A vertex operator approach for form factors of Belavin's $\mathbb{Z}/\boldsymbol{n}\mathbb{Z}$-symmetric model*

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1 Introduction

$(\mathbb{Z}/n\mathbb{Z})$-symmetric model: n-state version of the eight-vertex model.
$R^{ij}_{kl} = 0$ unless $i + j = k + l$ (mod n) i.e., n^3-vertex model. (8V for $n = 2$)
$R^{i+p\,j+p}_{k+p\,l+p} = R^{ij}_{kl}$. i.e., n^2 different weights. (a, b, c, d for $n = 2$)

Difficulty of elliptic vertex models
The difficulty results from the violation of spin sum conservation.

On correlation functions

- Lashkevich-Pugai's construction for 8V model.
- Kojima-Konno-Weston's construction for higher spin analogue of 8V model.
- Quano's construction for higher rank version of 8V model.

On form factors

- Lashkevich's construction for 8V model.
- Quano's construction for higher rank version of 8V model. (present talk)

2 Basic definitions

2.1 Vertex-face correspondence between Belavin's vertex model and dual face model

Let $V = \mathbb{C}^n = \langle \varepsilon_0, \cdots, \varepsilon_{n-1} \rangle_{\mathbb{C}}$. Then the R-matrix $R(v)$ acts on $V \otimes V$.

The dual face model: $A^{(1)}_{n-1}$-model. Let

$$\mathfrak{h}^* = \bigoplus_{\mu=0}^{n-1} \mathbb{C}\omega_\mu, \qquad \omega_\mu := \sum_{\nu=0}^{\mu-1} \bar{\varepsilon}_\nu, \qquad \bar{\varepsilon}_\mu = \varepsilon_\mu - \frac{1}{n} \sum_{\mu=0}^{n-1} \varepsilon_\mu. \tag{2.1}$$

Then $(a, b) \in \mathfrak{h}^{*2}$ is called admissible if $b = a + \exists \bar{\varepsilon}_\mu$.

For $(a, b, c, d) \in \mathfrak{h}^{*4}$, the Boltzmann weight $W \left[\begin{array}{cc|c} c & d & \\ b & a & v \end{array} \right]$ of the $A^{(1)}_{n-1}$-model vanishes unless $(a, b), (a, d), (b, c)$ and (d, c) are admissible.

Intertwining vectors from regime III of $A^{(1)}_{n-1}$-model to the principal regime of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model:

$$R(v_{12}) t^d_a(v_1) \otimes t^c_d(v_2) = \sum_b t^c_b(v_1) \otimes t^b_a(v_2) W \left[\begin{array}{cc|c} c & d & \\ b & a & v_{12} \end{array} \right]. \tag{2.2}$$

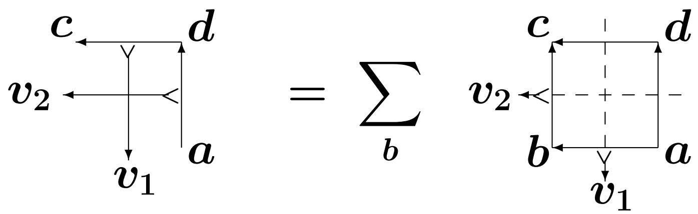


Fig 1. Vertex-face correspondence.

Dual intertwining vectors:

$$\sum_{\mu=0}^{n-1} t^*_\mu(v)_a^{a'} t^\mu(v)^a_{a''} = \delta^{a'}_{a''}, \quad \sum_{\nu=0}^{n-1} t^\mu(v)^a_{a-\bar{\varepsilon}_\nu} t^*_{\mu'}(v)_a^{a-\bar{\varepsilon}_\nu} = \delta^\mu_{\mu'}. \tag{2.3}$$

