INTEGRABILITY IN $\mathcal{N} = 4$ SUPER YANG MILLS: THE NON LINEAR INTEGRAL EQUATION APPROACH

Marco Rossi

Some papers on the subject:

- D. Bombardelli, D. Fioravanti, M. Rossi,
  Large spin corrections in N = 4 SYM sl(2): Still a linear integral equation,
  Nucl. Phys. B810 (2009) 460-490, arXiv:0802.0027 [hep-th];

- D. Fioravanti, P. Grinza, M. Rossi,
  The generalised scaling function: a systematic study,
  JHEP11 (2009) 037, arXiv:0808.1886 [hep-th];

- D. Fioravanti, M. Rossi,
  The high spin expansion of twist sector dimensions: the planar $\mathcal{N} = 4$ super Yang-Mills theory,
  AHEP, arXiv:1004.1081 [hep-th].

Main ideas

Integrability in $\mathcal{N} = 4$ SYM appears in form of Bethe Ansatz equations: anomalous dimensions are computable in terms of Bethe roots.

In general, instead of dealing with Bethe Ansatz equations, one can deal with nonlinear integral equations (NLIEs). A sector of $\mathcal{N} = 4$ SYM in which NLIEs are useful is the $sl(2)$ sector.

At first glance, there are complications:

- There are s real Bethe roots: OK
- Bethe roots localise on a finite interval $[-b, b]$: potential problem
- There are L holes: two outside $[-b, b]$, $L - 2$ in $[-b, b]$; L can be arbitrary large: potential problem

However, a modification of the traditional NLIE technique allows to comply with these points, at least in the high spin limit.

We will show how to:

- Write a NLIE for Bethe roots in a finite interval
- Properly take into account nonlinear terms arising from a large number of 'internal holes'

Non-linear integral equations in the $sl(2)$ sector

Bethe Ansatz equations [Beisert-Staudacher NPB 727]:

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L \left(\frac{1+\frac{g^2}{2x_k^{-2}}}{1+\frac{g^2}{2x_k^{+2}}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{s} \frac{u_k-u_j-i}{u_k-u_j+i} \left(\frac{1-\frac{g^2}{2x_k^+x_j^-}}{1-\frac{g^2}{2x_k^-x_j^+}}\right)^2 e^{2i\theta(u_k,u_j)} .$$

We define the counting function:

$$Z(u)=\Phi(u)-\sum_{k=1}^{s}\phi(u,u_k)\,,$$

where

$$\begin{aligned}
\Phi(u) &= -2L\arctan 2u - iL\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right) ,\\
\phi(u,v) &= 2\arctan(u-v)-2i\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right)+i\theta(u,v)\right] .
\end{aligned}$$

By definition $Z(u_k)=\pi(2N_k+1)$; there are also L 'extra' solutions (holes) $Z(u_h)=\pi(2M_h+1-L)$. Using the Cauchy theorem [Destri-De Vega NPB 438; Fioravanti,Mariottini,Quattrini,Ravanini PLB 390]:

$$\begin{aligned}
2\pi i\sum_{k=1}^{s} f(u_k) &= \int_{-b}^{b} dv f(v-i\epsilon)\frac{(-1)^L iZ'(v-i\epsilon)e^{iZ(v-i\epsilon)}}{1+(-1)^L\, e^{iZ(v-i\epsilon)}} + \\
&+ \int_{b}^{-b} dv f(v+i\epsilon)\frac{(-1)^L iZ'(v+i\epsilon)e^{iZ(v+i\epsilon)}}{1+(-1)^L\, e^{iZ(v+i\epsilon)}} + \\
&+ \int_{\epsilon}^{-\epsilon} idy f(-b+iy)\frac{(-1)^L iZ'(A+iy)e^{iZ(A+iy)}}{1+(-1)^L\, e^{iZ(-b+iy)}} + \\
&+ \int_{-\epsilon}^{\epsilon} idy f(b+iy)\frac{(-1)^L iZ'(b+iy)e^{iZ(b+iy)}}{1+(-1)^L\, e^{iZ(b+iy)}} - 2\pi i\sum_{h=1}^{L-2} f(u_h^{(i)}) .
\end{aligned}$$

