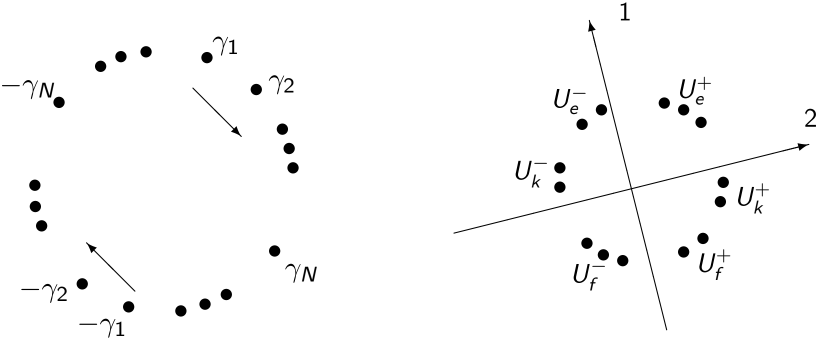
$$e_\gamma := [e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{-(\alpha,\beta)} e_\beta e_\alpha$$

$$e_{-\gamma} := [e_{-\beta}, e_{-\alpha}]_q = e_{-\beta} e_{-\alpha} - q^{(\alpha,\beta)} e_{-\alpha} e_{-\beta}$$

$$[e_{\pm\alpha}, e_{\pm\beta}]_q = \sum_{\{\gamma_j\},\{n_j\}} C^{\pm\{\gamma_j\}}_{\{n_j\}}(q)\ e^{n_1}_{\pm\gamma_1} e^{n_2}_{\pm\gamma_2} \cdots e^{n_m}_{\pm\gamma_m} \qquad \alpha, \beta \in \Sigma_+$$

$$\alpha \prec \gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_m \prec \beta \qquad \sum_j n_j \gamma_j = \alpha + \beta$$

Circle ordering



$$U_e^+ = \{E_i[n];\ n > 0\}, \quad U_k^+ = \{k_i^+[n];\ n \geq 0\}, \quad U_f^+ = \{F_i[n];\ n \geq 0\}$$
$$U_e^- = \{E_i[n];\ n \geq 0\}, \quad U_k^- = \{k_i^-[n];\ n \leq 0\}, \quad U_f^- = \{F_i[n];\ n < 0\}$$

$$\cdots \prec_c U_e^- \prec_c U_k^- \prec_c U_f^- \prec_c U_f^+ \prec_c U_k^+ \prec_c U_e^+ \prec_c \cdots$$

Universal transfer matrix for $U_q(\widehat{\mathfrak{gl}}_N)$

$$\mathcal{T}(t) = \mathrm{tr}\ \mathrm{L}^{+}(t) = \sum_{i=1}^{N} \mathrm{L}^{+}_{ii}(t)$$

$$\mathcal{T}(t) = \sum_{i=1}^{N} k_i^{+}(t) + \sum_{i<j} \mathrm{F}^{+}_{ji}(t) k_j^{+}(t) \mathrm{E}^{+}_{ij}(t) \in U_q(\mathfrak{b}^{+})$$

Let J be the left ideal of $U_q(\mathfrak{b}^{+})$, generated by all $E_i[n]$, $n > 0$

An element $u \in U_q(\mathfrak{b}^{+})$ is normal ordered if $u = u_1 \cdot u_2 \cdot u_3$ and $u_1 \in U_f^{+}$, $u_2 = U_k^{+}$, $u_3 = U_e^{+}$

Universal Bethe equations

Following theorem was proved in [Frappat, L., Khoroshkin, S., Pakuliak, S., Ragoucy, É. *Ann. Henri Poincaré* **10** (2009) 513–548] due to the special presentation of the UBV found in [Khoroshkin, S., Pakuliak, S. *SIGMA* **4** (2008)]

Theorem [FKPR]

$$\mathcal{T}(t)\cdot\mathcal{W}(\bar{t}_{[\bar{n}]}) \overset{J}{\sim} \mathcal{W}(\bar{t}_{[\bar{n}]})\left(\sum_{i=1}^{N} k_i^+(t)\prod_{a=1}^{n_i}\frac{q^{-1}t-qt_a^i}{t-t_a^i}\prod_{a=1}^{n_{i-1}}\frac{qt-q^{-1}t_a^i}{t-t_a^i}\right)$$

if the Universal Bethe equations of ABA are satisfied [ACDFR] (math-ph/0512037, 0510018, 0503014, 0411021)

