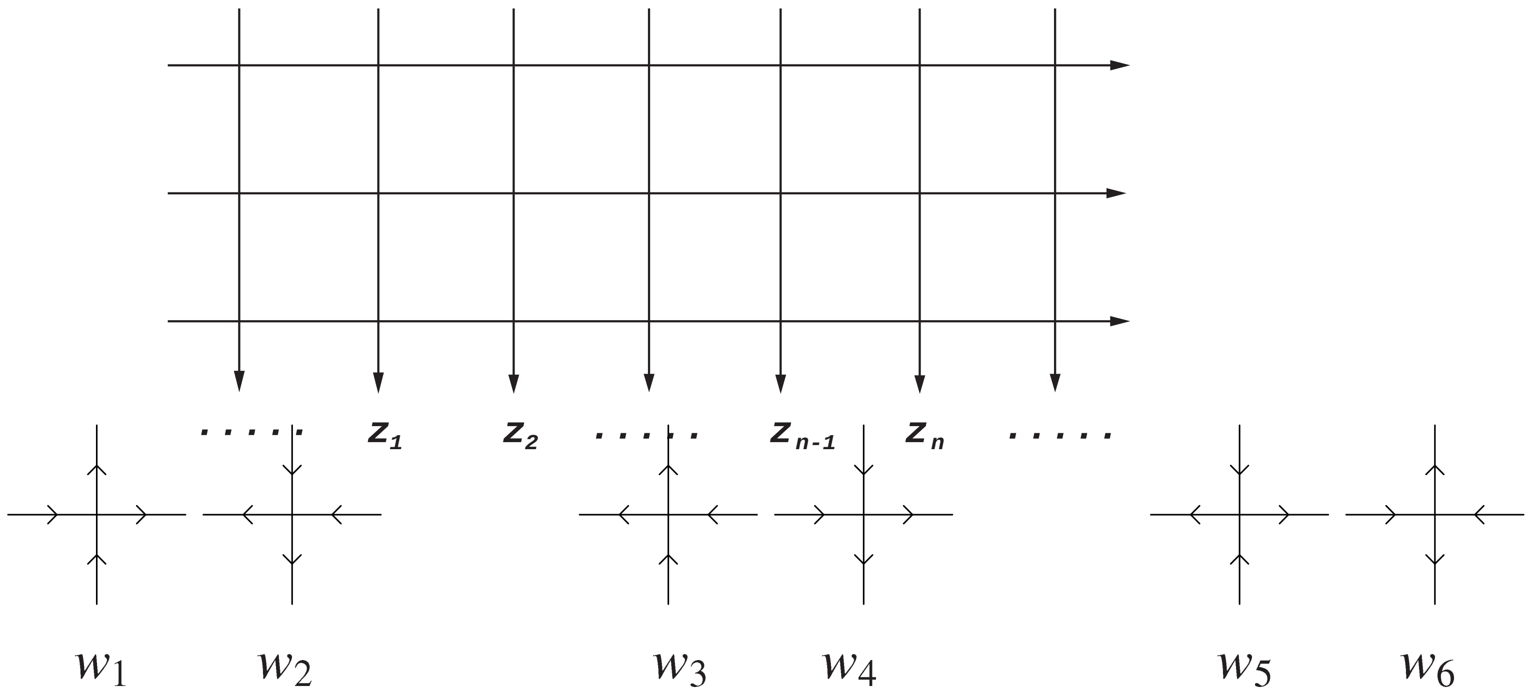
Correlation function and simplified TBA equations for XXZ chain

Minoru Takahashi

June/18/2010 Annecy

Algebraic Bethe ansatz for inhomogeneous six-vertex model



Vertices and their Boltzman weights of the six-vertex model. $N \times L$ lattice. N columns and L rows.

$$w_1 = w_2 = 1,\ w_3 = w_4 = b(\lambda - \xi_i),\ w_5 = w_6 = c(\lambda - \xi_i).$$

$$R(\lambda)=\begin{bmatrix}1, & 0, & 0, & 0\\ 0, & b(\lambda), & c(\lambda), & 0\\ 0, & c(\lambda), & b(\lambda), & 0\\ 0, & 0, & 0, & 1\end{bmatrix},\ b(\lambda)=\frac{\sinh\lambda}{\sinh(\lambda+\eta)},\ c(\lambda)=\frac{\sinh\eta}{\sinh(\lambda+\eta)}.$$

Monodoromy matrix

$$T(\lambda)=R_{0N}(\lambda-\xi_N)...R_{02}(\lambda-\xi_2)R_{01}(\lambda-\xi_1)=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}_{[0]}.$$

A, B, C, D are $2^N\times 2^N$ matrices. Transfer matrix is $\mathcal{T}(\lambda)=A(\lambda)+D(\lambda)$. At $L\to\infty$ the largest eigenvalue state is relevant.

$$Z=\mathrm{Tr}\mathcal{T}^L(\lambda)\simeq \text{Largest eigenvatue}^L.$$

Algebraic relations of operators A, B, C, D.

$$
\begin{aligned}
&[A(\mu), A(\lambda)] = [B(\mu), B(\lambda)] = [C(\mu), C(\lambda)] = [D(\mu), D(\lambda)] = 0,\\
&[C(\lambda), B(\mu)] = \frac{c(\lambda-\mu)}{b(\lambda-\mu)}(A(\lambda)D(\mu) - A(\mu)D(\lambda)),\\
&[D(\lambda), A(\mu)] = \frac{c(\lambda-\mu)}{b(\lambda-\mu)}(B(\lambda)C(\mu) - B(\mu)C(\lambda)),\\
&A(\mu)B(\lambda) = \frac{1}{b(\mu-\lambda)}B(\lambda)A(\mu) + \frac{c(\lambda-\mu)}{b(\lambda-\mu)}B(\mu)A(\lambda),\\
&D(\mu)B(\lambda) = \frac{1}{b(\lambda-\mu)}B(\lambda)D(\mu) + \frac{c(\mu-\lambda)}{b(\mu-\lambda)}B(\mu)D(\lambda),\\
&C(\lambda)A(\mu) = \frac{1}{b(\mu-\lambda)}A(\mu)C(\lambda) + \frac{c(\lambda-\mu)}{b(\lambda-\mu)}A(\lambda)C(\mu),\\
&C(\lambda)D(\mu) = \frac{1}{b(\lambda-\mu)}D(\mu)C(\lambda) + \frac{c(\lambda-\mu)}{b(\lambda-\mu)}D(\lambda)C(\mu).
\end{aligned}
$$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \ldots \begin{pmatrix}1\\0\end{pmatrix}$ is the vacuum state. This is an eigenstate of $\mathcal{T}$.

$$A(\mu)|0\rangle = |0\rangle, \quad D(\mu)|0\rangle = d(\mu)|0\rangle, \quad d(\mu) = \prod_{j=1}^{N} b(\mu - \xi_j).$$

$$|\psi\rangle = \prod_{j=1}^{M} B(\lambda_j)|0\rangle, \quad \langle\psi| = \langle 0| \prod_{j=1}^{M} C(\lambda_j)$$

are right and left eigenstates of $\mathcal{T}(\mu)$ with eigenvalue

$$\tau(\mu) = \prod b^{-1}(\lambda_j - \mu) + d(\mu) \prod b^{-1}(\mu - \lambda_j)$$

if Bethe equations are satisfied:

$$\frac{1}{d(\lambda_j)} \cdot \prod_{k \neq j} \frac{b(\lambda_j - \lambda_k)}{b(\lambda_k - \lambda_j)} = 1 \quad \text{for} \quad j = 1, 2, \ldots, M.$$

spin-$\frac{1}{2}$ anti-ferromagnetic Heisenberg chain

Homogeneous limit $\xi_j \to \eta/2$

$$\mathcal{H} = J \sum_{j=1}^{N} S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta(S_j^z S_{j+1}^z - \frac{1}{4}) = \frac{J}{2} \sinh \eta \frac{d}{d\lambda} \ln \mathcal{T}(\lambda)\Big|_{\lambda=0},$$

$$\Delta = \cosh \eta$$

XXZ chain hamiltonian. Case $\Delta = 1$ is called XXX chain. The eigenvectors are common with those of $\mathcal{T}(\lambda)$

$$E = \frac{J}{2} \sinh \eta \frac{d}{d\lambda} \ln \tau(\lambda)\Big|_{\lambda=0} = J \sum_{j=1}^{M} \frac{\sinh^2 \eta}{\cosh 2\lambda_j - \Delta}.$$

$$[S_k^\alpha, S_l^\beta] = \mathrm{i}\delta_{kl}\epsilon_{\alpha\beta\gamma} S_k^\gamma, \qquad \vec{S}_j = \frac{1}{2}\vec{\sigma}_j$$

$-1 < \Delta \leq 1$ massless region $\eta = i\zeta, 0 < \zeta < \pi,$

$1 < \Delta$ massive region

At ground state of $M = N/2$ and $N \to \infty$ we can calculate analytically distribution function of λ_j.

$$\rho(\lambda) = \frac{1}{2\zeta}\mathrm{sech}\frac{\pi\lambda}{\zeta}, \quad \text{for massless case}$$

In the XXX limit λ_j gather at origin. So we put $z_j = \lambda_j/(\zeta)$.

$$\rho(z) = \frac{1}{2}\mathrm{sech}\pi z, \quad E = -J/2\sum_j \frac{1}{z_j^2 + \frac{1}{4}}$$

In massive case λ_j distribute on imaginary axis. Putting $z_j = -i\lambda_j$ we have

$$\rho(z) = \sum_l \frac{1}{2\eta}\mathrm{sech}\frac{\pi(z-\pi l)}{\eta}, \quad E = -J\sum_j \frac{\sinh^2\eta}{\Delta - \cos 2z_j}$$

This function is doubly periodic and represented by elliptic function.

Known exact results for correlation functions

1. **Nearest-neighbor correlator**

$$\langle S_j^z S_{j+1}^z \rangle = \frac{1}{12} - \frac{1}{3}\ln 2 = -0.1477157268\cdots$$

from the ground state energy per site Hulthén (1938)

2. **Next nearest-neighbor correlator for XXX**

$$\langle S_j^z S_{j+2}^z \rangle = \frac{1}{12} - \frac{4}{3}\ln 2 + \frac{3}{4}\zeta(3) = 0.06067976995\cdots ,$$

from the ground state energy of the half-filled Hubbard model Takahashi (1977)

3. **The twisted four-body correlation function**

$$\langle (\boldsymbol{S}_j \times \boldsymbol{S}_{j+1}) \cdot (\boldsymbol{S}_{j+2} \times \boldsymbol{S}_{j+3}) \rangle_0 \quad = \quad \frac{1}{2}\ln 2 - \frac{3}{8}\zeta(3) = -0.104197748.$$

from third derivative of transfer matrix Muramoto and Takahashi(1999)

Strong coupling expansion of the Hubbard model

$$\mathcal{H}_0 = U \sum n_{i\uparrow} n_{i\downarrow}$$
$$\mathcal{H}_1 = -\sum_{\sigma} \sum_{i<j} t_{i,j} (c^{\dagger}_{i\sigma} c_{j\sigma} + c^{\dagger}_{j\sigma} c_{i\sigma}).$$

$$\mathcal{H}_{effective} = \sum_{i<j} \frac{t_{ij} t_{ji}}{U} (\sigma_i \cdot \sigma_j - 1)$$
$$+U^{-3} \Big[\sum_{i<j} t_{ij}^4 (1 - \sigma_i \cdot \sigma_j) + \sum_{i<k} t_{ij}^2 t_{jk}^2 (\sigma_i \cdot \sigma_k - 1) + \sum_{i<j<l, i<k, k \neq j,l}$$
$$t_{ij} t_{jk} t_{kl} t_{li} (5(\sigma_j \cdot \sigma_k)(\sigma_i \cdot \sigma_l) + 5(\sigma_i \cdot \sigma_j)(\sigma_k \cdot \sigma_l) - 5(\sigma_j \cdot \sigma_k)(\sigma_i \cdot \sigma_l)$$
$$-\sigma_i \cdot \sigma_j - \sigma_j \cdot \sigma_k - \sigma_k \cdot \sigma_l - \sigma_l \cdot \sigma_i - \sigma_i \cdot \sigma_k - \sigma_j \cdot \sigma_l + 1) \Big].$$

$$\mathcal{H} = -t \sum_{<ij>} \sum_{\sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + c^{\dagger}_{j\sigma} c_{i\sigma}) + U \sum_{i=1}^{N_a} c^{\dagger}_{i\uparrow} c_{i\uparrow} c^{\dagger}_{i\downarrow} c_{i\downarrow}$$

For one-dimensional half-filled case the effective Hamiltonian becomes

$$\frac{t^2}{U} \sum_i (4\mathbf{S}_i \cdot \mathbf{S}_{i+1} - 1) + \frac{t^4}{U^3} \sum_i \Big\{ 4(1 - 4\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + (4\mathbf{S}_i \cdot \mathbf{S}_{i+2} - 1) \Big\}$$
$$+ O(\frac{t^6}{U^5}).$$