Since $Z'(v) < 0$, for ϵ 'small':

$$\begin{aligned}\sum_{k=1}^{s} f(u_k) &= -\int_{-b}^{b} \frac{dv}{2\pi} f(v) Z'(v) + \mathrm{Im} \int_{-b}^{b} \frac{dv}{\pi} f(v - i\epsilon) \frac{d}{dv} \ln[1 + (-1)^L \, e^{iZ(v-i\epsilon)}] + \\ &+ \mathrm{Im} \int_{0}^{-\epsilon} \frac{dy}{\pi} f(-b + iy) \frac{d}{dy} \ln[1 + (-1)^L \, e^{iZ(-b+iy)}] + \\ &+ \mathrm{Im} \int_{-\epsilon}^{0} \frac{dy}{\pi} f(b + iy) \frac{d}{dy} \ln[1 + (-1)^L \, e^{iZ(b+iy)}] - \sum_{h=1}^{L-2} f(u_h^{(i)}) \, .\end{aligned}$$

In this formula all the $|e^{iZ(z)}| < 1$: this means that the ln can be replaced by their series expansions. We integrate term by term using

$$\int^{v} dv f(v - i\epsilon) \frac{d}{dv} e^{inZ(v-i\epsilon)} = e^{inx} \sum_{k=0}^{\infty} \left(\frac{i}{n} \right)^k \frac{d^k}{dx^k} f[Z^{-1}(x)] \Bigg|_{x=Z(v-i\epsilon)} ,$$

$$\int^{y} dy f(\pm b + iy) \frac{d}{dy} e^{inZ(\pm b+iy)} = e^{inx} \sum_{k=0}^{\infty} \left(\frac{i}{n} \right)^k \frac{d^k}{dx^k} f[Z^{-1}(x)] \Bigg|_{x=Z(\pm b+iy)} ,$$

in the hypothesis of convergence. We end up with

$$\begin{aligned}\sum_{k=1}^{s} f(u_k) &= -\int_{-b}^{b} \frac{dv}{2\pi} f(v) Z'(v) - \sum_{h=1}^{L-2} f(u_h^{(i)}) + NL \qquad (1) \\ NL &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{nL}}{\pi n} \Big[\sum_{k=0}^{\infty} \frac{i^{2k}}{n^{2k}} \sin nx \frac{d^{2k}}{dx^{2k}} f(Z^{-1}(x)) + \\ &+ \sum_{k=0}^{\infty} \frac{i^{2k}}{n^{2k+1}} \cos nx \frac{d^{2k+1}}{dx^{2k+1}} f(Z^{-1}(x)) \Big] \Bigg|_{x=Z(-b)}^{x=Z(b)} .\end{aligned}$$

Choosing the separator b such that $e^{iZ(\pm b)} = (-1)^L$ and summing over n:

$$NL = -\sum_{k=0}^{\infty} \frac{(2\pi)^{2k+1}}{(2k+2)!} B_{2k+2} \left(\frac{1}{2} \right) \left[\frac{\partial}{\partial x^{2k+1}} f(Z^{-1}(x)) \right] \Bigg|_{x=Z(-b)}^{x=Z(b)} . \qquad (2)$$

This procedure is effective if we have under control Z and f at the extremes $\pm b$. This is true in $sl(2)$ sector at high spin s.

Coming back to:

$$Z(u) = \Phi(u) - \sum_{k=1}^{s} \phi(u, u_k)\,,$$

When $s \to \infty$, $b = \frac{s}{2}\left(1 + O\left(\frac{1}{s}\right)\right)$,

$$\left.\frac{d^n}{dv^n}\phi(u,v)\right|_{\pm b} = O\left(\frac{1}{b^{n+1}}\right)\,, \quad \left.\frac{d^n}{dv^n}Z(v)\right|_{\pm b} = O\left(\frac{1}{b^{n}}\right)\,.$$

Therefore, $NL = O(1/s)$ and

$$Z(u) = \Phi(u) + \int_{-b}^{b} \frac{dv}{2\pi}\phi(u,v)Z'(v) + \sum_{h=1}^{L-2} \phi(u, u_h^{(i)}) + O\left((\ln s)^{-\infty}\right)\,.$$

Analogously for the anomalous dimension

$$\gamma(g,s,L) = \sum_{k=1}^{s} e(u_k) = -\int_{-b}^{b} \frac{dv}{2\pi} e(v)Z'(v) - \sum_{h=1}^{L-2} e(u_h^{(i)}) + O\left((\ln s)^{-\infty}\right)\,,$$

where

$$e(v) = \frac{ig^2}{x^+(v)} - \frac{ig^2}{x^-(v)}\,, \quad x^{\pm}(u) = \frac{u \pm i/2}{2}\left[1 + \sqrt{1 - \frac{2g^2}{(u \pm i/2)^2}}\right]\,.$$