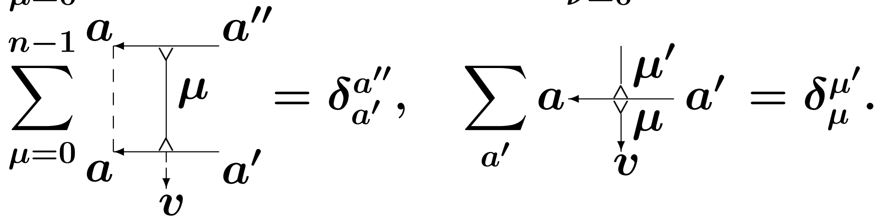


Fig 2. Dual intertwining vectors.

$$t^*(v_1)^b_c \otimes t^*(v_2)^a_b R(v_{12}) = \sum_d W\left[\begin{array}{cc|c} c & d & \\ b & a & v_{12}\end{array}\right] t^*(v_1)^a_d \otimes t^*(v_2)^d_c. \tag{2.4}$$

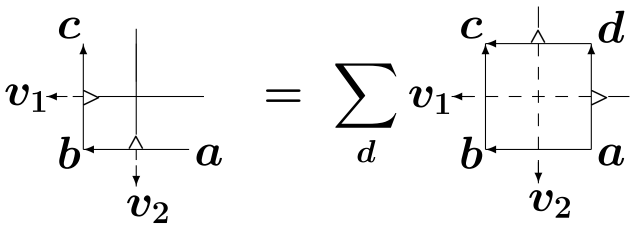


Fig 3. Dual vertex-face correspondence.

For fixed $r > 1$, let

$$S(v) = -R(v)|_{r\mapsto r-1}, \quad t'^*(u)^b_a := t^*(u)^b_a|_{r\mapsto r-1}, \quad W'\left[\begin{array}{cc|c} c & d & \\ b & a & \zeta\end{array}\right] = -\,W\left[\begin{array}{cc|c} c & d & \\ b & a & \zeta\end{array}\right]\Bigg|_{r\mapsto r-1}.$$

Then we have

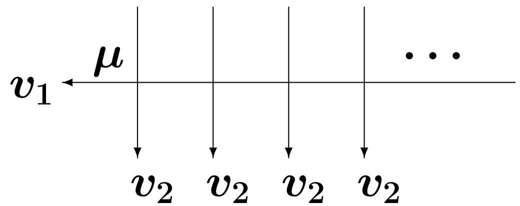
$$t'^*(v_1)^b_c \otimes t'^*(v_2)^a_b S(v_{12}) = \sum_d W'\left[\begin{array}{cc|c} c & d & \\ b & a & v_{12}\end{array}\right] t'^*(v_1)^a_d \otimes t'^*(v_2)^d_c. \tag{2.5}$$

3 Vertex operator algebra

3.1 Vertex operators for $(\mathbb{Z}/n\mathbb{Z})$-symmetric model

Introduce the type I vertex operator by the half-infinte transfer matrix

$$\Phi^{\mu}(v_1 - v_2) = \tag{3.1}$$



Op (3.1) is an intertwiner from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(i+1)}$, and satisfies

$$\Phi^{\mu}(v_1)\Phi^{\nu}(v_2) = \sum_{\mu',\nu'} R(v_1 - v_2)_{\mu'\nu'}^{\mu\nu}\Phi^{\nu'}(v_2)\Phi^{\mu'}(v_1). \tag{3.2}$$

Type II vertex operators:

$$\Psi_{\nu}^{*}(v_2)\Psi_{\mu}^{*}(v_1) = \sum_{\mu',\nu'} \Phi_{\mu'}^{*}(v_1)\Phi_{\nu'}^{*}(v_2)S(v_1 - v_2)_{\mu\nu}^{\mu'\nu'}, \tag{3.3}$$

$$\Phi^{\mu}(v_1)\Psi_{\nu}^{*}(v_2) = \chi(v_1 - v_2)\Psi_{\nu}^{*}(v_2)\Phi^{\mu}(v_1), \tag{3.4}$$

