Compton scattering

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The Compton effect describes the scattering off quasi-free atomic electrons :

$$\gamma + e \rightarrow \gamma' + e'$$

Each atomic electron acts as an independent cible; Compton effect is called incoherent scattering. Thus:

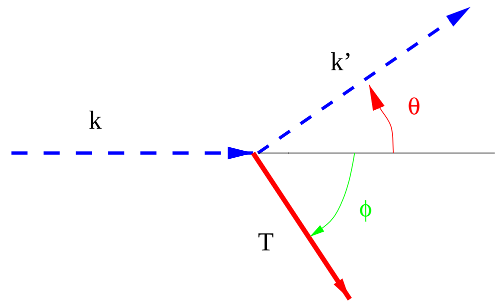
$$\text{cross section per atom} = Z \times \text{cross section per electron}$$

The inverse Compton scattering also exists: an energetic electron collides with a low energy photon which is blue-shifted to higher energy. This process is of importance in astrophysics.

Compton scattering is related to (e^+, e^-) annihilation by crossing symmetry.

kinematic

Assuming the initial electron free and at rest, the kinematic is given by energy-momentum conservation of two-boby scattering.



$$k' = \frac{k}{1+\kappa(1-\cos\theta)} \qquad where\ \kappa = \frac{k}{mc^2}$$

$$T = k - k'$$

$$\cot\phi = (1+\kappa)\tan(\theta/2)$$

limits $\theta = 0:\quad k'_{max} = k \quad T_{min} = 0 \quad \phi_{max} = \frac{\pi}{2}$

$\theta = \pi:\quad k'_{min} = k\frac{1}{2\kappa+1} \quad T_{max} = k\frac{2\kappa}{2\kappa+1} \quad \phi_{min} = 0$

energy spectrum

Under the same assumption,the unpolarized differential cross section per atom is given by the Klein-Nishina formula [Klein29] :

$$\frac{d\sigma}{dk'} = \frac{\pi\, r_e^2}{mc^2}\,\frac{Z}{\kappa^2}\,\left[\epsilon + \frac{1}{\epsilon} - \frac{2}{\kappa}\,\left(\frac{1-\epsilon}{\epsilon}\right) + \frac{1}{\kappa^2}\,\left(\frac{1-\epsilon}{\epsilon}\right)^2\right] \quad (1)$$

where

k' energy of the scattered photon ; $\epsilon = k'/k$

r_e classical electron radius

κ k/mc^2

total cross section per atom

$$\sigma(k) = \int_{k'_{min}=k/(2\kappa+1)}^{k'_{max}=k} \frac{d\sigma}{dk'}\, dk'$$

$$\sigma(k) = 2\pi\, r_e^2\, Z \left[\left(\frac{\kappa^2 - 2\kappa - 2}{2\kappa^3} \right) \ln(2\kappa+1) + \frac{\kappa^3 + 9\kappa^2 + 8\kappa + 2}{4\kappa^4 + 4\kappa^3 + \kappa^2} \right]$$

limits

$k \to \infty :$ $\quad \sigma$ goes to 0 : $\sigma(k) \sim \pi\, r_e^2\, Z\, \dfrac{\ln 2\kappa}{\kappa}$

$k \to 0 :$ $\quad \sigma \to \dfrac{8\pi}{3}\, r_e^2\, Z$ (classical Thomson cross section)

low energy limit

In fact, when $k \leq 100\ keV$ the binding energy of the atomic electron must be taken into account by a corrective factor to the Klein-Nishina cross section:

$$\frac{d\sigma}{dk'} = \left[\frac{d\sigma}{dk'}\right]_{KN} \times S(k,k')$$

See for instance [Cullen97] or [Salvat96] for derivation(s) and discussion of the *scattering function* S(k,k').

As a consequence, at very low energy, the total cross section goes to 0 like k^2. It also suppresses the forward scattering.