Working out the equation for

$$\sigma(u) = Z'(u)\,,$$

it is convenient to:

- pass to Fourier transform
- separate one loop ($\sigma_0(u)$) from higher loops ($\sigma_H(u)$) contributions
- define the auxiliary density

$$S(k) = \frac{\sinh\frac{|k|}{2}}{\pi|k|}\Big\{\hat{\sigma}_H(k) - \pi\frac{e^{-\frac{|k|}{2}}}{\sinh\frac{|k|}{2}}\sum_{h=1}^{L-2}\Big[\cos k u_{h,H}^{(i)} - \cos k u_{h,0}^{(i)}\Big]\Big\}\,.$$

When the spin s is large, $S(k)$ satisfies the equation

$$\begin{aligned} S(k) &= 4g^2 \ln s \hat{K}(\sqrt{2}gk,0) + 4g^2\int_0^{+\infty}\frac{dt}{e^t-1}\hat{K}^*(\sqrt{2}gk,\sqrt{2}gt) + \\ &+ \frac{L}{k}[1-J_0(\sqrt{2}gk)] + 4g^2\gamma_E\hat{K}(\sqrt{2}gk,0) + \\ &+ g^2(L-2)\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{1-e^{\frac{t}{2}}}{\sinh\frac{t}{2}} - \\ &- g^2\int_0^{+\infty} dt \hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{\sum_{h=1}^{L-2}\Big[\cos t u_h^{(i)} - 1\Big]}{\sinh\frac{t}{2}} - \\ &- g^2\int_0^{+\infty} dt e^{-\frac{t}{2}}\hat{K}(\sqrt{2}gk,\sqrt{2}gt)\frac{t}{\sinh\frac{t}{2}}S(t)\,, \end{aligned}$$

where $\hat{K}^*(t,t') = \hat{K}(t,t') - \hat{K}(t,0)$,

$$\hat{K}(t,t') = \frac{2}{tt'}\left[\sum_{n=1}^{\infty} n J_n(t)J_n(t') + 2\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}c_{2k+1,2l+2}(g)J_{2k}(t)J_{2l+1}(t')\right]\,.$$

Anomalous dimension $\gamma(g,s,L)$ is extracted from $S(k)$ [Kotikov,Lipatov, NPB 769]:

$$\gamma(g,s,L) = 2\lim_{k\to 0} S(k)\,.$$

Example: Minimal anomalous dimension state ($M_h = h$) in the limit

$$s \to \infty\,, \quad L \to \infty\,, \quad j = \frac{L-2}{\ln s} \quad \text{fixed}\,. \tag{3}$$

In this limit

$$S(k) = \sum_{r=-1}^{\infty} (\ln s)^{-r} \sum_{n=0}^{\infty} S^{(r,n)}(k) j^n + O\left((\ln s)^{-\infty}\right)$$

and, correspondingly, the anomalous dimension:

$$\gamma(g,s,L) = \ln s \sum_{n=0}^{\infty} f_n(g) j^n + \sum_{r=0}^{\infty} (\ln s)^{-r} \sum_{n=0}^{\infty} f_n^{(r)}(g) j^n + O\left((\ln s)^{-\infty}\right)\,.$$

The equation for $S(k)$ splits into equations for $S^{(r,n)}(k)$: e.g. for $r+n \geq 1$:

$$S^{(r,n)}(k) = -g^2 \int_0^{+\infty} dt \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \left. \frac{\sum_{h=1}^{L-2} \left[\cos t u_h^{(i)} - 1\right]}{\sinh \frac{t}{2}} \right|_{\frac{j^n}{(\ln s)^r}} -$$

$$- \quad g^2 \int_0^{+\infty} dt e^{-\frac{t}{2}} \hat{K}(\sqrt{2}gk, \sqrt{2}gt) \frac{t}{\sinh \frac{t}{2}} S^{(r,n)}(t)\,.$$