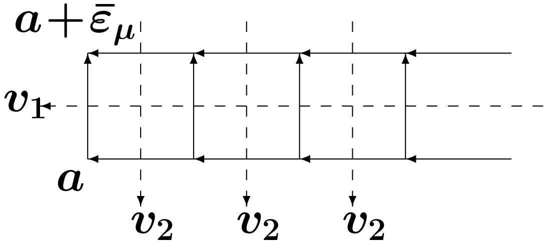
where $z = x^{2v}$ and

$$\chi(v) = z^{-\frac{n-1}{n}}\frac{(-xz;x^{2n})_{\infty}(-x^{2n-1}z^{-1};x^{2n})_{\infty}}{(-xz^{-1};x^{2n})_{\infty}(-x^{2n-1}z;x^{2n})_{\infty}}.$$

3.2 Vertex operatos for the $A^{(1)}_{n-1}$-model

Introduce the type I vertex operator by the half-infinte transfer matrix

$$\phi_a^{a+\bar{\varepsilon}_\mu}(v_1-v_2) = \qquad (3.5)$$



Op (3.5) is an intertwiner from $\mathcal{H}^{(i)}_{l,k}$ to $\mathcal{H}^{(i+1)}_{l,k+\bar{\varepsilon}_\mu}$, and satisfies

$$\Phi(v_1)^c_b \Phi(v_2)^b_a = \sum_d W\left[\begin{array}{cc|c} c & d & \\ & & v_{12} \\ b & a & \end{array}\right] \Phi(v_2)^c_d \Phi(v_1)^d_a. \qquad (3.6)$$

Bosonization of $\Phi(v_2)^b_a$ was given by Asai-Jimbo-Miwa-Pugai.

The type II vertex operators:

$$\Psi^*(v_2)^{\xi_c}_{\xi_d} \Psi^*(v_1)^{\xi_d}_{\xi_a} = \sum_{\xi_b} \Psi^*(v_1)^{\xi_c}_{\xi_b} \Psi^*(v_2)^{\xi_b}_{\xi_a} W'\left[\begin{array}{cc|c} \xi_c & \xi_d & \\ & & v_{12} \\ \xi_b & \xi_a & \end{array}\right], \qquad (3.7)$$

$$\Phi(v_1)^{a'}_a \Psi^*(v_2)^{\xi'}_{\xi} = \chi(v_{12}) \Psi^*(v_2)^{\xi'}_{\xi} \Phi(v_1)^{a'}_a. \qquad (3.8)$$

Bosonization of $\Psi^*(v)^{\xi'}_{\xi}$ was given by Furutsu-Kojima-Quano.

3.3 Tail operators and commutation relations

The intertwining operators between $\mathcal{H}^{(i)}$ and $\mathcal{H}^{(i)}_{l,k}$ ($k = l + \omega_i \pmod{Q}$):

$$\begin{aligned} T(u)^{\xi a_0} &= \prod_{j=0}^{\infty} t^{\mu_j}(-u)^{a_j}_{a_{j+1}} : \mathcal{H}^{(i)} \to \mathcal{H}^{(i)}_{l,k}, \\ T(u)_{\xi a_0} &= \prod_{j=0}^{\infty} t^*_{\mu_j}(-u)^{a_{j+1}}_{a_j} : \mathcal{H}^{(i)}_{l,k} \to \mathcal{H}^{(i)}, \end{aligned} \tag{3.9}$$

which satisfy

$$\rho^{(i)} = \sum_{\substack{k\equiv l+\omega_i \\ \pmod{Q}}} T(u)_{a\xi} \frac{\rho^{(i)}_{l,k}}{b_l} T(u)^{a\xi}, \qquad b_l = \left(\frac{(x^{2r};x^{2r})_\infty}{(x^{2r-2};x^{2r-2})_\infty} \right)^{(n-1)(n-2)/2} G'_\xi, \tag{3.10}$$

where

$$\rho^{(i)} = x^{2nH^{(i)}}, \qquad \rho^{(i)}_{l,k} = G_a x^{2nH^{(i)}_{l,k}} \ (k = a + \rho, l = \xi + \rho).$$