At X-rays energies the scattering function has little effect on the Klein-Nishina energy spectrum formula 1. In addition the Compton scattering is not the dominant process in this energy region.

total cross section per atom in Geant4

The total cross section has been parametrized [Geant3] :

$$\sigma(Z,\kappa) = \left[P_1(Z)\,\frac{\log(1+2\kappa)}{\kappa} + \frac{P_2(Z)+P_3(Z)\kappa+P_4(Z)\kappa^2}{1+a\kappa+b\kappa^2+c\kappa^3}\right]$$

where:

$$\begin{aligned} \kappa &= k/mc^2 \\ P_i(Z) &= Z(d_i + e_i Z + f_i Z^2) \end{aligned}$$

The fit was made over 511 data points chosen between:

$$1 \le Z \le 100 \qquad ; \qquad k \in [10\ \text{keV}, 100\ \text{GeV}]$$

The accuracy of the fit is estimated to be:

$$\frac{\Delta\sigma}{\sigma} = \begin{cases} \approx 10\% & \text{for } k \simeq 10\ \text{keV} - 20\ \text{keV} \\ \le 5-6\% & \text{for } k > 20\ \text{keV} \end{cases}$$

Mean free path

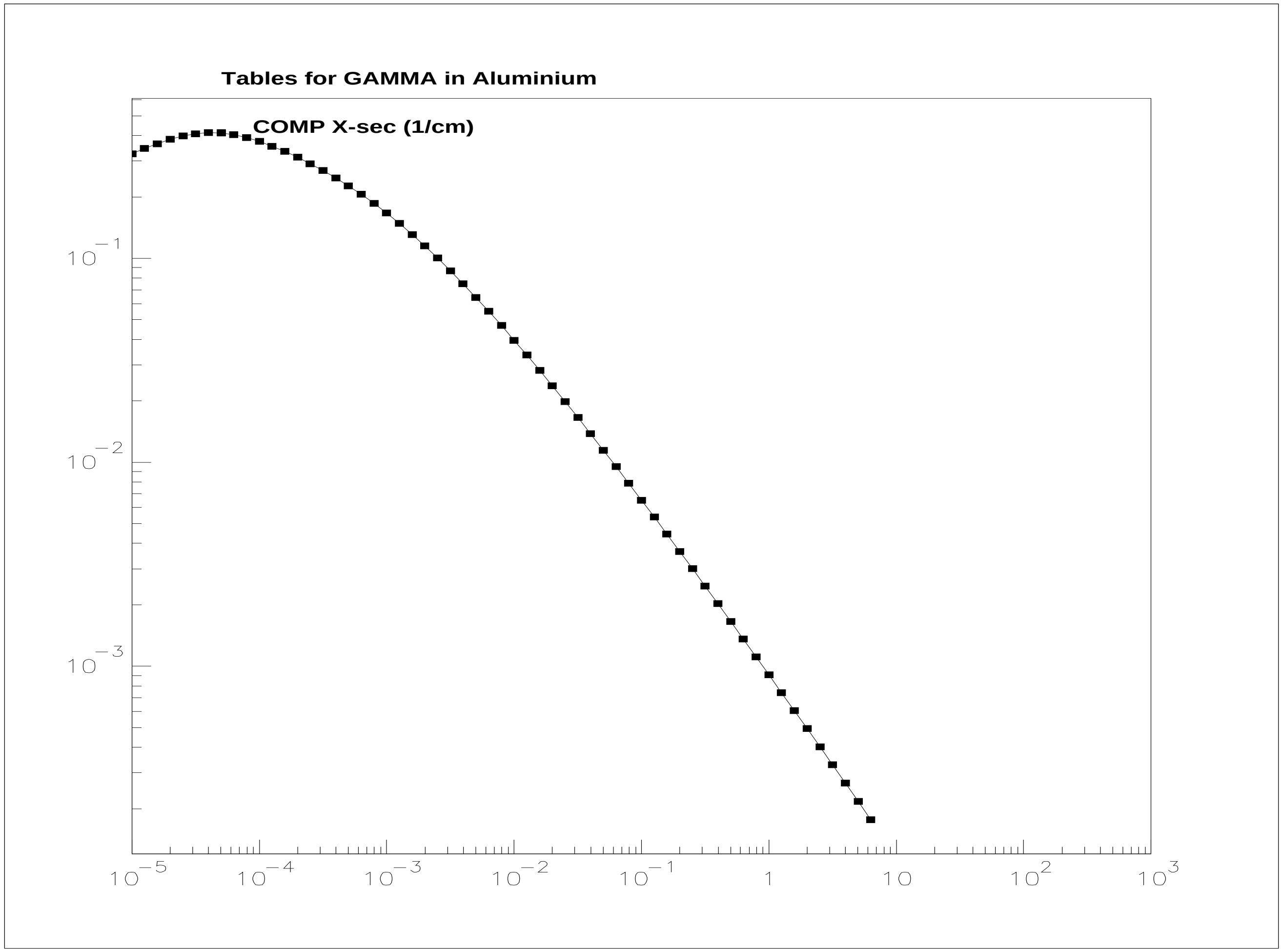
$$\lambda(k) = \left(\sum_i n_{ati} \cdot \sigma(Z_i, k) \right)^{-1}$$

