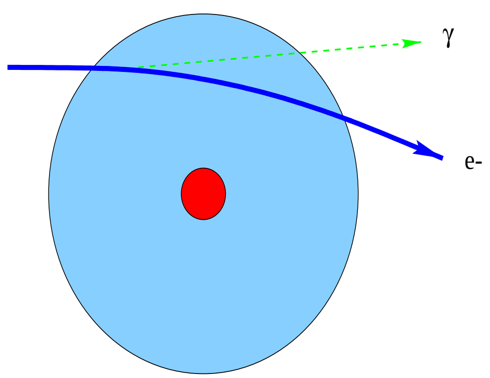
Bremsstrahlung

Bremsstrahlung

A fast moving charged particle is decelerated in the Coulomb field of atoms. A fraction of its kinetic energy is emitted in form of real photons.

The probability of this process is $\propto 1/M^2$ (M: masse of the particle) and $\propto Z^2$ (atomic number of the matter).

Above a few tens MeV, bremsstrahlung is the dominant process for e- and e+ in most materials. It becomes important for muons (and pions) at few hundred GeV.



differential cross section

The differential cross section is given by the Bethe-Heitler formula [Heitl57], corrected and extended for various effects:

- the screening of the field of the nucleus
- the contribution to the brems from the atomic electrons
- the correction to the Born approximation
- the polarisation of the matter (dielectric suppression)
- the so-called LPM suppression mechanism
- ...

See Seltzer and Berger for a synthesis of the theories [Sel85].

screening effect

Depending of the energy of the projectile, the Coulomb field of the nucleus can be more on less screened by the electron cloud.

A screening parameter measures the ratio of an 'impact parameter' of the projectile to the radius of an atom, for instance given by a Thomas-Fermi approximation or a Hartree-Fock calculation.

Then, screening functions are introduced in the Bethe-Heitler formula.

Qualitatively:

- at low energy → no screening effect
- at ultra relativistic electron energy → full screening effect

electron-electron bremsstrahlung

The projectile feels not only the Coulomb field of the nucleus (charge Ze), but also the fields of the atomic electrons (Z electrons of charge e).

The bremsstrahlung amplitude is roughly the same in both cases, except the charge.

Thus the electron cloud gives an additional contribution to the bremsstrahlung, proportional to Z (instead of Z^2).

Born approximation

The derivation of the Bethe-Heitler formula is based on perturbation theory, using plane waves for the electron. If the validity of the Born approximation:

$$\beta \gg \alpha Z$$

is violated for the initial and/or final velocity (low energy) the Coulomb waves would be used instead of the plane waves.

To correct for this, a Coulomb correction function is introduced in the Bethe-Heitler formula.

high energies regime : $E \gg mc^2/(\alpha Z^{1/3})$

Above few GeV the energy spectrum formula becomes simple :

$$\left.\frac{d\sigma}{dk}\right]_{Tsai} \approx 4\alpha\, r_e^2\, \frac{1}{k} \times \left\{ \left(\frac{4}{3} - \frac{4}{3}y + y^2\right) \left(Z^2 \left[L_{rad} - f(Z)\right] + Z L'_{rad}\right) \right\} \quad (1)$$

where

k	energy of the radiated photon ; $y = k/E$
α	fine structure constant
r_e	classical electron radius: $e^2/(4\pi\epsilon_0 mc^2)$
$L_{rad}(Z)$	$\ln(184.15/Z^{1/3})$ (for $Z \geq 5$)
$L'_{rad}(Z)$	$\ln(1194/Z^{2/3})$ (for $Z \geq 5$)
$f(Z)$	Coulomb correction function

limits of the energy spectrum

$k_{min} = 0$: In fact the infrared divergence is removed by the dielectric suppression mechanism, which is not shown in the formula 1.
For $k/E \leq 10^{-4}$: $d\sigma/dk$ becomes proportional to k [Antho96]

$k_{max} = E - mc^2 \approx E$: In this limit, the screening is incomplete, and the expression 1 of the cross section is not completely accurate.

mean rate of energy loss due to bremsstrahlung

$$-\frac{dE}{dx} = n_{at} \int_{k_{min}=0}^{k_{max}\approx E} k \, \frac{d\sigma}{dk} \, dk \tag{2}$$

n_{at} is the number of atoms per volume.

The integration immediately gives:

$$-\frac{dE}{dx} = \frac{E}{X_0} \tag{3}$$

with:

$$\frac{1}{X_0} \overset{def}{=} 4\alpha \, r_e^2 \, n_{at} \left\{ Z^2 \left[L_{rad} - f(Z) \right] + Z L'_{rad} \right\}$$

Radiation Length

The radiation length has been calculated by Y. Tsai [Tsai74]

$$\frac{1}{X_0} = 4\alpha\, r_e^2\, n_{at} \left\{ Z^2 \left[L_{rad} - f(Z) \right] + Z L'_{rad} \right\}$$

where

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r_e	classical electron radius
n_{at}	number of atoms per volume: $\mathcal{N}_{av}\rho/A$
$L_{rad}(Z)$	$\ln(184.15/Z^{1/3})$ (for $Z \geq 5$)
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$f(Z)$	Coulomb correction function

$$f(Z) = a^2[(1+a^2)^{-1} + 0.20206 - 0.0369a^2 + 0.0083a^4 - 0.002a^6 \cdots]$$

with $a = \alpha Z$

main conclusion: The relation 3 shows that the average energy loss per unit path length due to the bremsstrahlung increases linearly with the initial energy of the projectile.

equivalent:

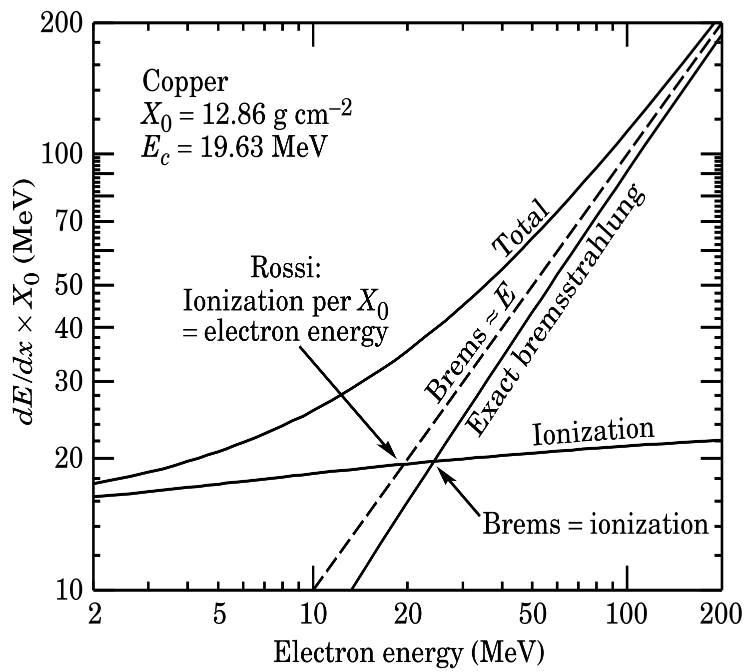
$$E(x) = E(0)\exp\left(-\frac{x}{X_0}\right)$$

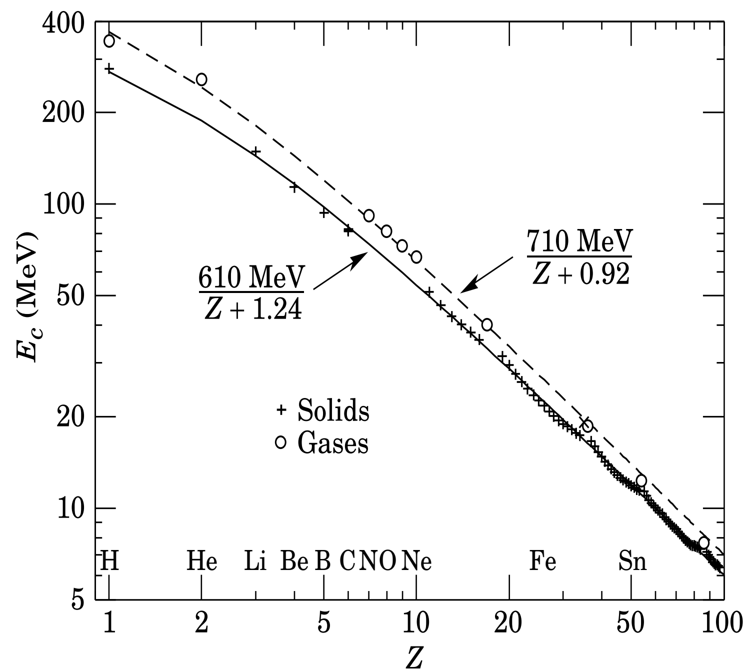
This is the exponential attenuation of the energy of the projectile by radiation losses.

critical energy

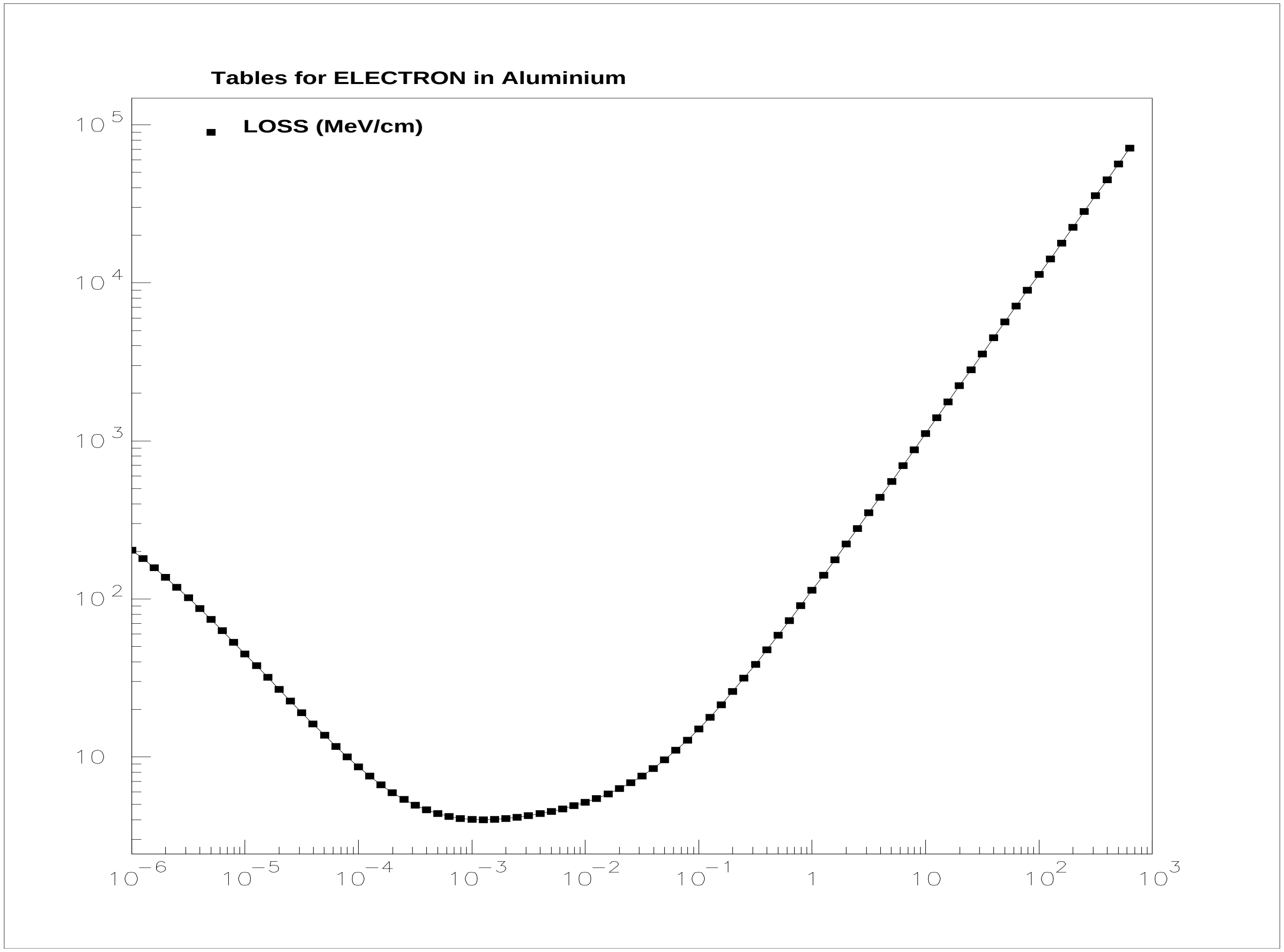
The total mean rate of energy loss is the sum of ionization and bremsstrahlung.