On the other hand exact ground state energy per site is expanded as

$$e = -4|t| \int_0^\infty \frac{J_0(\omega) J_1(\omega) d\omega}{\omega[1 + \exp(2U'\omega)]} = -4|t|[(\frac{1}{2})^2 \ln 2 \; U^{'-1}$$
$$-(\frac{1 \cdot 3}{2 \cdot 4})^2 \frac{\zeta(3)}{3} (1 - \frac{1}{2^2}) U^{'-3} + ...], \quad U' \equiv U/(4|t|).$$

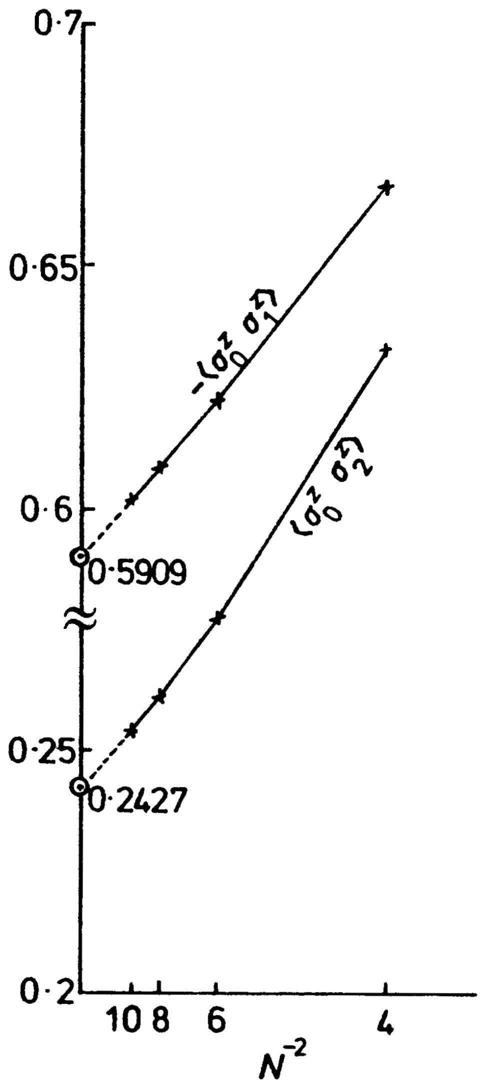


Figure 2. The first- and second-neighbour correlation functions of an antiferromagnetic ring with N atoms are plotted as functions of $1/N^2$. In the limit $N = \infty$, these values approach the theoretical values. Table 2 of Bonner and Fisher (1964) was used.

4. **Multiple-integral representations**

for more general XXZ model with an anisotropy parameter Δ

(a) $\Delta > 1, T = 0, H = 0$: Vertex operator approach $U_q(\hat{sl}(2))$
Jimbo, Miki, Miwa, Nakayashiki (1992)

(b) $-1 < \Delta < 1, T = 0, H = 0$: qKZ equation
Jimbo and Miwa (1996)

(c) **Rederivation by the quantum inverse scattering method.**
Generalization to the XXZ **model with a** *magnetic field* $T = 0$
Kitanine, Maillet, Terras (1998)

(d) **Integral formula for finite temperature and finite magnetic field**
Göhmann, Klümper, Seel (2004)

Correlation function for successive elementary block

$$\rho_n(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle\psi| \prod_{j=1}^{n} E_j^{\epsilon'_j,\epsilon_j} |\psi\rangle}{\langle\psi|\psi\rangle},$$

$$E_j^{+,+} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{[j]}, \quad E_j^{+,-} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_{[j]}, \quad E_j^{-,+} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_{[j]}, \quad E_j^{-,-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{[j]}.$$

Example. Emptiness Formation Probability (EFP) for XXX chain

$$P(n) \equiv \left\langle \left(S_1^z + \frac{1}{2}\right)\left(S_2^z + \frac{1}{2}\right)\cdots\left(S_n^z + \frac{1}{2}\right)\right\rangle = \rho_{n++\ldots+}^{++\ldots+} =$$

$$= (-\pi)^{\frac{n(n-1)}{2}} \, 2^{-n} \int_{-\infty}^{\infty} \mathrm{d}^n\lambda \prod_{a>b}^{n} \frac{\sinh \pi(\lambda_a - \lambda_b)}{\lambda_a - \lambda_b - \mathrm{i}} \prod_{j=1}^{n} \frac{\left(\lambda_j - \frac{\mathrm{i}}{2}\right)^{j-1} \left(\lambda_j + \frac{\mathrm{i}}{2}\right)^{n-j}}{\cosh^n \pi\lambda_j}$$

Other arbitrary correlation functions have similar integral representation.

"n-fold integral for the correlation over n-sites."

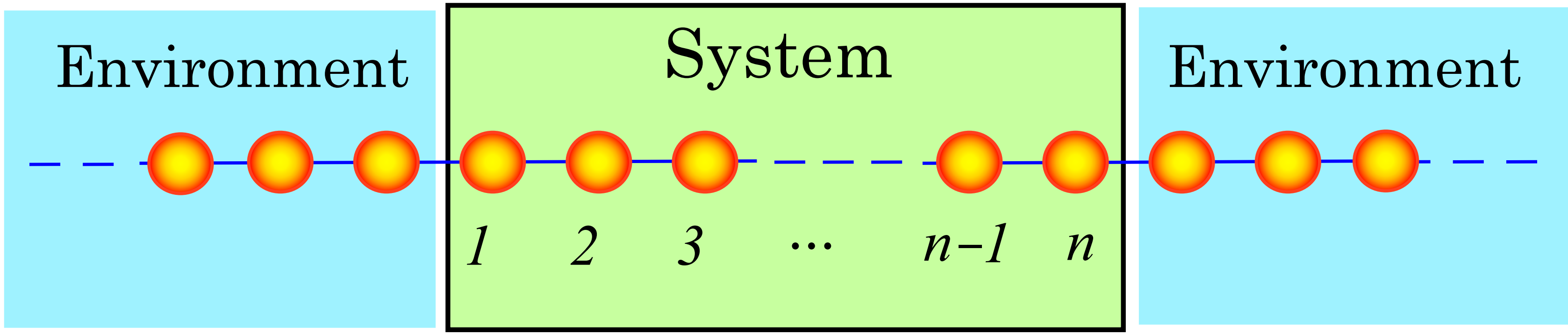


Figure 1: Finite sub-chain of length n in the infinite chain

Boos-Korepin method to evaluate integrals

1. Transform the integrand to a certain "*canonical form*" without changing the integral value.
2. Perform the integration using the residue theorem.

Example

$$P(3) \equiv \prod_{j=1}^{3} \int_{-\infty-\frac{i}{2}}^{\infty-\frac{i}{2}} \frac{\mathrm{d}\lambda_j}{2\pi i} U_3(\lambda_1, \lambda_2, \lambda_3) T_3(\lambda_1, \lambda_2, \lambda_3),$$

$$U_3(\lambda_1, \lambda_2, \lambda_3) \equiv \pi^6 \frac{\prod_{1\le k<j\le 3} \sinh\pi(\lambda_j - \lambda_k)}{\prod_{j=1}^{3} \sinh^3 \pi\lambda_j}$$

$$T_3(\lambda_1, \lambda_2, \lambda_3) \equiv \frac{(\lambda_1 + i)^2 \lambda_2 (\lambda_2 + i) \lambda_3^2}{(\lambda_2 - \lambda_1 - i)(\lambda_3 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)}$$

$$\frac{(\lambda_1 + i)^2(\lambda_2 + i)\lambda_2\lambda_3^2}{(\lambda_3 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)(\lambda_2 - \lambda_1 - i)}.$$

$$= \frac{i(\lambda_1 + i)^2(\lambda_2 + i)\lambda_2\lambda_3^2}{(\lambda_3 - \lambda_1 - i)(\lambda_2 - \lambda_1 - i)} + \frac{i(\lambda_1 + i)^2(\lambda_2 + i)\lambda_2\lambda_3^2}{(\lambda_3 - \lambda_1 - i)(\lambda_3 - \lambda_2 - i)}$$
$$- \frac{i(\lambda_1 + i)^2(\lambda_2 + i)\lambda_2\lambda_3^2}{(\lambda_3 - \lambda_2 - i)(\lambda_2 - \lambda_1 - i)}.$$

$$\text{first term} =\sim \lambda_1^2\lambda_2 - \frac{(\lambda_1 + i)^3\lambda_3}{\lambda_2 - \lambda_1 - i},$$

$$\text{second term} \sim \lambda_1^2\lambda_2 - \frac{(\lambda_1 + i)^3(\lambda_3 + i)}{\lambda_2 - \lambda_1 - i}.$$

$$\text{third term} \sim -\lambda_1^2\lambda_2 - i\frac{(\lambda_1+i)^3(\lambda_3+i)^2}{\lambda_2-\lambda_1-i} + i\frac{(\lambda_1+i)^3\lambda_3^2}{\lambda_2-\lambda_1-i}$$
$$-\frac{i(\lambda_1+i)^3\lambda_3^3}{(\lambda_3-\lambda_2-i)(\lambda_2-\lambda_1-i)}.$$

$$T_3 \sim -\lambda_2\lambda_3^2 - \frac{i(\lambda_1+i)^3\lambda_3^3}{(\lambda_3-\lambda_2-i)(\lambda_2-\lambda_1-i)}.$$
$$\sim -\lambda_2\lambda_3^2 - \frac{i\lambda_1^3(\lambda_3+i)^3}{(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)}.$$
$$\sim -\lambda_2\lambda_3^2 - \frac{3\lambda_1^2\lambda_3^2+3i\lambda_1\lambda_3^2+3i\lambda_1^2\lambda_3-\lambda_3^2-\lambda_3\lambda_1-\lambda_1^2}{\lambda_2-\lambda_1}.$$
$$\frac{\lambda_1^2}{\lambda_2-\lambda_1}f(\lambda_3) \sim \Big(\frac{-i\lambda_1+\frac{1}{3}}{\lambda_2-\lambda_1} - \frac{1}{3}(\lambda_1+i)\Big)f(\lambda_3)$$
$$T_3 \sim -2\lambda_2\lambda_3^2 + \frac{\frac{1}{3}-i\lambda_1-i\lambda_3-2\lambda_1\lambda_3}{\lambda_2-\lambda_1}.$$

Canonical form...Denominator is $\prod_{k=1}^{l}(\lambda_{2k-1}-\lambda_{2k})$, $l \le [n/2]$.
n dimensional integral is decomposed to one and two dimensional integrals.

Transform $T_3(\lambda_1,\lambda_2,\lambda_3)$ into a canonical form $T_3^{\mathrm{c}}(\lambda_1,\lambda_2,\lambda_3)$

$$T_3(\lambda_1,\lambda_2,\lambda_3) \sim T_3^{\mathrm{c}}(\lambda_1,\lambda_2,\lambda_3) = P_0^{(3)} + \frac{P_1^{(3)}}{\lambda_2-\lambda_1}$$

$$P_0^{(3)} = -2\lambda_2\lambda_3^2, \quad P_1^{(3)} = \frac{1}{3} - i\lambda_1 - i\lambda_3 - 2\lambda_1\lambda_3$$

Perform the integration

$$\begin{aligned}
J_0^{(3)} &= \prod_{j=1}^{3}\int_{-\infty-\frac{i}{2}}^{\infty-\frac{i}{2}}\frac{\mathrm{d}\lambda_j}{2\pi i}U_3(\lambda_1,\lambda_2,\lambda_3)P_0^{(3)} = \frac{1}{4}, \\
J_1^{(3)} &= \prod_{j=1}^{3}\int_{-\infty-\frac{i}{2}}^{\infty-\frac{i}{2}}\frac{\mathrm{d}\lambda_j}{2\pi i}U_3(\lambda_1,\lambda_2,\lambda_3)\frac{P_1^{(3)}}{\lambda_2-\lambda_1} = -\ln 2 + \frac{3}{8}\zeta(3), \\
P(3) &= J_0^{(3)} + J_1^{(3)} = \frac{1}{4} - \ln 2 + \frac{3}{8}\zeta(3)
\end{aligned}$$

Thus second neighbor correlator was rederived from the integral formula.