$$\frac{k_i^+(t_a^i)}{k_{i+1}^+(t_a^i)}\prod_{b\neq a}^{n_i}\frac{q^{-1}t_a^i-qt_b^i}{qt_a^i-q^{-1}t_b^i}\prod_{b=1}^{n_{i-1}}\frac{qt_a^i-q^{-1}t_b^{i-1}}{t_a^i-t_b^{i-1}}\prod_{b=1}^{n_{i+1}}\frac{t_a^i-t_b^{i+1}}{q^{-1}t_a^i-qt_b^{i+1}}=1$$

$$a=1,\dots,n_i,\quad i=1,\dots,N-1$$

Integral formulas for the UBV and calculation of the scalar products

Integral formulas for the UBV in case of $U_q(\widehat{\mathfrak{sl}}_2)$

$$P^+(F(s)) = F^+(s) = \oint \frac{dy}{y} \frac{1}{1-y/s}\; F(y)$$

$$P^{+*}(E(\sigma)) = E^+(\sigma) = \oint \frac{d\nu}{\nu} \frac{\nu/\sigma}{1-\nu/\sigma}\; E(\nu)$$

Projection of the same root currents

$$P^+\left(F(s_1)\cdots F(s_b)\right) = F^+(s_1)F^+(s_2;s_1)\cdots F^+(s_b;s_{b-1},\ldots,s_1)$$

$$P^{+*}\left(E(\sigma_b)\cdots E(\sigma_1)\right) = E^+(\sigma_b;\sigma_{b-1},\ldots,\sigma_1)\cdots E^+(\sigma_2;\sigma_1)E_i^+(\sigma_1)$$

$$F^+(s_k;s_{k-1},\ldots,s_1) = F^+(s_k) - \sum_{\ell=1}^{k-1}\frac{s_k}{s_\ell}\;\varphi_{s_\ell}(s_k;s_{k-1},\ldots,s_1)F^+(s_\ell)$$

$$E^+(\sigma_k;\sigma_{k-1},\ldots,\sigma_1) = E^+(\sigma_k) - \sum_{\ell=1}^{k-1}\varphi_{\sigma_\ell}(\sigma_k;\sigma_{k-1},\ldots,\sigma_1)F^+(\sigma_\ell)$$

$$\varphi_{s_\ell}(s_k;s_{k-1},\ldots,s_1) = \prod_{j=1,\ j\neq\ell}^{k-1}\frac{s_k-s_j}{s_\ell-s_j}\prod_{j=1}^{k-1}\frac{qs_\ell-q^{-1}s_j}{qs_k-q^{-1}s_j}$$

Integral formulas for the projections

$$P^{+}\left(F(s_1)\cdots F(s_b)\right)=\prod_{1\le i<j\le b}\frac{s_i-s_j}{q^{-1}s_i-qs_j}$$
$$\times\int\frac{dy_1}{y_1}\cdots\frac{dy_b}{y_b}\;F(y_1)\cdots F(y_b)Y(s_1,\ldots,s_b;y_1,\ldots,y_b)$$
$$P^{+*}\left(E(\sigma_b)\cdots E(\sigma_1)\right)=\prod_{1\le i<j\le b}\frac{\sigma_i-\sigma_j}{q^{-1}\sigma_i-q\sigma_j}$$
$$\times\int\frac{d\nu_1}{\nu_1}\cdots\frac{d\nu_b}{\nu_b}\;E(\nu_b)\cdots E(\nu_1)Z(\sigma_1,\ldots,\sigma_b;\nu_1,\ldots,\nu_b)$$

$$Y(t_1,\ldots,t_n;x_1,\ldots,x_n)=\prod_{i=1}^{n}\frac{1}{1-x_i/t_i}\prod_{j=1}^{i-1}\frac{q^{-1}-qx_i/t_j}{1-x_i/t_j}$$

$$Z(t_1,\ldots,t_n;x_1,\ldots,x_n)=Y(t_1,\ldots,t_n;x_1,\ldots,x_n)\prod_{i=1}^{n}\frac{x_i}{t_i}$$

Factorization property of the projection

There is a simple analytical proof of the factorization of the projection

$$P^+\left(F(s_1)F(s_2)\right) = F^+(s_1)F^+(s_2) - P^+\left(F(s_1)F^-(s_2)\right)$$