Tail operator Λ is defined by

$$\Lambda(u)^{\xi' a'}_{\xi\, a} = T(u)^{\xi' a'} T(u)_{\xi\, a} : \mathcal{H}^{(i)}_{l,k} \to \mathcal{H}^{(i)}_{l'k'}, \tag{3.11}$$

where $k = a + \rho$, $l = \xi + \rho$, $k' = a' + \rho$, and $l' = \xi' + \rho$. Let

$$L\left[\begin{array}{cc|c} a'_0 & a'_1 & \\ a_0 & a_1 & \end{array} u\right] := \sum_{\mu=0}^{n-1} t^*_\mu(-u)^{a_1}_{a_0} t^\mu(-u)^{a'_0}_{a'_1}. \tag{3.12}$$

Then we have

$$\Lambda(u)^{\xi' a_0'}_{\xi\, a_0} = \prod_{j=0}^{\infty} L\left[\begin{array}{cc} a_j' & a_{j+1}' \\ a_j & a_{j+1} \end{array}\middle|\, u\right] =$$

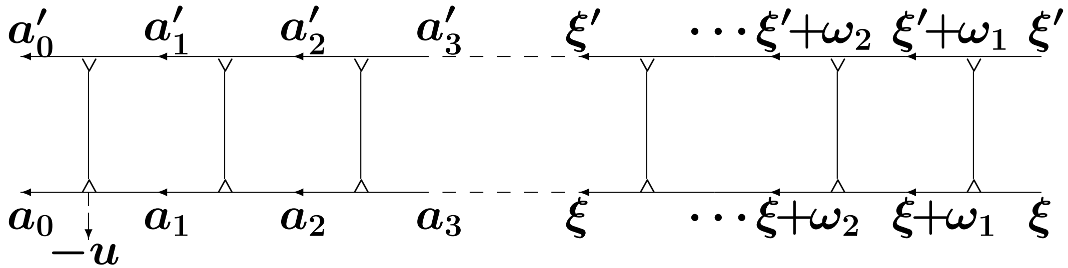


Fig 4. Tail op $\Lambda(u)^{\xi' a_0'}_{\xi\, a_0}$. The upper/lower half stands for $T(u)^{\xi' a_0'}/T(u)_{\xi\, a_0}$.

From the intertwining relations we have

$$\Lambda(u)^{\xi' c}_{\xi\, b}\Phi(v)^{b}_{a} = \sum_{d} L\left[\begin{array}{cc} c & d \\ b & a \end{array}\middle|\, u-v\right]\Phi(v)^{c}_{d}\Lambda(u)^{\xi' d}_{\xi\, a}. \tag{3.13}$$

Consider the algebra

$$\Psi^*(v)^{\xi_c}_{\xi_d}\Lambda(u)^{\xi_c\, a'}_{\xi_a\, a} = \sum_{\xi_b} L'\left[\begin{array}{cc} \xi_c & \xi_d \\ \xi_b & \xi_a \end{array}\middle|\, u'-v\right]\Lambda(u)^{\xi_c\, a'}_{\xi_b\, a}\Psi^*(v)^{\xi_b}_{\xi_a}, \tag{3.14}$$

where $u' = u + \Delta u$, and

$$L'\left[\begin{array}{cc} \xi_c & \xi_d \\ \xi_b & \xi_a \end{array}\middle|\, u\right] = L\left[\begin{array}{cc} \xi_c & \xi_d \\ \xi_b & \xi_a \end{array}\middle|\, u\right]\Bigg|_{r\mapsto r-1}. \tag{3.15}$$

We should find a free field representation of $\Lambda(u)^{\xi' a'}_{\xi\, a}$ and fix the constant Δu that solve (3.13) and (3.14).

4 Free filed realization

4.1 Bosons

Let $B_m^j\,(1 \leqslant j \leqslant n-1, m \in \mathbb{Z}\backslash\{0\})$ be the bosons introduced by Feigin-Frenkel and Awata-Kubo-Odake-Shiraishi, and let P_α, Q_β $(\alpha, \beta \in \mathfrak{h}^*)$ be the zero-modes. They satisfy appropriate commutation relations.