For $P(4) = \rho_{4+++++}^{++++}$ the integrand is

$$\frac{(\lambda_1 + i)^3(\lambda_2 + i)^2\lambda_2(\lambda_3 + i)\lambda_3^2\lambda_4^3}{(\lambda_{43} - i)(\lambda_{42} - i)(\lambda_{41} - i)(\lambda_{32} - i)(\lambda_{31} - i)(\lambda_{21} - i)},$$

Here we put $\lambda_{ab} \equiv \lambda_a - \lambda_b$. Canonical form is $P_0^{(4)} + P_1^{(4)}/\lambda_{21} + P_2^{(4)}/(\lambda_{21}\lambda_{43})$.

$$P_0^{(4)} = -\frac{34}{5}\lambda_2\lambda_3^2\lambda_4^3,$$

$$P_1^{(4)} = \lambda_1^2\,(30\,\lambda_3^2\lambda_4^3 + 30\,i\lambda_3\lambda_4^3 - 16\,\lambda_4^3 + 18\,\lambda_3\lambda_4^2 + 8\,\lambda_4) +$$
$$\lambda_1\,(30\,i\lambda_3^2\lambda_4^3 + 30\,\lambda_3\lambda_4^3 - 16\,i\lambda_4^3 + 18\,i\lambda_3\lambda_4^2 - 4\,\lambda_4^2 + 4\,i\lambda_4) -$$
$$20\,\lambda_3^2\lambda_4^3 - 20\,i\lambda_3\lambda_4^3 + \frac{54}{5}\lambda_4^3 - \frac{42}{5}\lambda_3\lambda_4^2 - \frac{43}{10}i\lambda_4,$$

$$P_2^{(4)} = 2\lambda_1^2\lambda_3^2 + 4i\lambda_1\lambda_3^2 - \frac{3}{2}\lambda_3^2 - \frac{3}{2}\lambda_1\lambda_3 - i\lambda_3 + \frac{1}{5}.$$

$$\begin{aligned} P(4) \quad &= \tfrac{1}{5} - 2\ln 2 + \tfrac{173}{60}\zeta(3) - \tfrac{11}{6}\ln 2 \cdot \zeta(3) - \tfrac{51}{80}\zeta^2(3) \\ &\qquad -\tfrac{55}{24}\zeta(5) + \tfrac{85}{24}\ln 2 \cdot \zeta(5) \end{aligned}$$

In 2003, Sakai, Shiroishi, Nishiyama, Takahashi calculated ρ^{+-+-}_{+-+-} by Boos–Korepin method and obtained *all* the correlation functions on 4 lattice sites. Especially, Third-neighbor correlator is

$$\left\langle S^z_j S^z_{j+3} \right\rangle = \frac{1}{12} - 3\ln 2 + \frac{37}{6}\zeta(3) - \frac{14}{3}\ln 2\cdot\zeta(3) - \frac{3}{2}\zeta(3)^2 - \frac{125}{24}\zeta(5) + \frac{25}{3}\ln 2\cdot\zeta(5) = -0.05024862725\cdots$$

Similarly the other correlation functions for $n = 4$ are

$$\left\langle S^x_j S^x_{j+1} S^z_{j+2} S^z_{j+3} \right\rangle = \frac{1}{240} + \frac{1}{12}\ln 2 - \frac{91}{240}\zeta(3) + \frac{1}{6}\ln 2\cdot\zeta(3) + \frac{3}{80}\zeta(3)^2 + \frac{35}{96}\zeta(5) - \frac{5}{24}\ln 2\cdot\zeta(5)$$

$$\left\langle S^x_j S^z_{j+1} S^x_{j+2} S^z_{j+3} \right\rangle = \frac{1}{240} - \frac{1}{6}\ln 2 + \frac{77}{120}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5)$$

$$\left\langle S^x_j S^z_{j+1} S^z_{j+2} S^x_{j+3} \right\rangle = \frac{1}{240} - \frac{1}{4}\ln 2 + \frac{169}{240}\zeta(3) - \frac{5}{12}\ln 2\cdot\zeta(3) - \frac{3}{20}\zeta(3)^2 - \frac{65}{96}\zeta(5) + \frac{5}{6}\ln 2\cdot\zeta(5)$$

$$\left\langle S^z_j S^z_{j+1} S^z_{j+2} S^z_{j+3} \right\rangle = \left\langle S^x_j S^x_{j+1} S^z_{j+2} S^z_{j+3} \right\rangle + \left\langle S^x_j S^z_{j+1} S^x_{j+2} S^z_{j+3} \right\rangle + \left\langle S^x_j S^z_{j+1} S^z_{j+2} S^x_{j+3} \right\rangle$$

From these we reproduce the twisted four-body correlations.

In a similar way, $P(5)$ was calculated after very tedious calculations.

$$
\begin{aligned}
P(5) = {} & \frac{1}{6} - \frac{10}{3}\ln 2 + \frac{281}{24}\zeta(3) - \frac{45}{2}\ln 2 \cdot \zeta(3) - \frac{489}{16}\zeta(3)^2 \\
& - \frac{6775}{192}\zeta(5) + \frac{1225}{6}\ln 2 \cdot \zeta(5) - \frac{425}{64}\zeta(3) \cdot \zeta(5) - \frac{12125}{256}\zeta(5)^2 \\
& + \frac{6223}{256}\zeta(7) - \frac{11515}{64}\ln 2 \cdot \zeta(7) + \frac{42777}{512}\zeta(3) \cdot \zeta(7)
\end{aligned}
$$

But the direct integral of other correlations for five sites is almost impossible.

Generalization to XXZ-model

In 2003, Kato, Shiroishi, Takahashi, Sakai have generalized the Boos-Korepin method to the XXZ models with an anisotropy parameter

Example : Multiple integral representation for $P(n)$ in the case of $|\Delta| \leq 1$

$$P(n) = (-\nu)^{-\frac{n(n-1)}{2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}x_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}x_n}{2\pi} \prod_{a>b} \frac{\sinh(x_a - x_b)}{\sinh\left((x_a - x_b - \mathrm{i}\pi)\,\nu\right)}$$

$$\times \prod_{j=1}^{n} \frac{\sinh^{n-j}\left(\left(x_j + \frac{\mathrm{i}\pi}{2}\right)\nu\right) \sinh^{j-1}\left(\left(x_j - \frac{\mathrm{i}\pi}{2}\right)\nu\right)}{\cosh^n x}$$

here $\Delta = \cos(\pi\nu)$ Jimbo and Miwa (1996)

“Similar integral representations for any arbitrary correlation functions”

Nearest-neighbor correlation functions

$$\langle S_j^x S_{j+1}^x \rangle = \frac{1}{4\pi s_1}\zeta_\nu(1) + \frac{c_1}{4\pi^2}\zeta'_\nu(1), \quad \langle S_j^z S_{j+1}^z \rangle = \frac{1}{4} - \frac{c_1}{2\pi s_1}\zeta_\nu(1) - \frac{1}{2\pi^2}\zeta'_\nu(1)$$

Next nearest-neighbor correlation functions

$$\langle S_j^x S_{j+2}^x \rangle = \frac{1}{2\pi s_2}\zeta_\nu(1) + \frac{c_2}{4\pi^2}\zeta'_\nu(1) - \frac{3(1-c_2)c_2}{8\pi s_2}\zeta_\nu(3) - \frac{{s_1}^2}{8\pi^2}\zeta'_\nu(3),$$

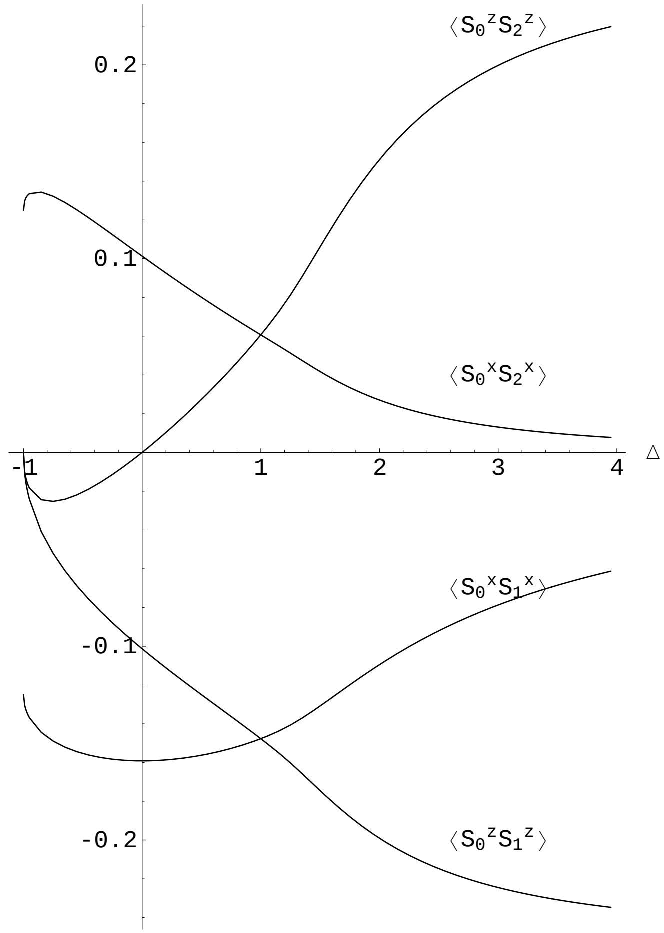
$$\langle S_j^z S_{j+2}^z \rangle = \frac{1}{4} - \frac{1+2c_2}{\pi s_2}\zeta_\nu(1) - \frac{1}{2\pi^2}\zeta'_\nu(1) + \frac{3s_1}{4\pi c_1}\zeta_\nu(3) + \frac{1-c_2}{8\pi^2}\zeta'_\nu(3)$$

$$c_j := \cos \pi j \nu, \quad s_j := \sin \pi j \nu$$

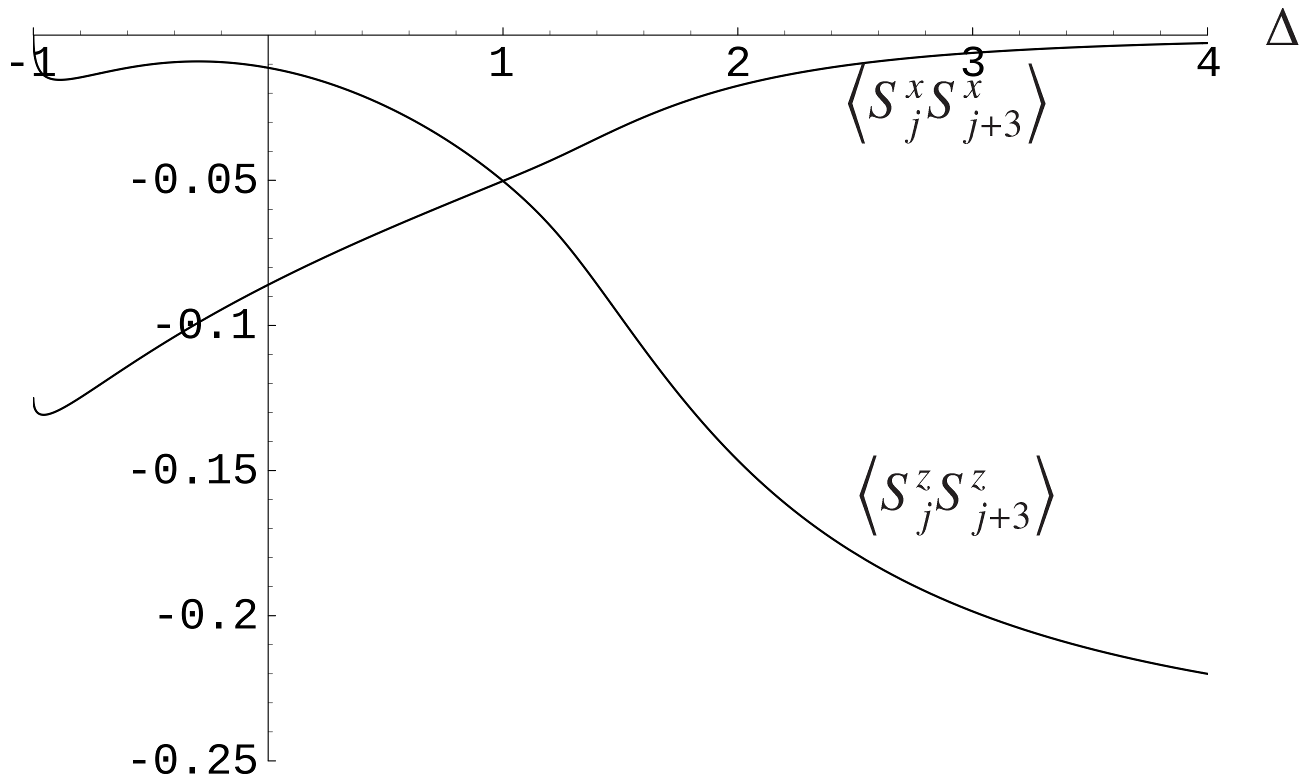
$$\zeta_\nu(j) := \int_{-\infty-\frac{\pi i}{2}}^{\infty-\frac{\pi i}{2}} \mathrm{d}x \frac{1}{\sinh x}\frac{\cosh \nu x}{\sinh^j \nu x} \quad \zeta'_\nu(j) := \int_{-\infty-\frac{\pi i}{2}}^{\infty-\frac{\pi i}{2}} \mathrm{d}x \frac{1}{\sinh x}\frac{\partial}{\partial \nu}\frac{\cosh \nu x}{\sinh^j \nu x}$$

Replacing $\nu \to i\eta/\pi$, we can also get the correlation functions
in the massive region ($\Delta = \cosh \eta > 1$) Takahashi, Kato, Shiroishi (2003)

Nearest-neighbor and next nearest neighbor correlation functions for the XXZ chain. We calculated $\langle S^z_j S^z_{j+1}\rangle$, $\langle S^x_j S^x_{j+1}\rangle$, $\langle S^z_j S^z_{j+2}\rangle$ and $\langle S^x_j S^x_{j+2}\rangle$



The third-neighbor correlation functions for the XXZ chain.



