Weak asymmetry limit of the Bethe Ansatz Equations for the asymmetric exclusion process (ASEP)

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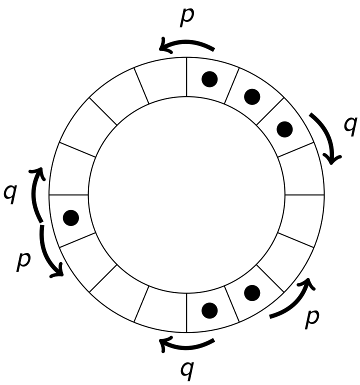
The model : the asymmetric exclusion process



- ring geometry of L sites,
- fixed number N of particles,
- jump rates p to the right, q to the left.

$$W = \sum_{i=1}^{L} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

Quantity of interest : the current Q_t

$$Q_t = \#\text{jumps of particle to the right} - \#\text{jumps of particle to the left}$$

Generating function:

$$\langle e^{sQ_t} \rangle \underset{t\to\infty}{\propto} e^{\mu_1(s)t}$$

where $\mu_1(s)$ is the eigenvalue with largest real part of the modfied transition matrix:

$$\widehat{W}_s = \sum_{i=1}^{L} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & pe^{s} & 0 \\ 0 & qe^{-s} & -p & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

Similar to the Hamiltonian of the asymmetric *XXZ* spin chain

but. . .

Bethe equations & hydrodynamical scalings

$$\psi(\mathbf{x}_<) = \sum_{\sigma \in \mathcal{S}_N} A_\sigma(\{z_i\}) \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

$$\boxed{z_k^L = (-1)^{N-1} \prod_{j \neq k} \frac{pe^s + qe^{-s} z_j z_k - (p+q) z_k}{pe^s + qe^{-s} z_j z_k - (p+q) z_j}}$$

Hydrodynamical scalings:

$$\begin{aligned} L &\to \infty \\ N/L &\to \rho \end{aligned}$$

$$\begin{aligned} \Delta &= \frac{p+q}{2\sqrt{pq}} \\ e^{2H} &= \sqrt{\frac{p}{q}} e^s \end{aligned}$$

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Hydrodynamical scalings:

$$\begin{aligned} L &\to \infty \\ N/L &\to \rho \\ p - q &\sim \frac{\nu}{L} \\ s &\sim \frac{\gamma}{L} \end{aligned}$$

$$\begin{aligned} \Delta &= \frac{p+q}{2\sqrt{pq}} \simeq 1 + \frac{\nu^2}{2L^2} \\ e^{2H} &= \sqrt{\frac{p}{q}} e^s \simeq 1 + \frac{\gamma + \nu}{L} \end{aligned}$$

Couplings vanishing as $L \to \infty$

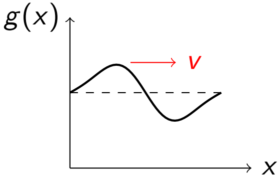
Why these scalings ?

From probability theory, existence of a nice hydrodynamical limit as $p - q = \nu/L$:

$$L\mu_1(\gamma/L) \to \sup_{g,J,v} \left(\gamma J - \int_0^1 \left[\frac{(J+v(g(x)-\rho)-\nu\sigma(g(x)))^2}{2\sigma(g(x))} + \frac{g'(x)^2}{8\sigma(g(x))} \right] dx \right)$$

Research of an optimal profile $g(x)$, with an optimal velocity v

1. $g(x)$: real density profile
2. v : macroscopic velocity
3. J : current
4. $\sigma(g) = g(1-g)$: conductivity



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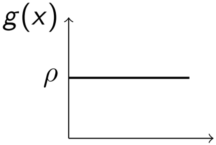
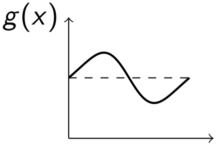


$\gamma > \gamma_c$, flat

$\widetilde{\mu}_1(\gamma) = \rho(1-\rho)(\gamma^2 + \gamma\nu)$



$\gamma < \gamma_c$, traveling wave

$\widetilde{\mu}_1(\gamma) =$ elliptic int.