Bosonic Fock spaces $\mathcal{F}_{l,k}$ generated by $B_{-m}^j(m > 0)$ over the vacuum $|l,k\rangle$:

$$|l,k\rangle \;=\; \exp\left(\sqrt{-1}(\beta_1 Q_k + \beta_2 Q_l)\right)|0,0\rangle, \qquad \beta_1 = -\sqrt{\frac{r-1}{r}},\ \beta_2 = \sqrt{\frac{r}{r-1}}.$$

4.2 Type I vertex operators

Free field rep. of the type I vertex operator for $0 \leqslant \mu \leqslant n-1$

$$\phi_\mu(v_0) \;=\; \oint_C \prod_{j=1}^{\mu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{\omega_1}(v_0) U_{-\alpha_1}(v_1) \cdots U_{-\alpha_\mu}(v_\mu) \prod_{j=0}^{\mu-1} f(v_{j+1}-v_j, K_{j\mu}) \prod_{\substack{j=0\\ j\neq\mu}}^{n-1} [K_{j\mu}]^{-1}, \tag{4.1}$$

satisfies (3.6). Here U_{ω_1} and $U_{-\alpha_\mu}$ are some basic operators, $z_j = x^{2v_j}$, $f(v,w) = \frac{[v+\frac{1}{2}-w]}{[v-\frac{1}{2}]}$, $K_{\mu\nu}|_{\mathcal{F}_{l,k}} = \langle \varepsilon_\mu - \varepsilon_\nu, k\rangle$, and $C : x|z_{j-1}| < |z_j| < x^{-1}|z_{j-1}|$.

In our previous work, we obtained the free field rep. of $\Lambda(u)_{\xi\, a}^{\xi\, a'}$ satisfying (3.13) for $\xi' = \xi$ and $\mu < \nu$:

$$\Lambda(u)_{\xi\, a-\bar{\varepsilon}_\nu}^{\xi\, a-\bar{\varepsilon}_\mu} = G_K \oint \prod_{j=\mu+1}^{\nu} \frac{dz_j}{2\pi\sqrt{-1}z_j} U_{-\alpha_{\mu+1}}(v_{\mu+1}) \cdots U_{-\alpha_\nu}(v_\nu) \prod_{j=\mu}^{\nu-1} f(v_{j+1} - v_j, K_{j\mu}) G_K^{-1}, \tag{4.2}$$

where $G_K = \prod_{0 \leqslant \mu < \nu \leqslant n-1} [K_{\mu\nu}]$.

4.3 Type II vertex operators

$$V_{-\alpha_j}(v) = e^{-\beta_2(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j}\log z)} : \exp\left(-\sum_{m\neq 0} \frac{1}{m}(A_m^j - A_m^{j+1})(x^j z)^{-m}\right) :, \tag{4.3}$$

$$V_{\omega_j}(v) = e^{\beta_2(\sqrt{-1}Q_{\omega_j} + P_{\omega_j}\log z)} : \exp\left(\sum_{m\neq 0} \frac{1}{m} \sum_{k=1}^{j} x^{(j-2k+1)m} A_m^k z^{-m}\right) :, \tag{4.4}$$

where

$$A_m^j = (-1)^m \frac{[rm]_x}{[(r-1)m]_x} B_m^j. \tag{4.5}$$

Free field rep. of the type II vertex operator for $0 \leqslant \mu \leqslant n-1$

$$\psi_\mu^*(v_0) = \oint_{C'} \prod_{j=1}^{\mu} \frac{-dz_j}{2\pi\sqrt{-1}z_j} V_{-\alpha_\mu}(v_\mu) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v_0) \prod_{j=0}^{\mu-1} f^*(v_j - v_{j+1}, 1 - L_{j\mu}) \tag{4.6}$$

satisfies (3.7) and (3.8). Here $f^*(v,w) = \frac{[v-\frac{1}{2}+w]'}{[v+\frac{1}{2}]'}$, $L_{\mu\nu}|_{\mathcal{F}_{l,k}} = \langle \varepsilon_\mu - \varepsilon_\nu, l \rangle$, and C' for z_j-integration encircles the poles at $z_j = x^{-1+2k(r-1)} z_{j-1}$ $(k \in \mathbb{Z}_{\geqslant 0})$, but not the poles at $z_j = x^{1-2k(r-1)} z_{j-1}$ $(k \in \mathbb{Z}_{\geqslant 0})$.