$\boxed{\text{Algebraic Approach, qKZ Relation}}$

“ Next problem is to calculate $\langle S_j^z S_{j+4}^z \rangle$ for XXX model”

In principle, it’s possible to calculate other five-dimensional integrals by use of Boos-Korepin method.

It, however, will take tremendous amount of time....

We propose a different method (“First Principle Approach”) and obtain analytical form of $\langle S_j^z S_{j+4}^z \rangle$

Generalization of the method by Boos, Korepin, Smirnov (2003) for $P(n)$

$$\lim_{z_i\to 0}\rho_{n,\epsilon_1,\ldots,\epsilon_n}^{\epsilon_1',\ldots,\epsilon_n'}(z_1,z_2,\cdots,z_n)=\langle E_{\epsilon_1}^{\epsilon_1'}\cdots E_{\epsilon_n}^{\epsilon_n'}\rangle$$

$$(E_\epsilon^{\epsilon'})_{s,s'}=\delta_{\epsilon,s}\delta_{\epsilon',s'},\qquad \epsilon,\epsilon'=\pm 1$$

"Algebraic Relations"

- **Translational Invariance**

$$\rho_{n,\epsilon_1,\ldots,\epsilon_n}^{\epsilon_1',\ldots,\epsilon_n'}(z_1+x,\ldots,z_n+x)=\rho_{n,\epsilon_1,\ldots,\epsilon_n}^{\epsilon_1',\ldots,\epsilon_n'}(z_1,\ldots,z_n)$$

- **Transposition, Negating and Reverse-Order Relations**

$$\rho_{n,\epsilon_1,\ldots,\epsilon_n}^{\epsilon_1',\ldots,\epsilon_n'}(z_1,\ldots,z_n)=\rho_{n,\epsilon_1',\ldots,\epsilon_n'}^{\epsilon_1,\ldots,\epsilon_n}(-z_1,\ldots,-z_n)$$

$$=\rho_{n,-\epsilon_1,\ldots,-\epsilon_n}^{-\epsilon_1',\ldots,-\epsilon_n'}(z_1,\ldots,z_n)=\rho_{n,\epsilon_n,\ldots,\epsilon_1}^{\epsilon_n',\ldots,\epsilon_1'}(-z_n,\ldots,-z_1)$$

- **Intertwining Relation**

$$R_{\tilde\epsilon_j'\tilde\epsilon_{j+1}'}^{\epsilon_j'\epsilon_{j+1}'}(z_j-z_{j+1})\rho_{\ldots\epsilon_{j+1},\epsilon_j\ldots}^{\ldots\tilde\epsilon_{j+1}',\tilde\epsilon_j'\ldots}(\ldots z_{j+1},z_j\ldots)=\rho_{\ldots\tilde\epsilon_j,\tilde\epsilon_{j+1}\ldots}^{\ldots\epsilon_j',\epsilon_{j+1}'\ldots}(\ldots z_j,z_{j+1}\ldots)R_{\epsilon_j\epsilon_{j+1}}^{\tilde\epsilon_j\tilde\epsilon_{j+1}}(z_j-z_{j+1})$$

$$R_{++}^{++}(z)=R_{--}^{--}(z)=1,\ R_{+-}^{+-}(z)=R_{-+}^{-+}(z)=\tfrac{z}{z+1},\ R_{-+}^{+-}(z)=R_{+-}^{-+}(z)=\tfrac{1}{z+1}.$$

- **Reduction Relation**

$$\rho^{+,\epsilon_2',\ldots,\epsilon_n'}_{n,+,\epsilon_2,\ldots,\epsilon_n}(z_1,z_2,\ldots,z_n)+\rho^{-,\epsilon_2',\ldots,\epsilon_n'}_{n,-,\epsilon_2,\ldots,\epsilon_n}(z_1,z_2,\ldots,z_n)$$
$$=\rho^{\epsilon_2',\ldots,\epsilon_n'}_{n-1,\epsilon_2,\ldots,\epsilon_n}(z_2,\ldots,z_n)$$

- **First Recurrent Relation**

$$\rho^{\epsilon_1',\epsilon_2',\ldots,\epsilon_n'}_{n,\epsilon_1,\epsilon_2,\ldots,\epsilon_n}(z+1,z,z_3\ldots,z_n)=-\delta_{\epsilon_1,-\epsilon_2}\epsilon_1'\epsilon_2\rho^{\epsilon_2',\epsilon_3',\ldots,\epsilon_n'}_{n-1,-\epsilon_1',\epsilon_3,\ldots,\epsilon_n}(z,z_3\ldots,z_n)$$
$$\rho^{\epsilon_1',\epsilon_2',\ldots,\epsilon_n'}_{n,\epsilon_1,\epsilon_2,\ldots,\epsilon_n}(z-1,z,z_3\ldots,z_n)=-\delta_{\epsilon_1',-\epsilon_2'}\epsilon_1\epsilon_2'\rho^{-\epsilon_1,\epsilon_3',\ldots,\epsilon_n'}_{n-1,\epsilon_2,\epsilon_3,\ldots,\epsilon_n}(z,z_3\ldots,z_n)$$

- **Second Recurrent Relation**

$$\lim_{z_1\to i\infty}\rho^{\epsilon_1',\epsilon_2',\ldots,\epsilon_n'}_{n,\epsilon_1,\epsilon_2,\ldots,\epsilon_n}(z_1,z_2,\ldots,z_n)=\delta_{\epsilon_1,\epsilon_1'}\frac{1}{2}\rho^{\epsilon_2',\ldots,\epsilon_n'}_{n-1,\epsilon_2,\ldots,\epsilon_n}(z_2,\ldots,z_n)$$

- **Identity Relations**

$$\sum_{\substack{\epsilon_1,\ldots,\epsilon_n\\ \sum_i\epsilon_i'=\sum_i\epsilon_i}}\rho^{\epsilon_1',\ldots,\epsilon_n'}_{n,\epsilon_1,\ldots,\epsilon_n}(z_1,\ldots,z_n)=\sum_{\substack{\epsilon_1',\ldots,\epsilon_n'\\ \sum_i\epsilon_i'=\sum_i\epsilon_i}}\rho^{\epsilon_1',\ldots,\epsilon_n'}_{n,\epsilon_1,\ldots,\epsilon_n}(z_1,\ldots,z_n)$$
$$=\rho^{+,\ldots,+}_{n,+,\ldots,+}(z_1,\ldots,z_n)=\rho^{-,\ldots,-}_{n,-,\ldots,-}(z_1,\ldots,z_n)$$

○ **We have calculated all the inhomogeneous correlation functions up to** $n \le 4$ **from the multiple integrals and confirmed these relations are fulfilled.**

Further we have found the inhomogeneous correlation functions can be represented in terms of ω**-function**

$$\omega(x) \equiv \frac{1}{2} + 2\sum_{k=1}^{\infty}(-1)^k k \frac{1-x^2}{k^2-x^2}$$
$$= \frac{1}{2} - 2(1-x^2)\sum_{k=0}^{\infty} x^{2k}\zeta_a(2k+1),$$

$$\zeta_a(x) \equiv \sum_{n=1}^{\infty}(-1)^{n-1}n^{-x} = (1-2^{1-x})\zeta(x), \quad \zeta_a(1) = \ln 2.$$

$$\omega(x+1) = -\frac{x(x+2)}{x^2-1}\omega(x) - \frac{3}{2}\frac{1}{1-x^2}, \quad \omega(-x) = \omega(x), \quad \omega(\pm i\infty) = 0.$$

Examples :

- $n = 1$

$$\rho^{+}_{1,+}(z_1) = \rho^{-}_{1,-}(z_1) = \frac{1}{2}, \;\; \rho^{-}_{1,+}(z_1) = \rho^{+}_{1,-}(z_1) = 0.$$

- $n = 2$

$$\rho^{++}_{2,++}(z_1, z_2) = \frac{1}{4} + \frac{1}{6}\omega(z_1 - z_2),$$

$$\rho^{+-}_{2,+-}(z_1, z_2) = -\frac{1}{6}\omega(z_1 - z_2), \;\; \rho^{-+}_{2,+-}(z_1, z_2) = \frac{1}{3}\omega(z_1 - z_2)$$

- $n = 3$

$$\begin{aligned}\rho^{+++}_{3,+++}(z_1, z_2, z_3) = \frac{1}{8} &+ A(z_1, z_2|z_3)\omega(z_1 - z_2) + A(z_1, z_3|z_2)\omega(z_1 - z_3) \\ &+ A(z_2, z_3|z_2)\omega(z_2 - z_3),\end{aligned}$$

$$\rho^{-++}_{3,-++}(z_1, z_2, z_3) = \rho^{++}_{2,++}(z_2, z_3) - \rho^{+++}_{3,+++}(z_1, z_2, z_3),$$

$$\rho^{-+-}_{3,-+-}(z_1, z_2, z_3) = \rho^{-+}_{2,-+}(z_1, z_2) - \rho^{-++}_{3,-++}(z_1, z_2, z_3), \ldots$$

$$A(z_1, z_2|z_3) = \frac{(z_1 - z_3)(z_2 - z_3) - 1}{12(z_1 - z_3)(z_2 - z_3)}$$

$$\rho_{n,\epsilon_1,\dots,\epsilon_n}^{\epsilon'_1,\dots,\epsilon'_n}(z_1,\dots,z_n) = \left(\prod_{j=1}^{n}\frac{\delta_{\epsilon_j,\epsilon'_j}}{2}\right) + \sum_{m=1}^{\left[\frac{n}{2}\right]}\sum_{1\le k_1<k_3<k_5\dots<k_{2m-1}<n,\ k_{2m}>k_{2m-1}} A_{\epsilon_1,\dots,\epsilon_n}^{\epsilon'_1,\dots,\epsilon'_n}(k_1,\cdots,k_{2m}|z_1,\dots,z_n)$$
$$\times\,\omega(z_{k_1}-z_{k_2})\cdots\omega(z_{k_{2m-1}}-z_{k_{2m}}),$$

$$A_{\epsilon_1,\dots,\epsilon_n}^{\epsilon'_1,\dots,\epsilon'_n}(k_1,\cdots,k_{2m}|z_1,\dots,z_n) = \frac{Q_{\epsilon_1,\dots,\epsilon_n}^{\epsilon'_1,\dots,\epsilon'_n}(k_1,\cdots,k_{2m}|z_1,\dots,z_n)}{\prod'_{i<j}(z_i-z_j)}$$ **: rational function of** $z_1,\dots,z_n$

Denominator is

$$\prod_{1\le j<k\le m}(z_{k_{2j-1}}-z_{k_{2l-1}})(z_{k_{2j-1}}-z_{k_{2l}})(z_{k_{2j}}-z_{k_{2l-1}})(z_{k_{2j}}-z_{k_{2l}})\prod_{l=1}^{2m}\Big(\prod_{i\neq k_1,k_2,\dots,k_{2m}}(z_l-z_i)\Big).$$

The total exponent for this is $4nm-2m^2-2m$**. The largest exponent for** z_i **is** $n-2$ **for** $i\in\{k_1,\dots,k_{2m}\}$ **and** $2m$ **for** $i\neq k_j$**. Numerator is also polynomials of** $z_1,\dots,z_n$ **which satisfies the same exponent conditions. Unknowns are the coefficients of each terms. Number increases drastically as** n,m **increases.**

By use of Algebraic relations, we have calculated all the correlation functions for $n=5$

$$\rho_{5,\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5}^{\epsilon_1'\epsilon_2'\epsilon_3'\epsilon_4'\epsilon_5'}(z_1,z_2,z_3,z_4,z_5),\ Q_{\epsilon_1,\ldots,\epsilon_5}^{\epsilon_1',\ldots,\epsilon_5'}(k_1,k_2|z_1,\ldots,z_5),$$
$$Q_{\epsilon_1,\ldots,\epsilon_5}^{\epsilon_1',\ldots,\epsilon_5'}(k_1,\ldots,k_4|z_1,\ldots,z_5)$$

Fourth-neighbor correlation function

$$\begin{aligned}\left\langle S_j^z S_{j+4}^z\right\rangle &= \frac{1}{12}-\frac{16}{3}\ln 2+\frac{145}{6}\zeta(3)-54\ln 2\cdot\zeta(3)-\frac{293}{4}\zeta(3)^2\\ &-\frac{875}{12}\zeta(5)+\frac{1450}{3}\ln 2\cdot\zeta(5)-\frac{275}{16}\zeta(3)\cdot\zeta(5)-\frac{1875}{16}\zeta(5)^2\\ &+\frac{3185}{64}\zeta(7)-\frac{1715}{4}\ln 2\cdot\zeta(7)+\frac{6615}{32}\zeta(3)\cdot\zeta(7)\\ &=0.034652776982\cdots\end{aligned}$$