Phase transition and **classical integrability for the macroscopic limit**

Bethe Ansatz approach (microscopic)

$T - Q$ approach (after change of variable $z_k = f(u_k)$):

$$T_{L,N}(Z)Q_N(Z) = e^{sL} p^N \left(Z\sqrt{\frac{p}{q}} + 1 \right)^L Q_N\left(\frac{q}{p}Z\right) + q^N \left(Z\sqrt{\frac{q}{p}} + 1 \right)^L Q_N\left(\frac{p}{q}Z\right) \tag{1}$$

As usual, resolvent:

$$W(Z) = \frac{1}{L}\sum_{i=1}^{N} \frac{1}{Z - u_k} \xrightarrow{\text{hydro. lim.}} \int_\Gamma \frac{\rho(u)du}{Z-u} = w(Z)$$

satisfying for the ground state:

$$t(z) = \cosh\left(\frac{\gamma}{2} + \rho\nu + \frac{\nu Z}{1+Z} - 2\nu Z w(Z) \right)$$

Not a quadratic equation in $w(Z)$
but use of the Riemann surface of $\cosh^{-1}$!

Properties of the cut of the resolvent

$$
\begin{aligned}
t(Z) &= \cosh \Phi(Z) \\
\Phi(Z) &= \frac{\gamma}{2} + \rho\nu + \frac{\nu Z}{1+Z} - 2\nu Z w(Z) \\
\rho &= \lim_{Z\to\infty} Z w(Z) \\
0 &= \gamma\rho + 2\rho\nu + 2\nu w(-1)
\end{aligned}
$$

+conditions :

1. $t(Z)$ holomorphic on $\mathbb{C} - \{-1\}$
2. $\Phi(Z)$ has the same cuts as $w(Z)$

Along a cut Γ:

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Along a cut Γ:

$$\Delta^{(\pm)}\Phi(Z) = \lim_{\epsilon\to 0, \epsilon\perp\Gamma} \Big(\Phi(Z+\epsilon) \pm \Phi(Z-\epsilon)\Big)$$

and necessarily :

$$\boxed{\Delta^{(+)}\Phi = 2i\pi m,\ m \in \mathbb{Z} \qquad \text{or} \qquad \Delta^{(-)}\Phi = 2i\pi n,\ n \in \mathbb{Z}^*}$$

Choice of $\Delta^{(\pm)}$, phase transition

1. Gaussian phase: one cut with $\Delta^{(+)} = 0$
   $\simeq$semi-circle/Marchenko-Pastor density on Γ and one recovers

$$\widetilde{\mu}_1^{\text{gauss}}(\gamma) = \rho(1-\rho)(\gamma^2 + \gamma\nu) \qquad (2)$$

2. transition : one cut with $\Delta^{(+)} = 0$ and one point with $\Delta^{(-)} = 2i\pi$

3. travelling wave phase: one cut $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$$\begin{cases} \Gamma_1 : & \Delta^{(+)}\Phi = 0 \\ \Gamma_2 : & \Delta^{(-)}\Phi = 2i\pi \\ \Gamma_3 : & \Delta^{(+)}\Phi = 0 \end{cases} \qquad \widetilde{\mu}_1^{\text{trav}}(\gamma) = \text{same elliptic integrals} \qquad (3)$$

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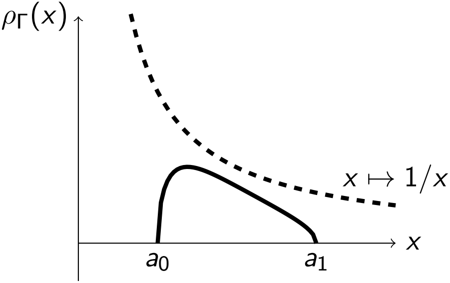
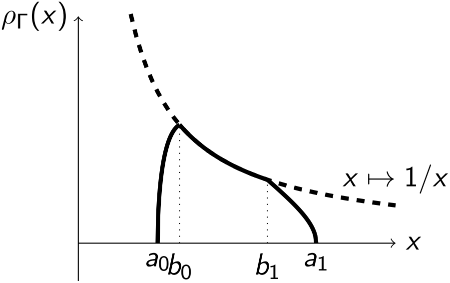
Bethe roots profile



Gaussian phase



Traveling wave phase

Conclusion

1. ASEP in hydro. regime (nice theory), XXZ with non-standard scaling couplings

2. non-quadratic equation for the resolvent: other Riemann surfaces and other types of cuts (piecewise structure)

3. relation with classical hydrodynamical integrable theory

Open questions :

1. relation between Bethe root density on the cut and the real traveling wave profile (spatial correlations?)

2. other models/generality ?

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