Using

$$F^-(s_2) = -\int \frac{dy}{y}\,\frac{s_2/y}{1-s_2/y}\,F(y)$$

from the commutation of the currents $F(s_1)$ and $F(y)$

$$P^+\left(F(s_1)F(s_2)\right) = F^+(s_1)F^+(s_2) + \frac{s_2}{s_1}\,\frac{X(s_1)}{q^{-1}s_1 - qs_2}$$

with unknown algebraic element $X(s_1)$. Setting $s_1 = s_2$ and using $F^2(s) = 0$

$$X(s_1) = (q-q^{-1})s_1\left(F^+(s_1)\right)^2$$

Calculation of the scalar products for $U_q(\widehat{\mathfrak{sl}}_2)$ UBV

$$\langle v' \cdot P^{+*}\left(E(\sigma_n)\cdots E(\sigma_1)\right), P^{+}\left(F(s_1)\cdots F(s_n)\right)\cdot v\rangle = \mathcal{S}(\bar\sigma,\bar s)\langle v', v\rangle$$

$$[E(\nu),F(y)] = (q-q^{-1})\delta(\nu/y)(\psi^+(y)-\psi^-(\nu))\,,\quad \psi^{\pm}(y) = k_1^{\pm}(y)/k_2^{\pm}(y)$$

$$E(\nu_n)\cdots E(\nu_1)\cdot F(y_1)\cdot F(y_n) \sim (q-q^{-1})^n \prod_{i<j}\frac{q^{-1}y_i - qy_j}{qy_i - q^{-1}y_j} \times \overline{\mathrm{Sym}}_{\bar y}\left(\prod_{i=1}^{n}\delta\left(\frac{y_i}{\nu_i}\right)\right)\prod_{j=1}^{n}\psi^+(y_j)$$

$$\psi^+(y)v = \lambda(y)v$$

$$\mathcal{S}(\bar\sigma,\bar s)=\frac{(q-q^{-1})^n}{n!}\oint\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n}\prod_{i<j}\frac{(y_i-y_j)^2}{(q^{-1}y_i-qy_j)(qy_i-q^{-1}y_j)}$$

$$\times\ \overline{\mathrm{Sym}}_{\bar s}\left(Y(\bar s,\bar y)\right)\ \overline{\mathrm{Sym}}_{\bar\sigma}\left(Z(\bar\sigma,\bar y)\right)\prod_{j=1}^{n}\lambda(y_i)$$

$$\overline{\mathrm{Sym}}_{\bar t}\left(Y(\bar t,\bar x)\right)=\prod_{i<j}^{n}\frac{t_i-t_j}{q^{-1}t_i-qt_j}\mathcal{Z}(\bar t,\bar x)$$

Izergin's determinant for PF of XXZ model with DWBC

$$\mathcal{Z}(\bar t,\bar x)=\frac{\prod_{i,j}(q^{-1}t_i-qx_j)}{\prod_{i<j}(t_i-t_j)(x_j-x_i)}\det\left|\frac{1}{(t_i-x_j)(q^{-1}t_i-qx_j)}\right|_{i,j=1,\dots,n}$$

Korepin-Izergin formula for the scalar products of the $U_q(\widehat{\mathfrak{sl}}_2)$ off-shell Bethe vectors

$$\mathcal{S}(\bar\sigma,\bar s)=\prod_{i<j}h(s_i,s_j)^{-1}h(\sigma_i,\sigma_j)^{-1}\sum_{\substack{\beta\cup\bar\beta=(1,\dots,n)\\ \gamma\cup\bar\gamma=(1,\dots,n)\\ \#\beta=\#\gamma=m\in\{0,1,\dots,n\}}}(-1)^{p(\beta,\bar\beta)+p(\gamma,\bar\gamma)}\times$$

$$\times\prod_{\substack{c\in\gamma\\ b\in\beta}}h(\sigma_c,s_b)\prod_{\substack{c\in\gamma\\ c'\in\bar\gamma}}h(\sigma_c,s_{c'})\prod_{\substack{b'\in\bar\beta\\ b\in\beta}}h(s_{b'},\sigma_b)\prod_{\substack{b'\in\bar\beta\\ c'\in\bar\gamma}}h(s_{b'},\sigma_{c'})\times$$

$$\times\det\left|\frac{1}{(\sigma_c-s_b)(q\sigma_c-q^{-1}s_b)}\right|_{b\in\beta,c\in\gamma}\det\left|\frac{1}{(s_{b'}-\sigma_{c'})(qs_{b'}-q^{-1}\sigma_{c'})}\right|_{b'\in\bar\beta,c'\in\bar\gamma}$$

$$\times\prod_{b\in\beta}\lambda(s_b)\prod_{c'\in\bar\gamma}\lambda(\sigma_{c'})$$

$$h(s,\sigma)=(q^{-1}s-q\sigma)$$

$$\mathcal{S}(\bar\sigma,\bar s)\sim\oint\prod_{i>j}^{n}\frac{(y_i-y_j)(y_j-y_i)}{(qy_i-y_jq^{-1})(qy_j-y_iq^{-1})}\cdot\mathcal{Z}_n(\bar s,\bar y)\tilde{\mathcal{Z}}_n(\bar\sigma,\bar y)\,dy_1\dots dy_n,$$