Formula by Boos,Jimbo,Miwa,Smirnov and Takeyama, (Lett. Math. Phys. 75 (2006) 201)

$$h_n(\epsilon_1, ..., \epsilon_n, \overline{\epsilon}_n, ..., \overline{\epsilon}_1)(z_1, ..., z_n) = (-1)^n(\prod_{j=1}^{n} \overline{\epsilon}_j)\rho_{n,\epsilon_1,...\epsilon_n}^{-\overline{\epsilon}_1,...-\overline{\epsilon}_n}(z_1, ..., z_n),$$

$$s_n = \prod_{j=1}^{n} \frac{1}{2}(|+\rangle_j|-\rangle_{\overline{j}} - |-\rangle_j|+\rangle_{\overline{j}}),$$

$$h_n = \exp \Omega_n \ s_n,$$

$$\Omega_n = \int \frac{d\mu}{2\pi i}\frac{d\nu}{2\pi i}\frac{g(\mu-\nu)U(\mu,\nu)V(\mu,\nu)}{\prod_{j=1}^{n}(\mu - z_j)(1-(\mu-z_j)^2)(\nu-z_j)(1-(\nu-z_j)^2)},$$

$$g(z) = -\omega(z)\frac{z}{(1-z^2)^2}.$$

h_n **and** s_n **are vectors in** $2n$ **Pauli spins space.** $\Omega_n, U(\mu,\nu)$ **and** $V(\mu,\nu)$ **are operators in this space.**

$$
\begin{aligned}
&U(\mu,\nu) = \mathrm{Tr}_{\mu-\nu} T_n^{(\mu-\nu)}\left(\frac{\mu+\nu}{2}\right),\\
&T_n(\lambda) = L_{\overline{1}}^{(0)}(\lambda - z_1 - 1)\ldots L_{\overline{n}}^{(0)}(\lambda - z_n - 1) L_n^{(0)}(\lambda - z_n)\ldots L_1^{(0)}(\lambda - z_1),\\
&L_j^{(0)}(\lambda) = (\lambda + \frac{1}{2}) I\,\sigma_j^0 + \frac{1}{2}(H\,\sigma_j^z + 2E\,\sigma_j^+ + 2F\,\sigma_j^-).
\end{aligned}
$$

Here I, H, E, F are $d \times d$ matrices

$$
I_{i,j} = \delta_{i,j},\quad H_{i,j} = (d+1-2i)\delta_{i,j},\quad E_{i,j} = (i-1)\delta_{i,j+1},\quad F_{i,j} = (d-i)\delta_{i+1,j}.
$$

These satisfy commutation relations

$$
[H,E] = -2E, [H,F] = 2F, [E,F] = -H.
$$

$\sigma_j^z, \sigma_j^+, \sigma_j^-$ **are Pauli operators in** $2n$ **spin space** $j = 1, \ldots, n, \overline{1}, \ldots, \overline{n}$**.**
$\mathrm{Tr}_{\mu-\nu}$ **is analytic continuation** $d \to \mu - \nu$**.**

At $d = 2$ we have

$$I = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}, \ H = \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0, 0 \\ 1, 0 \end{pmatrix}, \ F = \begin{pmatrix} 0, 1 \\ 0, 0 \end{pmatrix},$$

and

$$T_n^{(2)}(\lambda) = \begin{pmatrix} A(\lambda), B(\lambda) \\ C(\lambda), D(\lambda) \end{pmatrix},$$

$$V(\mu, \nu) = A(\mu)D(\nu) + D(\mu)A(\nu) - B(\mu)C(\nu) - C(\mu)B(\nu).$$

In the homogeneous limit $z_j \to 0$ this becomes

$$\Omega_n = \int \frac{d\mu}{2\pi i} \frac{d\nu}{2\pi i} \frac{g(\mu - \nu)U(\mu, \nu)V(\mu, \nu)}{\mu^n (1 - \mu^2)^n \nu^n (1 - \nu^2)^n}.$$

We calculated the cases $n = 1, 2, ..., 7$.

$$\langle S_1^z S_2^z \rangle = \frac{1}{12} - \frac{1}{3}\zeta_a(1) = -0.1477157268533151\cdots, (1938)$$

$$\langle S_1^z S_3^z \rangle = \frac{1}{12} - \frac{4}{3}\zeta_a(1) + \zeta_a(3) = 0.0606797699564353 0\cdots, (1977)$$

$$\begin{aligned}\langle S_1^z S_4^z \rangle =& \frac{1}{12} - 3\zeta_a(1) + \frac{74}{9}\zeta_a(3) - \frac{56}{9}\zeta_a(1)\zeta_a(3) - \frac{8}{3}\zeta_a(3)^2 - \frac{50}{9}\zeta_a(5) \\ &+ \frac{80}{9}\zeta_a(1)\zeta_a(5) = -0.0502486272572352 4\cdots, (2003)\end{aligned}$$

$$\begin{aligned}\langle S_1^z S_5^z \rangle =& \frac{1}{12} - \frac{16}{3}\zeta_a(1) + \frac{290}{9}\zeta_a(3) - 72\zeta_a(1)\zeta_a(3) - \frac{1172}{9}\zeta_a(3)^2 - \frac{700}{9}\zeta_a(5) \\ &+ \frac{4640}{9}\zeta_a(1)\zeta_a(5) - \frac{220}{9}\zeta_a(3)\zeta_a(5) - \frac{400}{3}\zeta_a(5)^2 + \frac{455}{9}\zeta_a(7) \\ &- \frac{3920}{9}\zeta_a(1)\zeta_a(7) + 280\zeta_a(3)\zeta_a(7) = 0.0346527769827281 6\cdots. (2005)\end{aligned}$$

$$
\begin{aligned}
\langle S_1^z S_6^z \rangle =& \frac{1}{12} - \frac{25}{3}\zeta_a(1) + \frac{800}{9}\zeta_a(3) - \frac{1192}{3}\zeta_a(1)\zeta_a(3) - \frac{15368}{9}\zeta_a(3)^2 \\
& - 608\zeta_a(3)^3 - \frac{4228}{9}\zeta_a(5) + \frac{64256}{9}\zeta_a(1)\zeta_a(5) - \frac{976}{9}\zeta_a(3)\zeta_a(5) \\
& + 3648\zeta_a(1)\zeta_a(3)\zeta_a(5) - \frac{3328}{3}\zeta_a(3)^2\zeta_a(5) - \frac{76640}{3}\zeta_a(5)^2 \\
& + \frac{66560}{3}\zeta_a(1)\zeta_a(5)^2 + \frac{12640}{3}\zeta_a(3)\zeta_a(5)^2 + \frac{6400}{3}\zeta_a(5)^3 + \frac{9674}{9}\zeta_a(7) \\
& + 56952\zeta_a(3)\zeta_a(7) - \frac{225848}{9}\zeta_a(1)\zeta_a(7) - \frac{116480}{3}\zeta_a(1)\zeta_a(3)\zeta_a(7) \\
& - \frac{35392}{3}\zeta_a(3)^2\zeta_a(7) + 7840\zeta_a(5)\zeta_a(7) - 8960\zeta_a(3)\zeta_a(5)\zeta_a(7) \\
& - \frac{66640}{3}\zeta_a(7)^2 + 31360\zeta_a(1)\zeta_a(7)^2 - 686\zeta_a(9) \\
& + 18368\zeta_a(1)\zeta_a(9) - 53312\zeta_a(3)\zeta_a(9) + 35392\zeta_a(1)\zeta_a(3)\zeta_a(9) \\
& + 16128\zeta_a(3)^2\zeta_a(9) + 38080\zeta_a(5)\zeta_a(9) - 53760\zeta_a(1)\zeta_a(5)\zeta_a(9) \\
=& -0.03089036664760932\cdots \\
\langle S_1^z S_7^z \rangle =& 0.02444673832795890\cdots, \\
\langle S_1^z S_8^z \rangle =& -0.0224982227633722\cdots . \text{(2006)}
\end{aligned}
$$

Von Neumann entropy (Entanglement Entropy)

$$S(n) \equiv -\mathrm{tr}\rho_n \log_2 \rho_n = -\sum_{\alpha=1}^{2^n} \omega_\alpha \log_2 \omega_\alpha,$$

Table 1: von Neumann entropy $S(n)$ of a finite sub-chain of length n

$S(1)$	$S(2)$	$S(3)$	$S(4)$
1	1.3758573262887466	1.5824933209573855	1.7247050949099274

$S(5)$	$S(6)$	$S(7)$
1.833704916848315	1.922358833819333	1.997129812895912

History of thermodynamics of 1D solvable model

· Repulsive δ–function boson	Yang & Yang (1969)
· XXX and XXZ ($\|\Delta\|>1$)	Takahashi Gaudin (1971)
· δ–function fermion	Takahashi Lai (1971)
· Hubbard model	Takahashi (1972)
· XXZ($\|\Delta\|<1$) & XYZ	Takahashi & Suzuki (1972)
· Kondo & Anderson model	Andrei, Wiegmann(1979)
· QTM for XXZ and Hubbard	Koma('87) Destri, de Vega, Klumper,
· QTM for XYZ	Suzuki, Akutsu Wadati (1990)
· Correlation Length	Yamada Takahashi Kluemper('91)

Low temperature thermodynamics

- *T*–Linear specific heat for XXZ model$(-1<\Delta<1)$
- $\sqrt{T}$ specific heat T^{-2} susceptibility for ferromagnetic XXX model
- Logarithmic anomary of susceptibility for antiferromagnetic XXX model

Simplification of thermodynamic Bethe-ansatz equations

cond-mat/0010486. P299-304, Physics and Combinatrix,

Proceeding of the Nagoya 2000 International Workshop, World Scientific (2001).

Thermodynamic Bethe ansatz equations for XXZ model at Δ≧1 [Gaudin ,Takahashi (1971)] is simplified to an integral equation which has one unknown function. [Takahashi(2001)]

This equation is analytically continued to Δ<1.

$$H(J,\Delta,h) = -J\sum_{l=1}^{N} S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta(S_l^z S_{l+1}^z - \frac{1}{4}) - 2h\sum_{l=1} S_l^z, h \geq 0,$$

$$\ln\eta_1(x) = \frac{2\pi J \sinh\phi}{T\phi}\mathbf{s}(x) + \mathbf{s} * \ln(1+\eta_2(x)),$$

$$\ln\eta_j(x) = \mathbf{s} * \ln(1+\eta_{j-1}(x))(1+\eta_{j+1}(x)), \quad j = 2,3,\ldots,$$

$$\lim_{l\to\infty}\frac{\ln\eta_l}{l} = \frac{2h}{T}.$$

$$\Delta = \cosh\phi, \quad Q \equiv \frac{\pi}{\phi}, \quad \mathbf{s}(x) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\mathrm{sech}\left(\frac{\pi(x-2nQ)}{2}\right), \quad \mathbf{s} * f(x) \equiv \int_{-Q}^{Q}\mathbf{s}(x-y)f(y)dy.$$

$$f = \frac{2\pi J \sinh\phi}{\phi}\int_{-Q}^{Q}\mathbf{a}_1(x)\mathbf{s}(x)dx - T\int_{-Q}^{Q}\mathbf{s}(x)\ln(1+\eta_1(x))dx, \quad \mathbf{a}_1(x) \equiv \frac{\phi\sinh\phi/(2\pi)}{\cosh\phi - \cos(\phi x).}$$

Takahashi-Suzuki equation for Δ=cos (π/n) (1972)

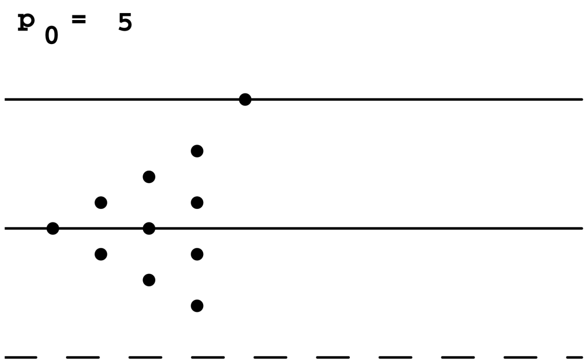
$$\ln(1+\eta_0(x)) = -2nJ\sin(\pi/n)T^{-1}\delta(x),$$