The *critical energy* E_c is the energy at which the two rates are equal. Above E_c the total energy loss rate is dominated by bremsstrahlung.





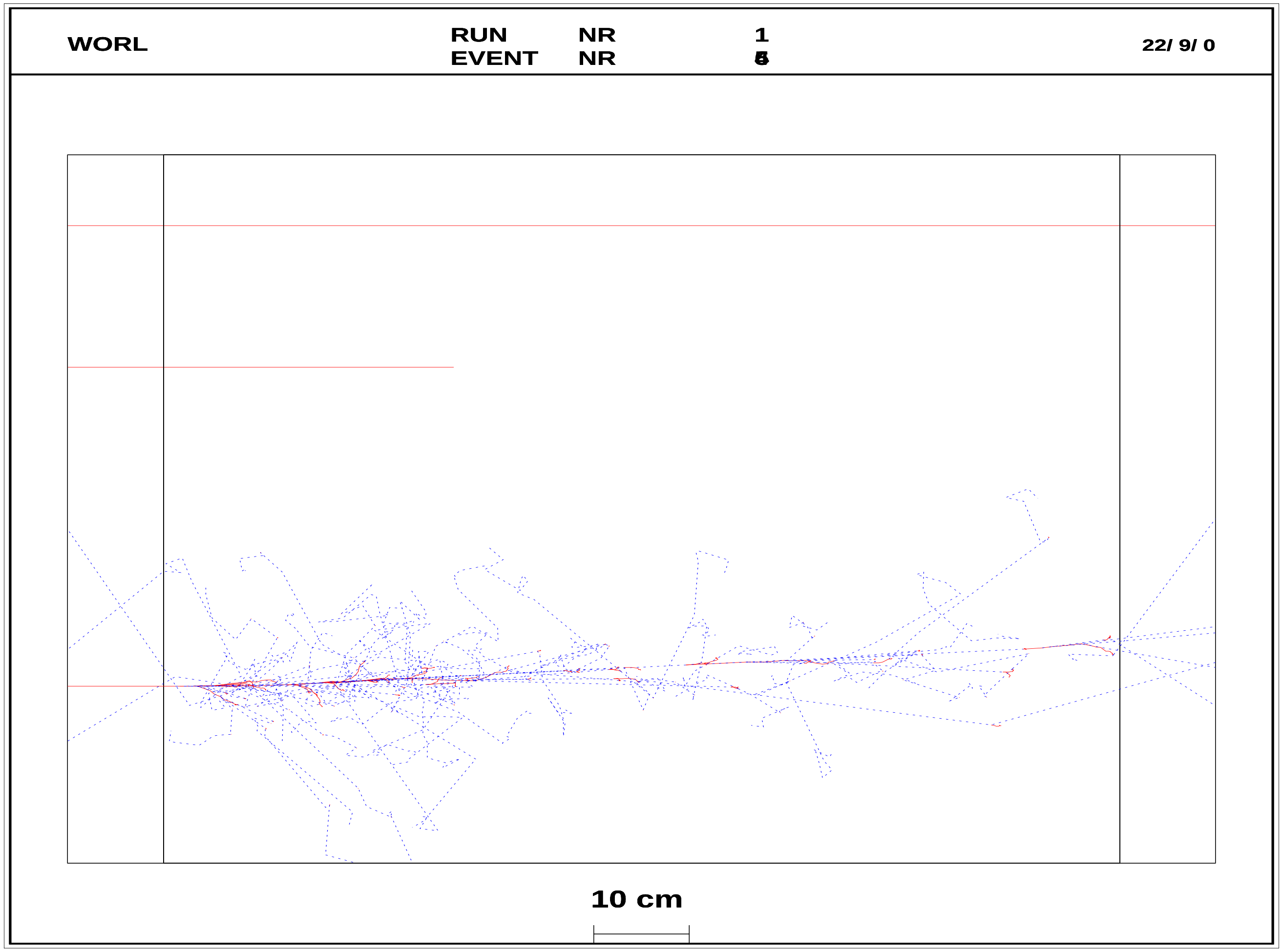
dE/dx for electrons



electron kinetic energy (GeV)

e^- 1 GeV in 1 meter of Aluminium.

Brems counted as continuous energy loss versus cascade development



fluctuations : Unlike the ionization loss which is quasicontinuous along the path length, almost all the energy can be emitted in one or two photons. Thus, the fluctuations on energy loss by bremstrahlung are large.

Energetic photons and truncated energy loss rate

One may wish to take into account separately the high-energy photons emitted above a given threshold k_{cut} (miss detection, explicit simulation ...).

Those photons must be excluded from the mean energy loss count.

$$-\frac{dE}{dx}\bigg]_{k<k_{cut}} = n_{at} \int_{k_{min}=0}^{k_{cut}} k \, \frac{d\sigma}{dk} \, dk \qquad (4)$$

n_{at} is the number of atoms per volume.