Now we have an infinite number of 'internal holes' filling the interval $[-c, c]$:

$$Z(c) = \frac{1}{2} \int_{-c}^{c} dv \sigma(v) = -\pi j \, \ln s\,.$$

However, the sum over the 'internal holes' is evaluated at the desired order using (1), (2):

$$\sum_{h=1}^{L-2} \left[\cos t u_h^{(i)} - 1\right] = -2 \int_{-\infty}^{+\infty} \frac{dk}{4\pi^2} \hat{\sigma}(k) \left[\frac{\sin(t+k)c}{t+k} - \frac{\sin kc}{k}\right] - \frac{\pi}{6} \frac{t \sin tc}{\sigma(c)} -$$

$$- \quad \frac{7\pi^3}{360} \frac{t^3 \sigma(c) \, \sin tc + 3t^2 \sigma_1(c) \, \cos tc + t \sigma_2(c) \, \sin tc}{(\sigma(c))^4} + O\left(\frac{j^3}{(\ln s)^3}\right)\,.$$

And, since in the limit (3) the separator c expands as

$$c = \sum_{r=0}^{\infty} (\ln s)^{-r} \sum_{n=1}^{\infty} c^{(r,n)} j^n \,,$$

one finally gets expressions for the various $f_n^{(r)}(g)$. For instance

$$\begin{aligned}
\frac{f_3(g)}{\sqrt{2}g} &= \frac{1}{3}\pi^3 \frac{\tilde{S}_1^{(1)}(g)}{[\sigma^{(-1,0)}]^2} \,, \quad \frac{f_4(g)}{\sqrt{2}g} = -\frac{2}{3}\pi^3 \frac{\tilde{S}_1^{(1)}(g)\sigma^{(-1,1)}}{[\sigma^{(-1,0)}]^3} ; \\
\frac{f_3^{(0)}(g)}{\sqrt{2}g} &= -\frac{2}{3}\pi^3 \frac{\sigma^{(0,0)}\tilde{S}_1^{(1)}(g)}{[\sigma^{(-1,0)}]^3} \,, \quad \frac{f_4^{(0)}(g)}{\sqrt{2}g} = 2\pi^3 \frac{\sigma^{(0,0)}\sigma^{(-1,1)}\tilde{S}_1^{(1)}(g)}{[\sigma^{(-1,0)}]^4} \,, \\
f_1^{(0)}(g) &= f_2^{(0)}(g) = 0; \quad \frac{f_1^{(1)}(g)}{\sqrt{2}g} = -\frac{\pi^2}{3} \frac{\tilde{S}_1^{(1)}(g)}{(\sigma^{(-1,0)})^2} \,, \quad \frac{f_2^{(1)}(g)}{\sqrt{2}g} = \frac{2\pi^3}{3} \frac{\sigma^{(-1,1)}\tilde{S}_1^{(1)}(g)}{(\sigma^{(-1,0)})^3} \,,
\end{aligned}$$

where $\sigma^{(r,n)}$ appear in the expansion of the density $\sigma(u)$ in $u = 0$:

$$\sigma(0) = \sum_{r=-1}^{\infty} \sum_{n=0}^{\infty} \sigma^{(r,n)} (\ln s)^{-r} j^n \,,$$

and $\tilde{S}_p^{(k)}$ are solutions of the systems

$$\begin{aligned}
\tilde{S}_{2p}^{(k)}(g) &= 2p \int_0^{+\infty} \frac{dh}{2\pi} h^{2k-1} \frac{J_{2p}(\sqrt{2}gh)}{\sinh \frac{h}{2}} - 4p \sum_{m=1}^{\infty} Z_{2p,m}(g)(-1)^m \tilde{S}_m^{(k)}(g) \,, \\
\tilde{S}_{2p-1}^{(k)}(g) &= (2p-1) \int_0^{+\infty} \frac{dh}{2\pi} h^{2k-1} \frac{J_{2p-1}(\sqrt{2}gh)}{\sinh \frac{h}{2}} - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,m}(g) \tilde{S}_m^{(k)}(g) \,,
\end{aligned}$$

where

$$Z_{n,m}(g) = \int_0^{+\infty} \frac{dt}{t} \frac{J_n(\sqrt{2}gt) J_m(\sqrt{2}gt)}{e^t - 1} \,.$$

- $S^{(r,n)}$ depend on $S^{(r',n')}$, with $r' \leq r, n' \leq n-1$: this allows iterative solution.

- Weak and strong coupling expansions of $f_n^{(r)}(g)$ are explicitly computed.