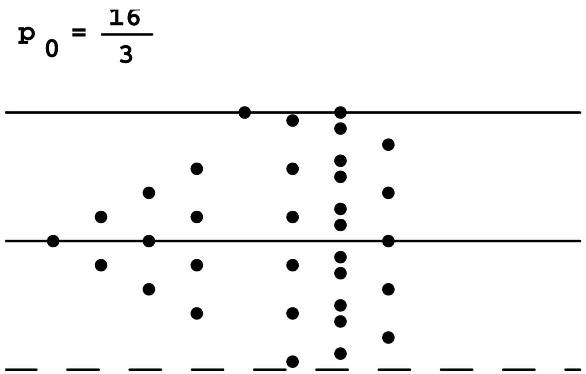
$$\ln\eta_j(x) = s_1 * \ln(1+\eta_{j-1}(x))(1+\eta_{j+1}(x));\ j=1,2,\cdots,n-3,$$

$$\ln\eta_{n-2}(x) = s_1 * \ln(1+\eta_{n-3}(x))\left(1+2\,\mathrm{ch}\frac{2nh}{T}\kappa(x)+\kappa^2(x)\right),$$

$$\ln\kappa(x) = s_1 * \ln(1+\eta_{n-2}(x)).$$

$$s_1(x) = \frac{1}{4}\mathrm{sech}(\frac{\pi}{2}x)$$



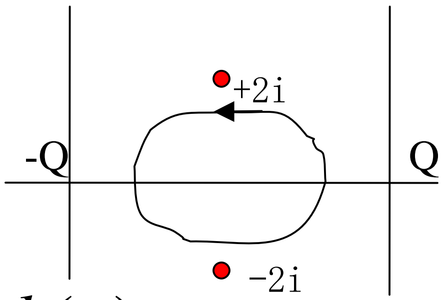


$$u(x) = 2\cosh(\frac{h}{T}) + \oint_c \frac{\phi}{2}(\cot\frac{\phi}{2}[x-y-2i]\exp[-\frac{2\pi J\sinh\phi}{T\phi}\mathbf{a}_1(y+i)]$$

$$+\cot\frac{\phi}{2}[x-y+2i]\exp[-\frac{2\pi J\sinh\phi}{T\phi}\mathbf{a}_1(y-i)])\frac{1}{u(y)}\frac{dy}{2\pi i},$$

$$f = -T\ln u(0).$$

New integral equation



$$g(x) = \mathbf{s} * h(x), \quad g(x+i) + g(x-i) = h(x).$$

$$\left(e^{\omega} + e^{-\omega}\right)\tilde{g}(\omega) = \tilde{h}(\omega), \qquad \tilde{g}(\omega) = \frac{1}{e^{\omega}+e^{-\omega}}\tilde{h}(\omega), \quad \omega = \frac{\pi}{Q}n.$$

$$\eta_1(x+i)\eta_1(x-i) = \exp\left[\frac{2\pi J \sinh\phi}{T\phi}\sum_n \delta(x-2nQ)\right](1+\eta_2(x)),$$

$$\eta_j(x+i)\eta_j(x-i) = (1+\eta_{j-1}(x))(1+\eta_{j+1}(x)), \quad j = 2,3,\ldots,$$

$$\lim_{l\to\infty}\frac{\ln\eta_l}{l} = \frac{2h}{T}.$$

$$\eta_j = \left(\frac{\sinh(j+1)h/T}{\sinh h/T}\right)^2 - 1.$$ Solution at J=0

$\eta_j(x)$ has singularities only at $\pm ji, \pm(j+2)i.$

$$1+\eta_j(x) = A_j(x-ji)\overline{A}_j(x+ji)B_j(x-(j+2)i)\overline{B}_j(x+(j+2)i).$$

$$\overline{A_j}(x) \equiv \overline{A_j(\overline{x})}, \quad \overline{B_j}(x) \equiv \overline{B_j(\overline{x})}.$$

Note that $\overline{A_j}(x)$ is analytic. $\overline{A_j(x)}$ is not.

$$\eta_j(x+i)\eta_j(x-i)=$$

$$A_{j-1}(x-(j-1)i)\overline{A_{j-1}}(x+(j-1)i)B_{j-1}(x-(j+1)i)\overline{B_{j-1}}(x+(j+1)i)$$

$$\times A_{j+1}(x-(j+1)i)\overline{A_{j+1}}(x+(j+1)i)B_{j+1}(x-(j+3)i)\overline{B_{j+1}}(x+(j+3)i).$$

Assume that

$$\eta_j(x)=X(x-ji)Y(x+ji)Z(x-(j+2)i)W(x+(j+2)i),$$

$$\eta_j(x+i)\eta_j(x-i)\ is$$

$$X(x-(j-1)i)Y(x+(j-1)i)X(x-(j+1)i)Y(x+(j+1)i)$$

$$Z(x-(j+1)i)W(x+(j+1)i)Z(x-(j+3)i)W(x+(j+3)i).$$

So we have $X\sim A_{j-1}, Y\sim\overline{A_{j-1}}, Z\sim B_{j+1}, W\sim\overline{B_{j+1}}.$

$$\eta_j(x)=A_{j-1}(x-ji)\overline{A_{j-1}}(x+ji)B_{j+1}(x-(j+2)i)\overline{B_{j+1}}(x+(j+2)i).$$

$$\frac{A_{j-1}(x)}{B_{j-1}(x)}=\frac{A_{j+1}(x)}{B_{j+1}(x)}.$$

$$\delta(x) = \frac{1}{2\pi}\left\{\frac{-i}{x-i\varepsilon} + \frac{i}{x+i\varepsilon}\right\},$$

$$A_0(x) = \exp\left(\frac{J\sinh\phi}{T\phi}\sum_n \frac{-i}{x-2nQ-i\varepsilon}\right) = \exp\left(\frac{J\sinh\phi}{2Ti}\cot\frac{\phi}{2}(x-i\varepsilon)\right), \quad B_0(x) = 1.$$

$$\frac{A_{2j}(x)}{B_{2j}(x)} = \frac{A_0(x)}{B_0(x)} = A_0(x), \quad \frac{A_{2j+1}(x)}{B_{2j+1}(x)} = \frac{A_1(x)}{B_1(x)} = A_0'(x).$$

$$1 = \lim_{j\to\infty}\frac{1+\eta_{2j+1}(x+(2j+1)i)}{\eta_{2j+1}(x+(2j+1)i)} = \lim_{j\to\infty}\frac{A_0'(x)\overline{A'}_0(x+(4j+2)i)}{A_0(x)\overline{A}_0(x+(4j+2)i)}$$
$$\times\left[\frac{B_{2j+1}(x)\overline{B_{2j+1}}(x+(4j+2)i)}{B_{2j}(x)\overline{B_{2j}}(x+(4j+2)i)}\cdot\frac{B_{2j+1}(x-2i)\overline{B_{2j+1}}(x+(4j+4)i)}{B_{2j+2}(x-2i)\overline{B_{2j+2}}(x+(4j+4)i)}\right].$$

$$\frac{A_0'(x)}{A_0(x)} = \frac{\overline{A_0}(i\infty)}{\overline{A'}_0(i\infty)} = \alpha.$$

$$A_{j-1}(x - ji)\overline{A_{j-1}}(x + ji)B_{j+1}(x-(j+2)i)\overline{B_{j+1}}(x+(j+2)i)+1$$
$$= A_j(x - ji)\overline{A_j}(x + ji)B_j(x-(j+2)i)\overline{B_j}(x+(j+2)i).$$

$$B_1(x-i)\overline{B_1}(x+i)B_1(x-3i)\overline{B_1}(x+3i) = \frac{1}{A_0(x-i)\overline{A_0}(x+i)} + B_2(x-3i)\overline{B_2}(x+3i).$$

$$\lim_{x\to i\infty} B_1(x-i)\overline{B_1}(x+i)B_1(x-3i)\overline{B_1}(x+3i) = (2\cosh h/T)^2,$$

These are sufficient to determine functions!!

$$u(x) = B_1(x-2i)\overline{B_1}(x+2i).$$

$$u(x+i) = \frac{1}{A_0(x-i)\overline{A_0}(x+i)u(x-i)} + \frac{B_2(x-3i)\overline{B_2}(x+3i)}{u(x-i)}.$$

$$u(x) = 2\cosh\left(\frac{h}{T}\right) + \sum_{j=1}^{\infty}\sum_{n}\frac{c_j}{(x-2nQ-2i)^j} + \sum_{j=1}^{\infty}\sum_{n}\frac{\overline{c_j}}{(x-2nQ-2i)^j}.$$

$$c_j = \oint\frac{(x-i)^{j-1}}{A_0(x-i)\overline{A_0}(x+i)u(x-i)}\frac{dx}{2\pi i} = \oint\frac{y^{j-1}}{A_0(y)\overline{A_0}(y+2i)u(y)}\frac{dy}{2\pi i}$$

$$\sum_{j=1}^{\infty}\oint\sum_{n}\frac{\exp\left[-\frac{2\pi J\sinh\phi}{T\phi}\mathsf{a}_1(y+i)\right]}{(x-2nQ-2i)^j}\frac{y^{j-1}}{u(y)}\frac{dy}{2\pi i}=\oint\sum_{n}\frac{\exp\left[-\frac{2\pi J\sinh\phi}{T\phi}\mathsf{a}_1(y+i)\right]}{x-y-2nQ-2i}\frac{1}{u(y)}\frac{dy}{2\pi i}$$

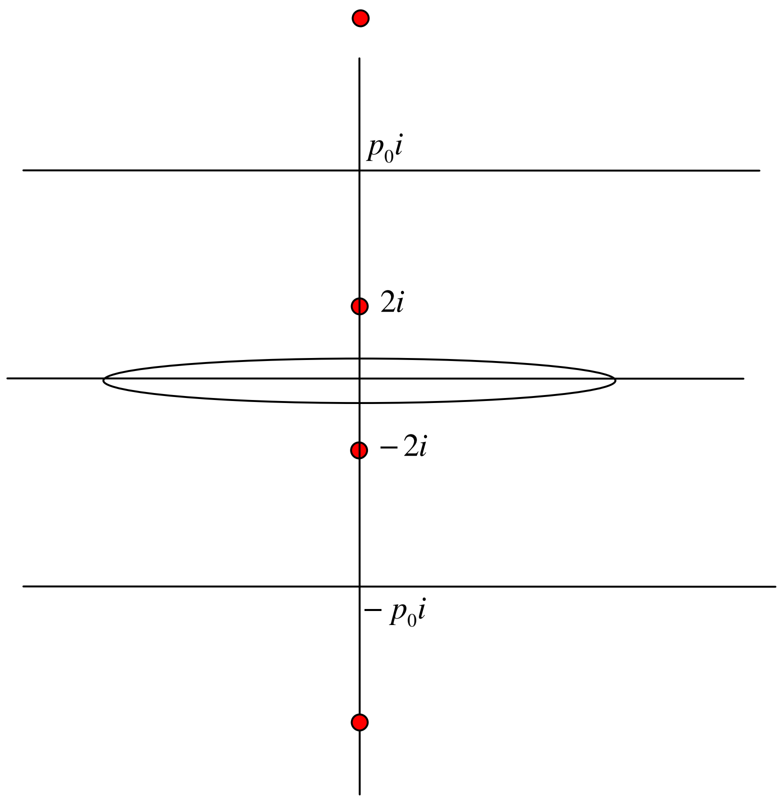
$$=\oint\frac{\phi}{2}\cot\frac{\phi}{2}(x-y-2i)\exp[-\frac{2\pi J\sinh\phi}{T\phi}\mathsf{a}_1(y+i)]\frac{1}{u(y)}\frac{dy}{2\pi i}.$$

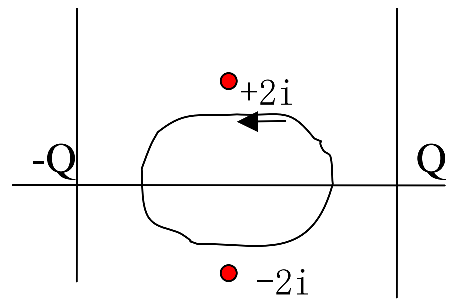
$$1+\eta_1(x)=\exp\left(\frac{2\pi J\sinh\phi}{T\phi}\mathsf{a}_1(x)\right)u(x+i)u(x-i).$$

$$f=-T\int_{-Q}^{Q}\mathsf{s}(x)\left[\ln u(x+i)+\ln u(x-i)\right]dx=-T\ln u(0).$$

Ising limit

$$u(0)=2\cosh(h/T)+\frac{1-\exp(-J_z/T)}{u(0)}.$$





$$u(x) = 2\cosh(\frac{h}{T}) + \oint_c \frac{\phi}{2}(\cot\frac{\phi}{2}[x-y-2i]\exp[-\frac{2\pi J\sinh\phi}{T\phi}\mathbf{a}_1(y+i)]$$

$$+\cot\frac{\phi}{2}[x-y+2i]\exp[-\frac{2\pi J\sinh\phi}{T\phi}\mathbf{a}_1(y-i)])\frac{1}{u(y)}\frac{dy}{2\pi\, i,}$$

$$f = -T\ln u(0).$$

New integral equation

Phys. Rev. Lett. **89** 117201.