We also introduce another type of basic operators:

$$W_{-\alpha_j}(v) = e^{-\beta_0(\sqrt{-1}Q_{\alpha_j} + P_{\alpha_j}\log(-1)^r z)} : \exp\left(-\sum_{m \neq 0} \frac{1}{m}(O_m^j - O_m^{j+1})(x^j z)^{-m} \right) :, \tag{4.7}$$

where $\beta_0 = \frac{1}{\sqrt{r(r-1)}}$, $(-1)^r$ implies $\exp(\pi\sqrt{-1}r)$ and

$$O_m^j = \frac{[m]_x}{[(r-1)m]_x} B_m^j. \tag{4.8}$$

Note that

$$W_{-\alpha_j}\left(v + \tfrac{r}{2} - \tfrac{\sqrt{-1}\pi}{2\epsilon}\right) V_{-\alpha_{j\pm1}}(v) = 0 = V_{-\alpha_{j\pm1}}(v) W_{-\alpha_j}\left(v - \tfrac{r}{2} - \tfrac{\sqrt{-1}\pi}{2\epsilon}\right), \tag{4.9}$$

$$W_{-\alpha_j}\left(v + \tfrac{r}{2} - \tfrac{\sqrt{-1}\pi}{2\epsilon}\right) V_{\omega_j}(v) = 0 = V_{\omega_j}(v) W_{-\alpha_j}\left(v - \tfrac{r}{2} - \tfrac{\sqrt{-1}\pi}{2\epsilon}\right). \tag{4.10}$$

4.4 Free field realization of tail operators

Consider (3.14) for $(\xi_a, \xi_d, \xi_c) = (\xi, \xi, \xi + \bar{\varepsilon}_{n-1})$, and $(a, a') \to (a_{n-2}, a_{n-1}) = (a + \bar{\varepsilon}_{n-2}, a + \bar{\varepsilon}_{n-1})$:

$$\Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_{\xi\, a_{n-2}}^{\xi\, a_{n-1}} = \sum_{\mu=0}^{n-1} L' \left[\begin{array}{cc} \xi + \bar{\varepsilon}_{n-1} & \xi \\ \xi + \bar{\varepsilon}_{\mu} & \xi \end{array} \middle| u' - v \right] \Lambda(v)_{\xi+\bar{\varepsilon}_{\mu}\, a_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\, a_{n-1}} \Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_{\mu}}. \tag{4.11}$$

This equation can be rewritten as follows:

$$\begin{aligned} &\Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_{n-1}} \Lambda(u)_{\xi\, a_{n-2}}^{\xi\, a_{n-1}} - \Lambda(u)_{\xi+\bar{\varepsilon}_{n-1}\, a_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\, a_{n-1}} \Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_{n-1}} \\ = &\sum_{\mu=0}^{n-2} \frac{[u' - v + \xi_{\mu\, n-1}]'}{[u' - v]'} \prod_{j \neq \mu} \frac{[\xi_{n-1\, j} + 1]'}{[\xi_{\mu j} + 1]'} \Lambda(v)_{\xi+\bar{\varepsilon}_{\mu}\, a_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\, a_{n-1}} \Psi^*(v)_{\xi}^{\xi+\bar{\varepsilon}_{\mu}}. \end{aligned} \tag{4.12}$$

Since the tail operators on LHS of (4.12) are diagonal components with respect to the ground state sectors, we have

$$\begin{aligned} &\text{LHS of (4.12)} = G_K \oint_C \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-1} \frac{-dz_j}{2\pi\sqrt{-1}z_j} \\ \times\ &[V_{-\alpha_{n-1}}(v_{n-1}), U_{-\alpha_{n-1}}(v')] V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \\ \times\ &f(v' - u, K_{n-2\, n-1}) \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - L_{jn-1}) G_K^{-1}, \end{aligned} \tag{4.13}$$