Then, the truncated total cross-section for emitting 'hard' photons is:

$$\sigma(E, k_{cut} \leq k \leq k_{max}) = \int_{k_{cut}}^{k_{max} \approx E} \frac{d\sigma}{dk} \, dk \qquad (5)$$

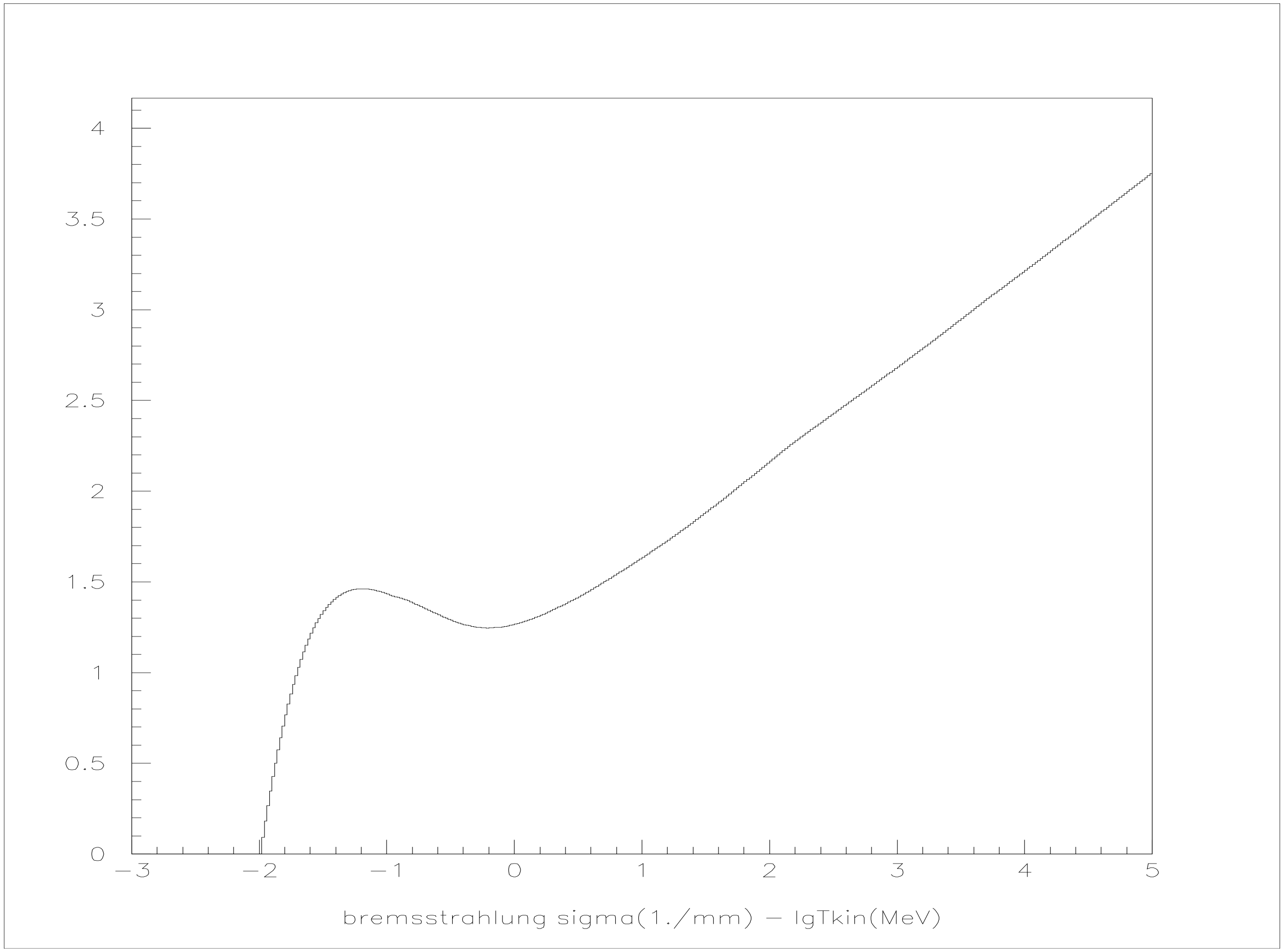
In the high energies regime, one can use the complete screened expression 1 of $d\sigma/dk$ to integrate 5. This gives:

$$\text{for } E \gg k_{cut} : \quad \sigma_{br}(E, k \geq k_{cut}) \approx \frac{4}{3} \frac{1}{n_{at} \, X_0} \ln\left[\frac{E}{k_{cut}}\right] \tag{6}$$

The bremsstrahlung total cross section increases as $\ln E$.

(in fact, taking into account the LPM effect leads to $\sigma_{br} \propto 1/\sqrt{E}$ when $E > E_{lpm}$)

number of interactions per mm in Lead (cut 10 keV)



electron kinetic energy (ln(MeV))