High temperature expansion at $\Delta=1$

TABLE I. Series coefficients α_n for the high-temperature expansion of the specific heat $C = \sum_n \frac{\alpha_n}{n!} \left(\frac{J}{4T}\right)^n$ at $h = 0$.

n	α_n	n	α_n
0	0	26	484455914465376683487755420408217600
1	0	27	−1592964364128699671723658807556964352
2	6	28	−1659222341377723674454893065936371187712
3	−36	29	53827694891210973745020673240061454581760
4	−360	30	5090517962961447184851808942927438864711680
5	7200	31	−388446833192725659973817494776649147157053440
6	15120	32	−11028658525378274359384407389662010654796546048
7	−1848672	33	2255854109806569120380670028755308714386167693312
8	11426688	34	−18878702580622989070078793482363993425701807063040
9	594846720	35	−11721570087037734701356860480896473609272542720163840
10	−11558004480	36	527183038642469386328859769728396518893185382125404160
11	−199812856320	37	52548749252010967993948480669499712309299856992951599104
12	10106191180800	38	−5382365237582398925074954773487741035075672601159589167104
13	19376365252608	39	−150281021589219619860159284209265140804955107276364175114240
14	−9289795522775040	40	44482678475307391762958213932359681737961800852665438128046080
15	121944211136778240	41	−670778300712303276022754187872671936481343744506621812675706880
16	8791781390116945920	42	−323311185126253530334911142992092649497388429937499549362387156992
17	−310402124957945954304	43	19271391500067613736198673193545354611765664770995927250862568636416
18	−7225535925744106143744	44	1963797073102024140530884388201619017857423297479642613447074848440320
19	643407197363813620776960	45	−2617574495013913833349154989821467694962901907072496780634929173522022400
20	96147483542540314214400	46	−6715036186134671522475926929150627328836680429076585020863949295740518400
21	−1279121513829538179364945920	47	2897640509780835688484069216581936412870887902144153768250804439297575354368
22	27962069861743501862336200704	48	−67884583842448252705729493380589916284543590592089243545625816663055684075520
23	2398518627113966015427501883392	49	−27839667354545349510646207322267839343449407617219100508752156300912706663219200
24	−129834725539335848980192847462400	50	2130333568970965233678580974509426707048585535358286474483865352948915543579033600
25	−3493877000064415911285457158144000	51	216827879657653769500650387534438914339017251205042192428944424756474567924803174400

Derivation from QTM

Takahashi, Shiroishi and Klumper,

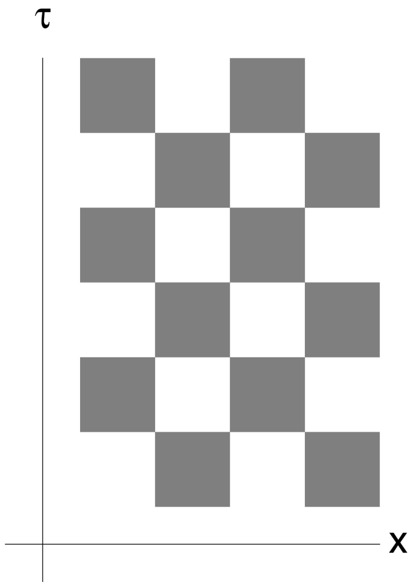
Equivalence of TBA and QTM

J. Phys. A: Math. Gen 34 L187-194,

Cond mat/0102027

Quantum transfer matrix and fusion hierarchy models

$$Z = \sum_{\{\sigma\}} \prod_{j=1}^{N} \prod_{i=1}^{M} A(\sigma_{2i+j,j}\sigma_{2i+j+1,j};\sigma_{2i+j,j+1}\sigma_{2i+j+1,j+1,})$$



$$A(\sigma_1\sigma_2;\sigma_1'\sigma_2') = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & c & b' & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

$$Z = Tr\mathrm{T}^N, \quad \mathrm{T}(\sigma_1,\sigma_{2,\ldots,}\sigma_{2M};\sigma_1',\sigma_{2,\ldots,}'\sigma_{2M}')$$

$$\equiv A(\sigma_1\sigma_2;\sigma_{2M}'\sigma_1')A(\sigma_3\sigma_4;\sigma_2'\sigma_3')\ldots A(\sigma_{2M-1}\sigma_{2M};\sigma_{2M-2}'\sigma_{2M-1}').$$

$$a = \exp(-\frac{J\Delta}{2MT})\sinh(\frac{J}{2MT}), \quad b = \exp(\frac{-h}{MT}),$$

$$b' = \exp(\frac{h}{MT}), \quad c = \exp(-\frac{J\Delta}{2MT})\cosh(\frac{J}{2MT}).$$

Consider an inhomogeneous six-vertex model with the following column dependent Boltzman weights:

$$\mathrm{T} = Tr\mathrm{R}_1(\sigma_{1,}\sigma_1')\mathrm{R}_2(\sigma_{2,}\sigma_2')\ldots\mathrm{R}_L(\sigma_{L,}\sigma_L'),$$

$$\mathrm{R}_l(++) = \begin{pmatrix} a_l & 0 \\ 0 & b_l \end{pmatrix}, \quad \mathrm{R}_l(+-) = \begin{pmatrix} 0 & 0 \\ c_l & 0 \end{pmatrix},$$

$$\mathrm{R}_l(-+) = \begin{pmatrix} 0 & c_l \\ 0 & 0 \end{pmatrix}, \quad \mathrm{R}_l(--) = \begin{pmatrix} b_l' & 0 \\ 0 & a_l \end{pmatrix}.$$

$$\mathrm{a}_l = \rho_l \mathbf{h}(v + v_l + \eta)$$

$$b_l = \rho_l \omega^{-1} \mathbf{h}(v + v_l - \eta)$$

$$b_l' = \rho_l \omega\, \mathbf{h}(v + v_l - \eta)$$

$$c_l = \rho_l \mathbf{h}(2\eta), \quad l = 1, \ldots, L.$$

Wave function and eigenvalue

$$\left|\Psi\right\rangle = \sum f(y_1, y_2, \ldots, y_k)\sigma^-_{y_1}\sigma^-_{y_2}\ldots\sigma^-_{y_k}\left|0\right\rangle,$$

$$f(y_1 y_2, \ldots, y_k) = \sum_P A(P)\prod_{j=1}^{k} F(y_i; u_{Pj}),$$

$$F(y;u) \equiv \omega^y \prod_{l=1}^{y-1} \mathsf{h}(u + v_l + \eta) \prod_{l=y+1}^{L} \mathsf{h}(u + v_l - \eta),$$

$$A(P) = \varepsilon(P)\sum_{j<l} \mathsf{h}(u_{Pj} - u_{Pl} - 2\eta).$$

$$\frac{\varphi(u_j+\eta)}{\varphi(u_j-\eta)} = -\omega^{-L}\prod_{m=1}^{k}\frac{\mathsf{h}(u_j - u_m + 2\eta)}{\mathsf{h}(u_j - u_m + 2\eta)},$$

$$\phi(v) = \prod_{l=1}^{L} \rho_l \mathsf{h}(v + v_l).$$

$$T_1(v) = \omega^{-L+k}\varphi(v-\eta)\frac{Q(v+2\eta)}{Q(v)} + \omega^k \varphi(v+\eta)\frac{Q(v-2\eta)}{Q(v)},$$

$$Q(v) = \prod_{j=1}^{k} \mathsf{h}(v - u_j).$$

$$a_l = c_l = 1, \quad b_l = b_l' = 0 \quad \text{for even } l,$$

$$a_l = \exp\left(-\frac{J\Delta}{2MT}\right)\sinh\left(\frac{J}{2MT}\right), \quad b_l = \exp\left(\frac{-h}{MT}\right),$$

$$b_l' = \exp\left(\frac{h}{MT}\right), \quad c_l = \exp\left(-\frac{J\Delta}{2MT}\right)\cosh\left(\frac{J}{2MT}\right) \quad \text{for odd } l.$$

$$\rho_l = 1/h(2\eta), v_l = \eta \quad \text{for even } l$$

$$\rho_l = \frac{\sqrt{bb'}}{\mathsf{h}(v_l - \eta)}, \quad \frac{\mathsf{h}(v_l + \eta)}{\mathsf{h}(v_l - \eta)} = \frac{a}{\sqrt{bb'}} = \exp\left(-\frac{J\Delta}{2MT}\right)\sinh\left(\frac{J}{2MT}\right) \quad \text{for odd } l.$$

$$L = 2M, \quad \omega = \exp\left(\frac{h}{MT}\right), \quad v = 0, \quad \frac{\mathsf{h}'(2\eta)}{\mathsf{h}'(0)} = \frac{\sinh\left(\frac{J\Delta}{MT}\right)}{\sinh\left(\frac{J}{MT}\right)},$$

$$\phi(v)=\left(\frac{\mathsf{h}(v+\eta)\mathsf{h}(v+2\alpha_M-\eta)}{\mathsf{h}(2\eta)\mathsf{h}(2\alpha_M-2\eta)}\right)^M,$$

$$\frac{\phi(u_j+\eta)}{\phi(u_j-\eta)}=-e^{-2h/T}\prod_{m=1}^{M}\frac{h(u_j-u_m+2\eta)}{h(u_j-u_m-2\eta)}.$$ Bethe ansatz equation,

$$Q(v)=\prod_{m=1}^{M}h(v-u_m).$$

For a solution of Bethe ansatz equations we have an eigenvalue;

$$\mathsf{T}_1(v)=e^{-h/T}\phi(v-\eta)\frac{Q(v+2\eta)}{Q(v)}+e^{-h/T}\phi(v+\eta)\frac{Q(v-2\eta)}{Q(v)}.$$

$$f=-T\lim_{M\to\infty}\ln\mathsf{T}_1(0).$$

Fusion models

$$\mathsf{T}_j(v)\equiv\sum_{t=0}^{j}e^{-(j-2l)h/T}\phi(v-(j-2l)\eta)\frac{Q(v+(j+1)\eta)Q(v-(j+1)\eta)}{Q(v+(2l-j+1)\eta)Q(v+(2l-j-1)\eta)}.$$

$$\mathsf{T}_j(v+\eta)\mathsf{T}_j(v-\eta)=\varphi(v+(j+1)\eta)\varphi(v-(j+1)\eta)+\mathsf{T}_{j+1}(v)\mathsf{T}_{j-1}(v),$$

$$\mathsf{T}_0(v)\equiv\varphi(v).$$

Derivation of Gaudin-Takahashi equation

$$\mathsf{h}(u) = \sin u, \quad \eta = i\tilde{\phi}/2, \quad \tilde{\phi} = \cosh^{-1}\left(\frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)}\right),$$

$$\alpha_M = \frac{i}{2}\tanh^{-1}\left(\tanh\tilde{\phi}\tanh\frac{J\Delta}{2MT}\right),$$

$$\tilde{\phi} = \phi, \quad M\alpha_M = iJ\sinh\phi/(4T).$$

$$Q(x) = \prod_{j=1}^{M}\sin\frac{\tilde{\phi}}{2}(x - x_j), \quad \varphi(x) = \left(\frac{\sin\frac{\tilde{\phi}}{2}(x+i)\sin\frac{\tilde{\phi}}{2}(x-(1-2u_M)i)}{\sinh\tilde{\phi}\sinh\tilde{\phi}(1-u_M)}\right)^M,$$

$$u_M = \alpha_M/\eta, \quad x_j = iu_j/\eta.$$

Set of entire functions

$$\mathsf{T}_j(x) \equiv \sum_{t=0}^{j} e^{-(j-2l)h/T} \varphi(x-(j-2l)i) \frac{Q(x+(j+1)i)Q(x-(j+1)i)}{Q(x+(2l-j+1)i)Q(x+(2l-j-1)i)}.$$

Define new functions which has poles;

$$\tilde{\mathsf{T}}_j(x) \equiv \mathsf{T}_j(x) \left(\frac{\sinh(\tilde{\phi})\sinh\tilde{\phi}(1-u_M)}{\sin\frac{\tilde{\phi}}{2}(x+(j+1)i)\sin\frac{\tilde{\phi}}{2}(x-(j+1-2u_M)i)} \right)^M .$$

These satisfies;

$$\tilde{\mathsf{T}}_j(x+i)\tilde{\mathsf{T}}_j(x-i) = \mathbf{b}_j(x) + \tilde{\mathsf{T}}_{j-1}(x)\tilde{\mathsf{T}}_{j+1}(x),$$

$$\mathbf{b}_j(x) = \left(\frac{\sin\frac{\tilde{\phi}}{2}(x+(j+2u_M)i)\sin\frac{\tilde{\phi}}{2}(x-ji)}{\sin\frac{\tilde{\phi}}{2}(x+ji)\sin\frac{\tilde{\phi}}{2}(x-(j-2u_M)i)} \right)^M ,$$

$$\tilde{\mathsf{T}}_j(\pm i\infty) = \frac{\sinh(j+1)h/T}{\sinh h/T}.$$

$$Y_j(x) = \frac{\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x)}{\mathbf{b}_j(x)}, \quad j = 1,2,\dots$$

$$Y_1(x-i)Y_1(x+i) = 1 + Y_2(x),$$

$$Y_j(x+i)Y_j(x-i) = (1 + Y_{j-1}(x))(1 + Y_{j+1}(x)), \qquad j = 2,3,\dots,$$

$$\lim_{l \to \infty} \frac{\ln\ Y_l(x)}{l} = \frac{2h}{T}.$$

$$\ln\ Y_{\mathrm{j}}(x) = \mathbf{s} * (\ln(1 + Y_{j-1}) + \ln(1 + Y_{j+1})), \qquad j \geq 2.$$

$$\tilde{T}_2(x+i)\tilde{T}_2(x-i) = \mathbf{b}_2(x)(1 + Y_2(x)),$$

$$\ln\ \tilde{T}_2(x) = \mathbf{s} * (\ln\ \mathbf{b}_2(x) + \ln(1 + Y_2(x))).$$

$$\ln Y_1(x) = -\ln \mathbf{b}_1(x) + \mathbf{s} * \ln \mathbf{b}_2(x) + \mathbf{s} * \ln(1 + Y_2(x)).$$

$$\mathsf{b}_j(x) = \lim_{M\to\infty} \exp\left[M \ln \frac{\sin\frac{\tilde\phi}{2}(x + (j + 2u_M)i)\sin\frac{\tilde\phi}{2}(x - ji)}{\sin\frac{\tilde\phi}{2}(x + ji)\sin\frac{\tilde\phi}{2}(x - (j - 2u_M)i)} \right]$$

$$= \exp\left(-\frac{2\pi J \sinh\phi}{\phi T} \mathsf{a}_j(x) \right), \quad \mathsf{a}_j(x) \equiv \frac{\phi \sinh j\phi/(2\pi)}{\cosh j\phi - \cos(\phi x)},$$