where $z_j = x^{2v_j}$ and $z' = x^{2v'}$. By using the commutation relation

$$[V_{-\alpha_j}(v), U_{-\alpha_j}(v')] = \frac{\delta(\frac{z}{-xz'}) - \delta(\frac{z'}{-xz})}{(x^{-1}-x)zz'} : V_{-\alpha_j}(v)U_{-\alpha_j}(v') :, \tag{4.14}$$

the integral for z_{n-1} of (4.13) can be evaluated by the residues at $z_{n-1} = -x^{\pm 1}z'$. Then the result is

$$\begin{aligned}\mathrm{LHS}|_{\mathcal{F}_{\xi+\rho, a+\bar{\varepsilon}_{n-2}+\rho}} &= \frac{(-1)^n}{x^{-1}-x} G_K \left(\oint_{x^{-r}C} - \oint_{x^r C} \right) \frac{dz'}{2\pi\sqrt{-1}z'} \oint_{C'} \prod_{j=1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ &\times \ F(v') W_{-\alpha_{n-1}}(v') V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) V_{\omega_1}(v) \prod_{j=0}^{n-3} f^*(v_j - v_{j+1}, 1 - \xi_{j\,n-1}) G_K^{-1},\end{aligned} \tag{4.15}$$

where

$$F(v') = \frac{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon} - \xi_{n-2\,n-1}]'}{[v_{n-2} - v' + \frac{r}{2} - \frac{\pi\sqrt{-1}}{2\epsilon}]'} \frac{[v' - u - \frac{r+1}{2} - a_{n-2\,n-1}]}{[v' - u - \frac{r+1}{2}]}. \tag{4.16}$$

The z'-integral of (4.15) can be evaluated by the residues at $z' = -x^r z_{n-2}$, $x^{-r+1+2u}$. The former residue vanishes because of (4.9). Thus, we have

$$
\begin{aligned}
(4.15) \ = \ & \oint_{C'} \prod_{j=1}^{n-2} \frac{-dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \tfrac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_1}(v_1) \\
& \times \ V_{\omega_1}(v) \prod_{j=0}^{n-2} f^*(v_j - v_{j+1}, 1 - \xi_{j\,n-1}) \frac{[a_{n-2\,n-1}]}{(x^{-1}-x)(x^{2r};x^{2r})_\infty^3} \frac{G_{a+\bar{\varepsilon}_{n-1}}}{G_{a+\bar{\varepsilon}_{n-2}}}.
\end{aligned}
\tag{4.17}
$$

On (4.17), we should read as $v_{n-1} = u + \frac{\pi\sqrt{-1}}{2\epsilon}$.

Equating (4.17) and the RHS of (4.12), we find $\Delta u = -\frac{n-1}{2} + \frac{\pi\sqrt{-1}}{2\epsilon}$, and the free filed representation of the tail operator

$$
\begin{aligned}
& \Lambda(v)_{\xi+\bar{\varepsilon}_\mu\, a+\bar{\varepsilon}_{n-2}}^{\xi+\bar{\varepsilon}_{n-1}\, a+\bar{\varepsilon}_{n-1}} = \frac{(-1)^{n-\mu}[a_{n-2\,n-1}]}{(x^{-1}-x)(x^{2r};x^{2r})_\infty^3} \frac{[\xi_{n-1\,\mu}+1]'}{[1]'} G_K {G'_L}^{-1} \\
& \times \ \oint_{C'} \prod_{j=\mu+1}^{n-2} \frac{dz_j}{2\pi\sqrt{-1}z_j} W_{-\alpha_{n-1}}\left(u - \tfrac{r-1}{2}\right) V_{-\alpha_{n-2}}(v_{n-2}) \cdots V_{-\alpha_{\mu+1}}(v_{\mu+1}) \\
& \times \ \prod_{j=\mu+1}^{n-2} f^*(v_j - v_{j+1}, L_{\mu j}) G_K^{-1} G'_L.
\end{aligned}
\tag{4.18}
$$

We omit detail but we can construct free field reps of any $\Lambda(v)_{\xi a}^{\xi' a'}$.