final state

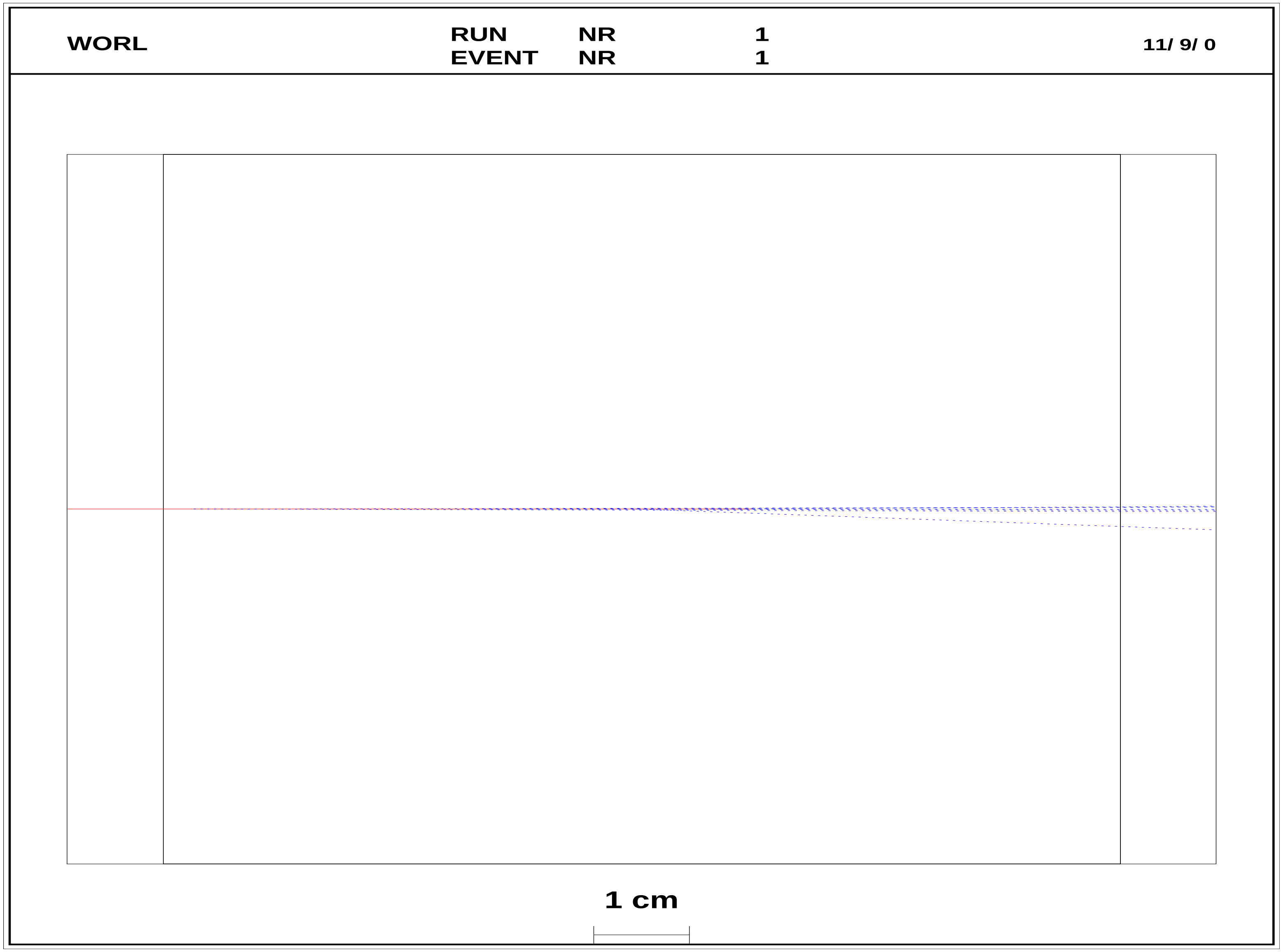
The polar angle θ of γ is defined w. r. to the direction of the parent electron. The energy-angle is sampled from a density function suggested by Urban [geant3], as an approximation of the Tsai distribution [Tsai74] :

$$\forall u \in [0,\ \infty[\ : \quad f(u) = \frac{9a^2}{9+d}\left[ue^{-au} + d\ ue^{-3au}\right]$$

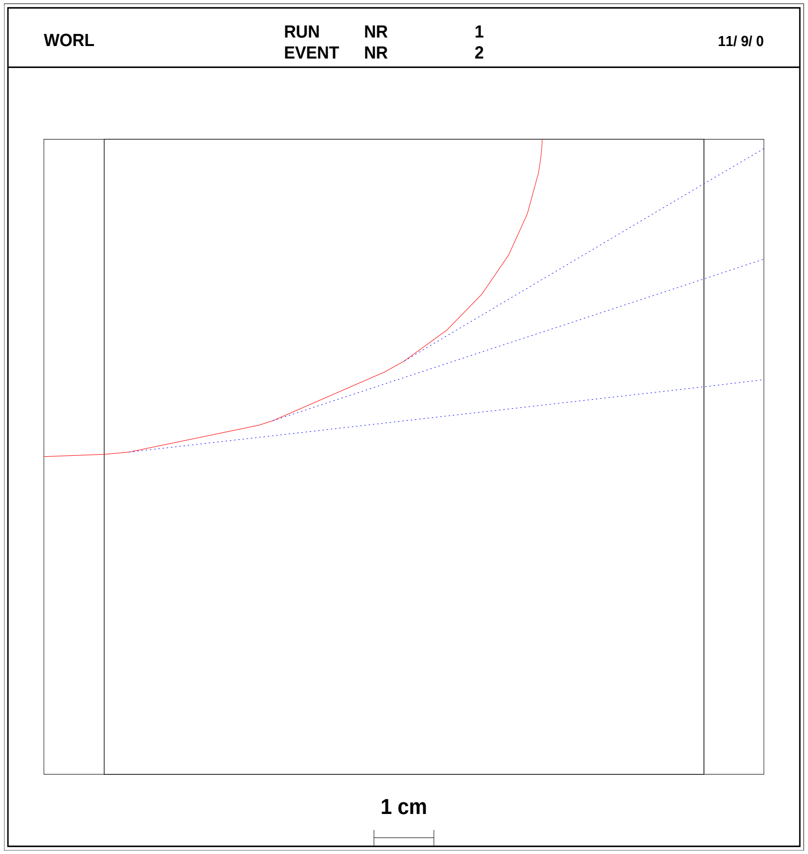
with : $\theta = (mc^2/E)\ u$

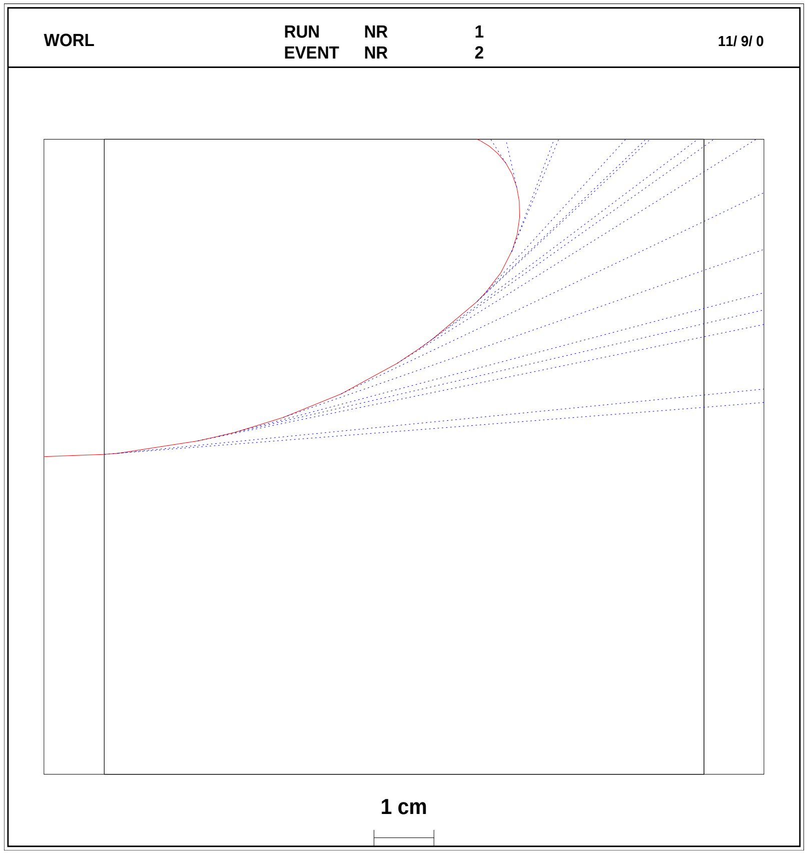
The azimuthal angle ϕ is generated isotropically.