Then we find $\lim_{M\to\infty} Y_j(x) = \eta_j(x).$

$$\ln \tilde{\mathsf{T}}_1(x) = \mathsf{s} * \ln[(1 + Y_1(x))/\mathsf{b}_1(x)]$$

$$u_j(x) \equiv \lim_{M\to\infty} \tilde{\mathsf{T}}_{\mathrm{j}}(x),$$

$$u_1(x+i)u_1(x-i) = \mathsf{b}_1(x) + u_2(x).$$

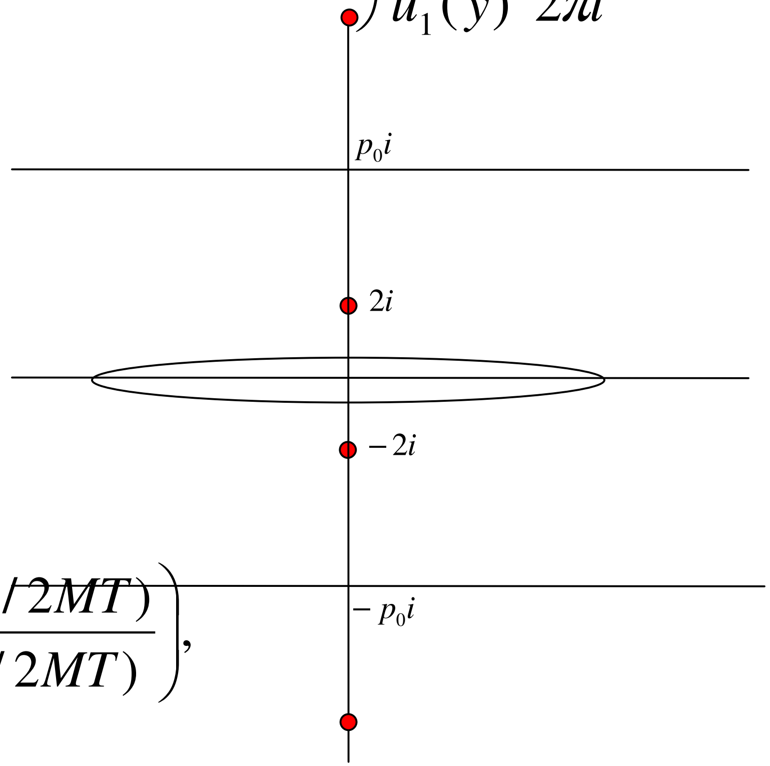
We construct equations for $u_j(x)$.

$$u_1(x+i) = \mathsf{b}_1(x)/u_1(x-i) + u_2(x)/u_1(x-i),$$

$$u_1(x) = 2\cosh h/T$$

$$+\oint_c \frac{\phi}{2}\left(\cot\frac{\phi}{2}[x-y-2i]\mathbf{b}_1(y+i) + \cot\frac{\phi}{2}[x-y+2i]\mathbf{b}_1(y-i) \right)\frac{1}{u_1(y)}\frac{dy}{2\pi i}.$$

$$f = -T\ln u_1(0).$$



Case Δ<1

$$h(u) = \sinh u, \quad \eta = i\tilde{\theta}/2, \quad \tilde{\theta} = \cos^{-1}\left(\frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)}\right),$$

$$\alpha_M = \frac{i}{2}\tanh^{-1}(\tan\tilde{\theta}\tanh\frac{J\Delta}{2MT}).$$

$$\tilde{\theta} = \cos^{-1}\Delta, \quad M\alpha_M = iJ\sin\theta/(4T).$$

$$Q(x)=\prod_{j=1}^{M}\sinh\frac{\tilde{\theta}}{2}(x-x_j),\quad \varphi(x)=\left(\frac{\sinh\frac{\tilde{\phi}}{2}(x+i)\sinh\frac{\tilde{\phi}}{2}(x-(1-2u_M)i}{\sin\tilde{\theta}\sin\tilde{\theta}(1-u_M)}\right)^M .$$

$$\tilde{\mathsf{T}}_j(x)\equiv\mathsf{T}_j(x)\left(\frac{\sin(\tilde{\theta})\sin\tilde{\theta}(1-u_M)}{\sinh\frac{\tilde{\phi}}{2}(x+(j+1)i)\sinh\frac{\tilde{\phi}}{2}(x-(j+1-2u_M)i}\right)^M .$$

$$\tilde{\mathsf{T}}_1(x+i)\tilde{\mathsf{T}}_1(x-i)=b_1(x)+\tilde{\mathsf{T}}_2(x),$$

$$b_1(x)=\left(\frac{\sinh\frac{\tilde{\phi}}{2}(x+(1+2u_M)i)\sinh\frac{\tilde{\phi}}{2}(x-i)}{\sinh\frac{\tilde{\phi}}{2}(x+i)\sinh\frac{\tilde{\phi}}{2}(x-(1-2u_M)i)}\right)^M . \qquad \tilde{\mathsf{T}}_1(\pm\infty)=2\cosh h/T.$$

$$b_1(x)=\exp\left(-\frac{2\pi J\sin\theta}{\theta T}a_1(x)\right),\quad a_1(x)\equiv\frac{\theta\sin\theta/(2\pi)}{\cosh(\theta x)-\cos\theta}.$$

$$\tilde{T}_1(x) = 2\cosh\left(\frac{h}{T}\right) + \sum_{j=1}^{\infty}\sum_{n}\frac{c_j}{(x-2np_0 i-2i)^j} + \sum_{j=1}^{\infty}\sum_{n}\frac{\overline{c_j}}{(x-2np_0 i-2i)^j}.$$

$$c_j = \oint \frac{(x-j)^{j-1} b_1(x)}{\tilde{T}_1(x-i)}\frac{dx}{2\pi i} = \oint \frac{y^{j-1} b_1(y+i)}{\tilde{T}_1(y)}\frac{dy}{2\pi i}.$$

$$\sum_{j=1}^{\infty}\oint\sum_{n}\frac{b_1(y+i)}{(x-2np_0 i-2i)^j}\frac{y^{j-1}}{\tilde{T}_1(y)}\frac{dy}{2\pi i} = \oint\sum_{n}\frac{b_1(y+i)}{x-y-2np_0 i-2i}\frac{1}{\tilde{T}_1(y)}\frac{dy}{2\pi i}$$

$$= \oint\frac{\theta}{2}\coth\frac{\theta}{2}(x-y-2i)\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y+i)\right]\frac{1}{\tilde{T}_1(y)}\frac{dy}{2\pi i}.$$

$$u(x) = 2\cosh\left(\frac{h}{T}\right) + \oint_c\frac{\theta}{2}(\coth\frac{\theta}{2}[x-y-2i]\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y+i)\right]$$

$$+\coth\frac{\theta}{2}[x-y+2i]\exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y-i)\right]\frac{1}{u(y)}\frac{dy}{2\pi i},$$

$$f = -T\ln u(0).$$

Summary

1. **Correlation functions of spin-$\frac{1}{2}$ Heisenberg chain at the ground state can be calculated *analytically* when there is no magnetic field.**

2. **For $\Delta = 1$ (XXX model), correlations are expressed in terms of $\ln 2$ and $\zeta(2k+1)$.**

3. **For general Δ, correlations are expressed as a polynomial of certain one-dimensional integrals.**

4. **We have calculated all the correlation functions**

   up to $n = 7$ for XXX model
   up to $n = 4$ for XXZ model

5. **About the two-point function for XXX chain we calculated 6-th neighbor and 7-th neighbor correlators using generating function method.**

Conclusion

1) Thermodynamic Bethe ansatz equations are simplified to an equation which has only one unknown function.

2) Gaudin Takahashi equations and Takahashi Suzuki equations are redrived from quantum transfer matrix formulation.

3) $|\Delta| \geqq 1$ and $|\Delta|<1$ cases are treated by a single integral equation.

4) This equation is useful for high temperature expansion.

Future Problems

1)Extention to other solvable models (XYZ chain, Hubbard chain,…).

2)Analytic solution for XY case($\Delta=0$).

References

[1] L. Hulthén, Ark. Mat. Astron. Fys. A 26 (1938) 1.

[2] M. Takahashi, J. Phys. C 10 (1977) 1289.

[3] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, Phys. Lett. A 168, (1992) 256.

[4] A. Nakayashiki, Int. J. Mod. Phys. A 9, (1994) 5673.

[5] M. Jimbo and T. Miwa, J. Phys. A 29 (1996) 2923.

[6] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B 567 (2000) 554.

[7] H.E. Boos and V.E. Korepin, J. Phys. A 34 (2001) 5311.

[8] H.E. Boos and V.E. Korepin, *Integrable models and Beyond*, edited by M. Kashiwara and T. Miwa (Birkhäuser, Boston, 2002); hep-th/0105144.

[9] H.E. Boos, V.E. Korepin, Y. Nishiyama and M. Shiroishi, J. Phys. A 35 (2002) 4443.

[10] H.E. Boos, V.E. Korepin and F.A. Smirnov, Nucl. Phys. B. 658 (2003) 417.

[11] K. Sakai, M. Shiroishi, Y. Nishiyama and M. Takahashi, Phys. Rev. E 67 (2003) 065101(R).

[12] G. Kato, M. Shiroishi, M. Takahashi, K. Sakai, J. Phys. A: Math. Gen. 36 L337-L344 (2003)

[13] M. Takahashi, G. Kato, M. Shiroishi: J. Phys. Soc. Jpn. 73 245-253 (2004).

[14] G. Kato, M. Shiroishi, M. Takahashi, K. Sakai, J. Phys. A: Math. Gen. 37 (2004) 5097.

[15] H.E. Boos, M. Shiroishi and M. Takahashi, Nucl. Phys. B 712 p.573-599 (2005).

[16] G. Göhmann, A. Klümper and A. Seel, J. Phys. A37 (2004) 7625.

[17] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, Algebra and Analysis 17 (2005) 115.

[18] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, Commun. Math. Phys. 261 (2006) 245.

[19] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, J. Phys. A 38 (2005) 7629.

[20] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, Lett. Math. Phys. 75 (2006) 201.

[21] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, hep-th/0601132.

[22] H. Boos, M. Jimbo, T. Miwa, F. Smirnov and Y. Takeyama, hep-th/0606280.

Koma, T. *Prog. Theor. Phys.* **81**, (1989) 783.
Takahashi, M. *Phys. Rev. B* **44** (1991) 12382.
Takahashi, M. *Phys. Rev. B.* **43** (1991) 5788.
Suzuki, J. ,Akutsu, Y., and Wadati, M., *J. Phys. Soc. Japan,* **59** (1990) 2667
Takahashi, M. *Prog. Theor. Phys.* **47** (1972) 69.
Koma, T. *Prog. Theor. Phys.* **83** (1990) 655.
Takahashi, M. (1971) *Prog. Theor. Phys.* **46**, 401.
Gaudin, M. (1971) *Phys. Rev. Lett.* **26**, 1301.
Klumper A. (1992) *Ann. Physik* **1**, 540
Kuniba, A. Sakai, K. and Suzuki, J. *Nucl. Phys. B* **525** (1998) 597.
Takahashi M. and Suzuki M., *Prog. Theor. Phys.* **46** (1972) 2187.

Klumper, A. *Z. Phys. B* **91** (1993) 507.
Schlottmann, P. *Phys. Rev. B* **36** (1987) 5177.
Juttner, G., Klumper, A. and Suzuki, J. *Nucl. Phys. B* **487** (1997) 650.
Juttner, G., Klumper, A. and Suzuki, J. *Nucl. Phys. B* **512** (1998) 581.
Juttner, G., Klumper, A. and Suzuki, J. *Nucl. Phys. B* **522** (1998) 471.

Takahashi, M. (1999) Thermodynamics of One-Dimensional Solvable Models, Cambridge University Press.

Takahashi, M. (1971) *Prog. Theor. Phys.* **46**, 401.

Gaudin, M. (1971) *Phys. Rev. Lett.* **26**, 1301.

Takahashi, M. and Suzuki, M. (1972) *Prog. Theor. Phys.* **46**, 2187.

Kuniba, A. , Sakai, K. and Suzuki, J. (1998) *Nucl. Phys. B* **525**, 597.

Shiroishi, M. and Takahashi M. (2002) *Phys. Rev. Lett.* **89** 117201.

Tsuboi, Z. and Takahashi M. (2005) *JPSJ* **74** 898.