$$\tilde{\mathcal{Z}}_n(\bar\sigma,\bar y)=\frac{\prod_{i,j=1}^{n}(q\sigma_i-q^{-1}y_j)}{\prod_{i>j}^{n}(\sigma_i-\sigma_j)(y_j-y_i)}$$

$$\times\det_n\left|\frac{\lambda(y_j)}{(\sigma_i-y_j)(q^{-1}\sigma_i-qy_j)}+\frac{\kappa}{(\sigma_i-y_j)(q\sigma_i-q^{-1}y_j)}\prod_{k=1}^{n}\frac{q\sigma_k-q^{-1}y_j}{q^{-1}\sigma_k-qy_j}\right|$$

$$\lambda(\sigma_i)=\kappa\prod_{k=1}^{n}\frac{q\sigma_k-q^{-1}\sigma_i}{q^{-1}\sigma_k-q\sigma_i},$$

Slavnov's determinant formula for the scalar products

$$\mathcal{S}(\bar s,\bar\sigma)\sim\frac{\prod_{i,j=1}^{n}(q^{-1}\sigma_i-qs_j)}{\prod_{i>j}^{n}(\sigma_i-\sigma_j)(s_j-s_i)}$$

$$\times\det_n\left|\frac{\psi(s_j)}{(\sigma_i-s_j)(q^{-1}\sigma_i-qs_j)}+\frac{\kappa}{(\sigma_i-s_j)(q\sigma_i-q^{-1}s_j)}\prod_{k=1}^{n}\frac{q\sigma_k-q^{-1}s_j}{q^{-1}\sigma_k-qs_j}\right|$$

Scalar products of the $U_q(\widehat{\mathfrak{gl}}_3)$ UBV

Khoroshkin, S., Pakuliak, S. *Theor. Math. Phys.* **145** (2005) n1 1373–1399

$$
\begin{aligned}
P^+\left(F_1(t_1)\cdots F_1(t_a)F_2(s_1)\cdots F_2(s_b)\right) &= \sum_{k=0}^{\min\{a,b\}} \frac{1}{k!(a-k)!(b-k)!} \\
\times\, \overline{\mathrm{Sym}}_{\bar t,\bar s} \Big(P^+ &\left(F_1(t_1)\cdots F_1(t_{a-k})F_{3,1}(t_{a-k+1})\cdots F_{3,1}(t_a)\right) \\
&\times\ P^+\left(F_2(s_{k+1})\cdots F_2(s_b)\right) Z(t_a,\ldots,t_{a-k+1};s_k,\ldots,s_1)\Big)
\end{aligned}
$$

$$
F_{3,1}(t) = (q^{-1}-q)F_2(t)F_1(t) \in \overline{U}_F \qquad P^+\left(F_{3,1}(t)\right) = (q-q^{-1})\mathrm{F}^+_{3,1}(t)
$$

$$
\begin{aligned}
&P^+\left(F_1(t_1)\cdots F_1(t_a)F_2(s_1)\cdots F_2(s_b)\right) = \\
&= \oint \frac{dy_1}{y_1}\cdots\oint\frac{dy_b}{y_b}\oint\frac{dx_1}{x_1}\cdots\oint\frac{dx_a}{x_a}\,\mathbb{F}(\bar t,\bar s;\bar x,\bar y) \\
&\times F_2(y_1)\cdots F_2(y_b)\ F_1(x_a)\cdots F_1(x_1)
\end{aligned}
$$

Kernel of the integral for $U_q(\widehat{\mathfrak{gl}}_3)$ UBV

$$\mathbb{F}(\bar t,\bar s;\bar x,\bar y)=\overline{\mathrm{Sym}}_{\ \bar t,\bar s}\left(\sum_{k=0}^{\min\{a,b\}}\frac{(q^{-1}-q)^k}{k!(a-k)!(b-k)!}\prod_{k<i<j\le b}\frac{s_i-s_j}{q^{-1}s_i-qs_j}\right.$$

$$\times\prod_{\substack{1\le i<j\le a-k\\ a-k<i<j\le a}}\frac{t_i-t_j}{q^{-1}t_i-qt_j}Z(t_a,\dots,t_{a-k+1};s_k,\dots,s_1)\ Y(t_a,\dots,t_1;x_a,\dots,x_1)$$

$$\left.\times\ Y(x_a,\dots,x_{a-k+1},s_{k+1},\dots,s_b;y_1,\dots,y_b)\prod_{j=1}^{a-k}\prod_{i=k+1}^{b}\frac{q^{-1}-qy_i/x_j}{1-y_i/x_j}\right)$$