e^- 200 MeV in 10 cm Aluminium



e^- 200 MeV in 10 cm Aluminium (cut: 1 MeV, 10 keV). Field 5 tesla





Bremsstrahlung by high energy muon

The bremsstrahlung cross section is proportional to $1/m^2$. Thus the critical energy for muons scales as $(m_\mu/m_e)^2$, i.e. near the TeV.

Bremsstrahlung dominates other muon interaction processes in the region of catastrophic collisions ($y \geq 0.1$), at "moderate" muon energies - above knock–on electron production kinematic limit.

At high energies ($E \geq 1$ TeV) this process contributes about 40% to average muon energy loss.

See [Keln97] for up-to-date review and formulae.

formation length ([Antho96])

In the bremsstrahlung process the longitudinal momentum transfer from the nucleus to the electron can be very small. For $E \gg mc^2$ and $E \gg k$:

$$q_{long} \sim \frac{k(mc^2)^2}{2E(E-k)} \sim \frac{k}{2\gamma^2}$$

Thus, the uncertainty principle requires that the emission take place over a comparatively long distance :

$$f_v \sim \frac{2\hbar c\gamma^2}{k} \tag{7}$$

f_v is called the formation length for bremsstrahlung in vacuum.
It is the distance of coherence, or the distance required for the electron and photon to separate enough to be considered as separate particles. If anything happens to the electron or photon while traversing this distance, the emission can be disrupted.

dielectric suppression mechanism

The suppression is due to the photon interaction with the electron gas (Compton scattering) within the formation length.
The magnitude of the process can be evaluated using classical electromagnetism [Ter72].

A dielectric medium is characterized by his dielectric function:

$$\epsilon(k) = 1 - (\hbar\omega_p/k)^2 \qquad \text{with } \hbar\omega_p = \sqrt{4\pi n_{el} r_e^3}\, mc^2/\alpha$$

(n_{el} is the electron density of the medium $= Z n_{at}$.)

It can be shown that the formation length in this medium is reduced:

$$f_m \sim \frac{2\hbar c\gamma^2\, k}{k^2 + (\gamma\hbar\omega_p)^2} \tag{8}$$

The effect is factorized in the bremsstrahlung differential cross section:

$$\frac{d\sigma}{dk} = S_d(k) \left[\frac{d\sigma}{dk}\right]_{Tsai} \qquad (9)$$

S_d is called the suppression function :

$$S_d(k) \stackrel{def}{=} \frac{f_m(k)}{f_v(k)} = \frac{k^2}{k^2 + (\gamma\hbar\omega_p)^2} \qquad (10)$$

$S_d(k)$ is vanishing for $k^2 \ll (\gamma\hbar\omega_p)^2$ i.e. :

$$\frac{k}{E} \ll \frac{\hbar\omega_p}{mc^2} \quad (\sim 10^{-4},\ 10^{-5} \text{ in all materials}) \qquad (11)$$

Then,

$$S_d(k) \approx \frac{k^2}{(\gamma\hbar\omega_p)^2} \qquad (12)$$

which cancel the infrared divergence of the Bethe-Heitler formula.

Landau-Pomeranchuk-Migdal suppression mechanism

The electron can multiple scatter with the atoms of the medium while it is still in the formation zone. If the angle of multiple scattering, θ_{ms}, is greater than the typical emission angle of the emitted photon, $\theta_{br} = mc^2/E$, the emission is suppressed.

In the gaussian approximation : $\theta_{ms}^2 = \frac{2\pi}{\alpha}\frac{1}{\gamma^2}\frac{f_v(k)}{X_0}$ where f_v is the formation length in vacuum, defined in equation 7.

Writing $\theta_{ms}^2 > \theta_{br}^2$ show that suppression becomes signifiant for photon energies below a certain value, given by

$$\frac{k}{E} < \frac{E}{E_{lpm}} \tag{13}$$

E_{lpm} is a characteristic energy of the effect :

$$E_{lpm} = \frac{\alpha}{4\pi}\frac{(mc^2)^2}{\hbar c}\,X_0 \;=\frac{\alpha^2}{4\pi}\frac{mc^2}{r_e}\,X_0 \;\sim (7.7\,TeV/cm)\times X_0\;(cm) \tag{14}$$

The multiple scattering increases q_{long} by changing the direction and the momentum of the electron. One can show :

$$q_{long} = \left(\frac{k}{2\gamma^2}\right)\left[1 + \frac{(mc^2)^2}{2\hbar c\, E_{lpm}}\, f_m(k)\right]$$

where f_m is the formation length in the material.