5 Form factors

Form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model are defined as matrix elements of some local operators. Consider the local operator

$$\mathcal{O} = E^{(1)}_{\mu_1\mu'_1} \cdots E^{(N)}_{\mu_N\mu'_N}, \tag{5.1}$$

where $E^{(j)}_{\mu_j\mu'_j}$ is the matrix unit on the j-th site. The free field representation of $\mathcal{O}$ is given by

$$\hat{\mathcal{O}} = \Phi^*_{\mu_1}(u_1) \cdots \Phi^*_{\mu_N}(u_N)\Phi^{\mu'_N}(u_N) \cdots \Phi^{\mu'_1}(u_1). \tag{5.2}$$

The corresponding form factors with m 'charged' particles are given by

$$F^{(i)}_m(\mathcal{O}; v_1, \cdots, v_m)_{\nu_1\cdots\nu_m} = \frac{1}{\chi^{(i)}} \mathrm{Tr}_{\mathcal{H}^{(i)}} \left(\Psi^*_{\nu_1}(v_1) \cdots \Psi^*_{\nu_m}(v_m)\hat{\mathcal{O}}\rho^{(i)} \right), \tag{5.3}$$

where

$$\chi^{(i)} = \mathrm{Tr}_{\mathcal{H}^{(i)}}\, \rho^{(i)} = \frac{(x^{2n}; x^{2n})_\infty}{(x^2; x^2)_\infty}. \tag{5.4}$$

and $m \equiv 0 \pmod{n}$. Note that the local operator (5.1) commute with the type II vertex operators because of (5.2) and (3.4).

By using (3.10), we can rewrite (5.3) as follows:

$$
\begin{aligned}
& F_m^{(i)}(\mathcal{O}; v_1, \cdots, v_m)_{\nu_1 \cdots \nu_m} \\
= & \frac{1}{\chi^{(i)}} \sum_{\xi_1, \cdots, \xi_m} t'^{*}_{\nu_1}\left(v_1 - u + \tfrac{n-1}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right)_{\xi}^{\xi_1} \cdots t'^{*}_{\nu_m}\left(v_m - u + \tfrac{n-1}{2} - \tfrac{\pi\sqrt{-1}}{2\epsilon}\right)_{\xi_{m-1}}^{\xi_m} \\
\times & \sum_{\substack{k \equiv l + \omega_i \\ (\mathrm{mod}\ Q)}} \sum_{\substack{a_1 \cdots a_N \\ a'_1 \cdots a'_N}} t^{*}_{\mu_1}(u_1 - u)_{a_1}^{a} \cdots t^{*}_{\mu_N}(u_N - u)_{a_N}^{a_{N-1}} t^{\mu'_N}(u_N - u)_{a'_N}^{a_N} \cdots t^{\mu'_1}(u_1 - u)_{a'_1}^{a'_2} \\
\times & \mathrm{Tr}_{\mathcal{H}_{l,k}^{(i)}} \left(\Psi^*(v_1)_{\xi_1}^{\xi} \cdots \Psi^*(v_m)_{\xi_m}^{\xi_{m-1}} \Phi^*(u_1)_{a_1}^{a} \cdots \Phi^*(u_N)_{a_N}^{a_{N-1}} \Phi(u_N)_{a'_N}^{a_N} \cdots \Phi(u_1)_{a'_1}^{a'_2} \right. \\
\times & \left. \Lambda(u)_{\xi\, a}^{\xi_m a'_1} \frac{\rho_{l,k}^{(i)}}{b_l} \right),
\end{aligned}
\tag{5.5}
$$

where $k = a + \rho$, $l = \xi + \rho$. The expression (5.5) implies that the form factors of $(\mathbb{Z}/n\mathbb{Z})$-symmetric model can be expressed in terms of the type I and the type II vertex operators of $A_{n-1}^{(1)}$-model with an insertion of a non-local operator Λ. Free filed representations of the tail operators Λ's have been constructed in the present talk, besides all other operators Φ's, Φ^*'s and Ψ^*'s on (5.5) were given by Asai-Jimbo-Miwa-Pugai, and Furutsu-Kojima-Quano, respectively. Integral formulae can be therefore obtained for form factors of Belavin's $(\mathbb{Z}/n\mathbb{Z})$-symmetric model, in principle.