Example for $P^+\left(F_1(t)F_2(s)\right)$

$$P^+\left(F_1(t)F_2(s)\right) = \oint \frac{dy}{y} \oint \frac{dx}{x} \mathbb{F}(t,s;x,y)\; F_2(y)F_1(x)$$

$$\mathbb{F}(t,s;x,y) = Y(t;x)Y(s;y)\frac{q^{-1}-qy/x}{1-y/x} + (q^{-1}-q)Z(t,s)Y(t;x)Y(x;y)$$

$$\oint \frac{dy}{y}\, \frac{1}{1-y/s}\frac{q^{-1}-qy/x}{1-y/x}\; F_2(y)F_1(x) = \frac{q^{-1}x-qs}{x-s}F_2^+(s;x)F_1(x) = F_1(x)F_2^+(s)$$

$$(q^{-1}-q)F_2^+(x)F_1(x) = F_1(x)F_2[0] - qF_2[0]F_1(x)$$

$$P^+\left(F_1(t)F_2(s)\right) = \mathrm{F}^+_{2,1}(t)\mathrm{F}^+_{3,2}(s) + \frac{(q-q^{-1})s}{t-s}\mathrm{F}^+_{3,1}(t)$$

$$F_1^+(t) = \mathrm{F}^+_{2,1}(t), \quad F_2^+(s) = \mathrm{F}^+_{3,2}(s)$$

$$F_1^+(t)F_2[0] - qF_2[0]F_1^+(t) = P^+\left(F_{3,1}(t)\right) = (q-q^{-1})\mathrm{F}^+_{3,1}(t)$$

Exchange relation in quantum double

For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ from $\mathcal{D}(\mathcal{A}) = \mathcal{A} \oplus \mathcal{B}$

$$\langle a^{(2)}, b^{(2)} \rangle \; b^{(1)} \cdot a^{(1)} = a^{(2)} \cdot b^{(2)} \; \langle a^{(1)}, b^{(1)} \rangle$$

where $\Delta_{\mathcal{A}}(a) = a^{(1)} \otimes a^{(2)}$ and $\Delta_{\mathcal{B}}(b) = b^{(1)} \otimes b^{(2)}$.

Let

$$a = \mathcal{E} = \mathcal{E}(\bar{\mu}, \bar{\nu}) = E_1(\mu_1) \cdots E_1(\mu_a) \; E_2(\nu_b) \cdots E_2(\nu_1) \in \overline{U}_E$$

$$b = \mathcal{F} = \mathcal{F}(\bar{x}, \bar{y}) = F_2(y_1) \cdots F_2(y_b) F_1(x_a) \cdots F_1(x_1) \in \overline{U}_F$$

Commutators of the products of the total currents

$$\Delta^{(D)}\mathcal{E} = 1 \otimes \mathcal{E} + \mathcal{E}' \otimes \mathcal{E}'' \qquad \Delta^{(D)}\mathcal{F} = \mathcal{K}^{+} \otimes \mathcal{F} + \mathcal{F}' \otimes \mathcal{F}''$$

where $\varepsilon(\mathcal{E}') = 0$ and the elements $\mathcal{E}''$ contains at least one negative Cartan current $k_i^{-}(\tau)$. An element $\mathcal{K}^{+}$

$$\mathcal{K}^{+} = \prod_{i=1}^{a} \psi_1^{+}(x_i) \prod_{j=1}^{b} \psi_2^{+}(y_j)$$

Exchange relations modulo ideals

$$\mathcal{E} \cdot \mathcal{F} \sim \langle \mathcal{E}, \mathcal{F} \rangle \cdot \mathcal{K}^{+}$$

$$\langle a_1 a_2, b \rangle = \langle a_1 \otimes a_2, \Delta_{\mathcal{B}}(b) \rangle \,, \qquad \langle a, b_1 b_2 \rangle = \langle \Delta_{\mathcal{A}}(a), b_2 \otimes b_1 \rangle$$

$$\langle E_i(z), F_j(w) \rangle = (q - q^{-1}) \delta_{ij} \delta(z/w)$$

Conclusion

Using different algebraic realizations of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ we are able to construct all objects of the Nested Bethe Ansatz and, in principle, to calculate the scalar products of the Nested Bethe vectors. The answer for scalar products can be written as multiple sums of the integrals which can be calculated and we lack the determinant representations for these integrals.

Thank you for your attention