On the other hand the uncertainty principle says : $q_{long} = \hbar c / f_m$

These two equations can be solved in f_m:

$$f_m(k) \approx \frac{2\hbar c \gamma^2}{k}\sqrt{\frac{k\, E_{lpm}}{E^2}} \qquad \text{if } k\, E_{lpm} \ll E^2 \tag{15}$$

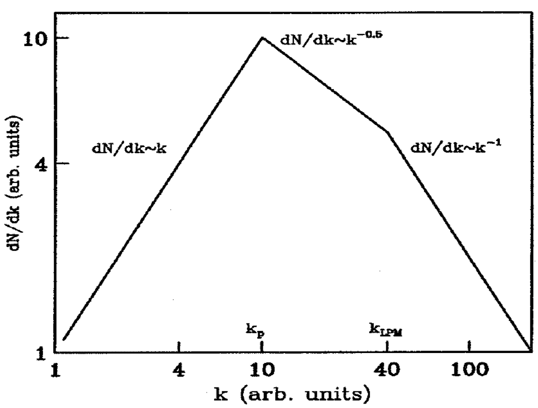
Hence, the suppression function S_{lpm} :

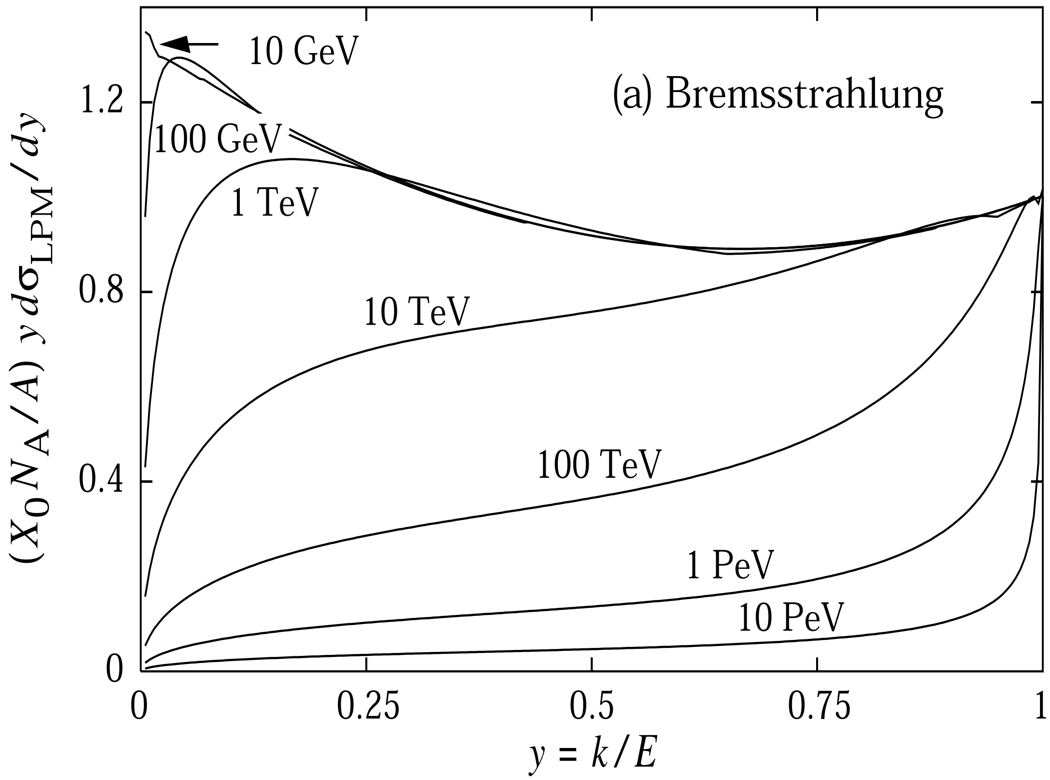
$$S_{lpm}(k) \stackrel{def}{=} \frac{f_m(k)}{f_v(k)} = \sqrt{\frac{k\, E_{lpm}}{E^2}} \equiv \sqrt{\frac{(k/E)}{(E/E_{lpm})}} \tag{16}$$

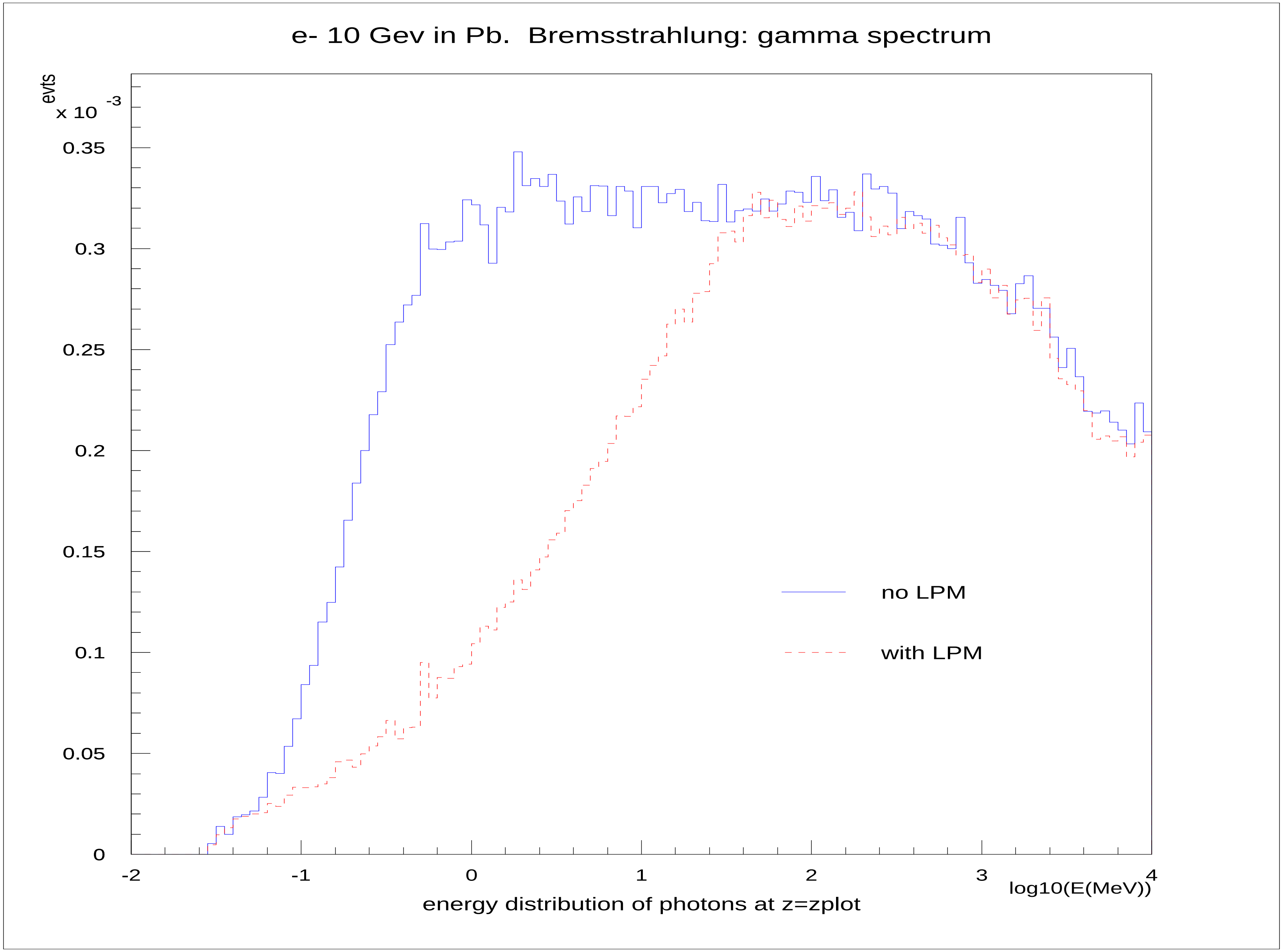
total suppression

The LPM and dielectric mechanism both reduce the effective formation length. The suppressions do not simply multiply. The total suppression follows [Gal64]

$$\frac{1}{S} = 1 + \frac{1}{S_d} + \frac{S}{S_{lpm}^2}$$

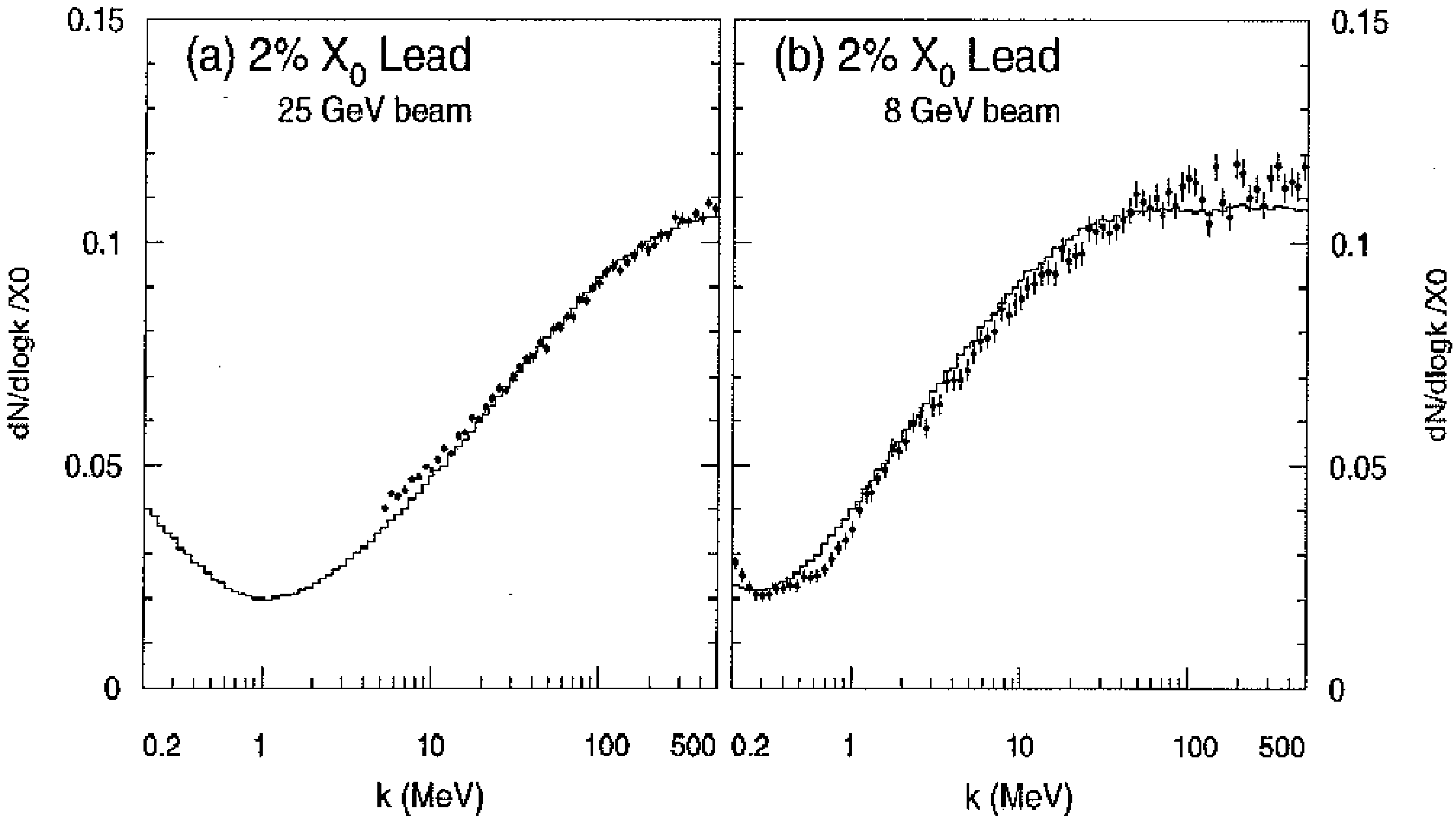

10
4
1
dN/dk (arb. units)
dN/dk~k
dN/dk~k^-0.5
dN/dk~k^-1
k_p
k_LPM
1 4 10 40 100
k (arb. units)


10 GeV
(a) Bremsstrahlung
100 GeV
1 TeV
10 TeV
100 TeV
1 PeV
10 PeV
1.2
0.8
0.4
0
0 0.25 0.5 0.75 1
$(X_0 N_A/A)\, y\, d\sigma_{\mathrm{LPM}}/dy$
$y = k/E$



e- 10 Gev in Pb. Bremsstrahlung: gamma spectrum
evts
x 10 -3
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-2
-1
0
1
2
3
4
log10(E(MeV))
energy distribution of photons at z=zplot
no LPM
with LPM

E146 (SLAC) data [Antho96]



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