n_{ati} : nb of atoms per volume of the i^{th} element in the material.

At initialization stage, the function `BuildPhysicsTables()` computes and tabulates :

- crossSectionPerAtom for all elements
- meanFreePath for all materials

number of interactions per cm in Aluminium



photon energy (GeV)

sample the gamma energy

The Klein-Nishina differential cross-section per atom is :

$$\frac{d\sigma}{d\epsilon} = \pi r_e^2 \, \frac{m_e c^2}{k} \, Z \left[\frac{1}{\epsilon} + \epsilon\right] \left[1 - \frac{\epsilon \sin^2\theta}{1+\epsilon^2}\right]$$

$$\epsilon = \frac{k'}{k} = \frac{1}{1+\kappa(1-\cos\theta)} \Longrightarrow \epsilon \in \left[\epsilon_0 = \frac{1}{1+2\kappa}, \; 1\right]$$

we can factorize :

$$\begin{aligned} \frac{d\sigma}{d\epsilon} & \simeq \left[\frac{1}{\epsilon} + \epsilon\right] \left[1 - \frac{\epsilon \sin^2\theta}{1+\epsilon^2}\right] = f(\epsilon) \cdot g(\epsilon) \\ & = [N_1 f_1(\epsilon) + N_2 f_2(\epsilon)] \cdot g(\epsilon) \end{aligned}$$

where:

$$N_1 = \ln(1/\epsilon_0) \quad ; \quad f_1(\epsilon) = 1/(N_1 \epsilon)$$

$$N_2 = (1-\epsilon_0^2)/2 \quad ; \quad f_2(\epsilon) = \epsilon/N_2$$

$f_1(\epsilon)$ and $f_2(\epsilon)$ are probability density functions on $\epsilon \in [\epsilon_0,\ 1]$

$$g(\epsilon) = \left[1 - \frac{\epsilon}{1+\epsilon^2}\sin^2\theta\right] \text{ is a rejection function : } 0 < g(\epsilon) \leq 1$$

Given a triplet of random numbers $\{r_a, r_b, r_c\}$ uniform in [0,1] :

1. choose $f_1(\epsilon)g_1(\epsilon)$ or $f_2(\epsilon)g_2(\epsilon)$ with r_a :
   if $r_a < N_1/(N_1 + N_2)$ select $f_1(\epsilon)$, else $f_2(\epsilon)$;
2. sample ϵ from the distributions f_1 or f_2 with r_b.
   for $f_1 : \epsilon = \exp(-N_1 r_b)$
   for $f_2 : \epsilon^2 = \epsilon_0^2 + (1 - \epsilon_0^2) r_b$
3. calculate $\sin^2\theta = t(2-t)$ where $t = (1-\epsilon)/(\kappa\epsilon)$
4. accept/reject ϵ with r_c : if $r_c \leq g(\epsilon)$ accept ϵ, else go to 1

compute the final kinematic

The function `PostStepDoIt()` samples ϵ

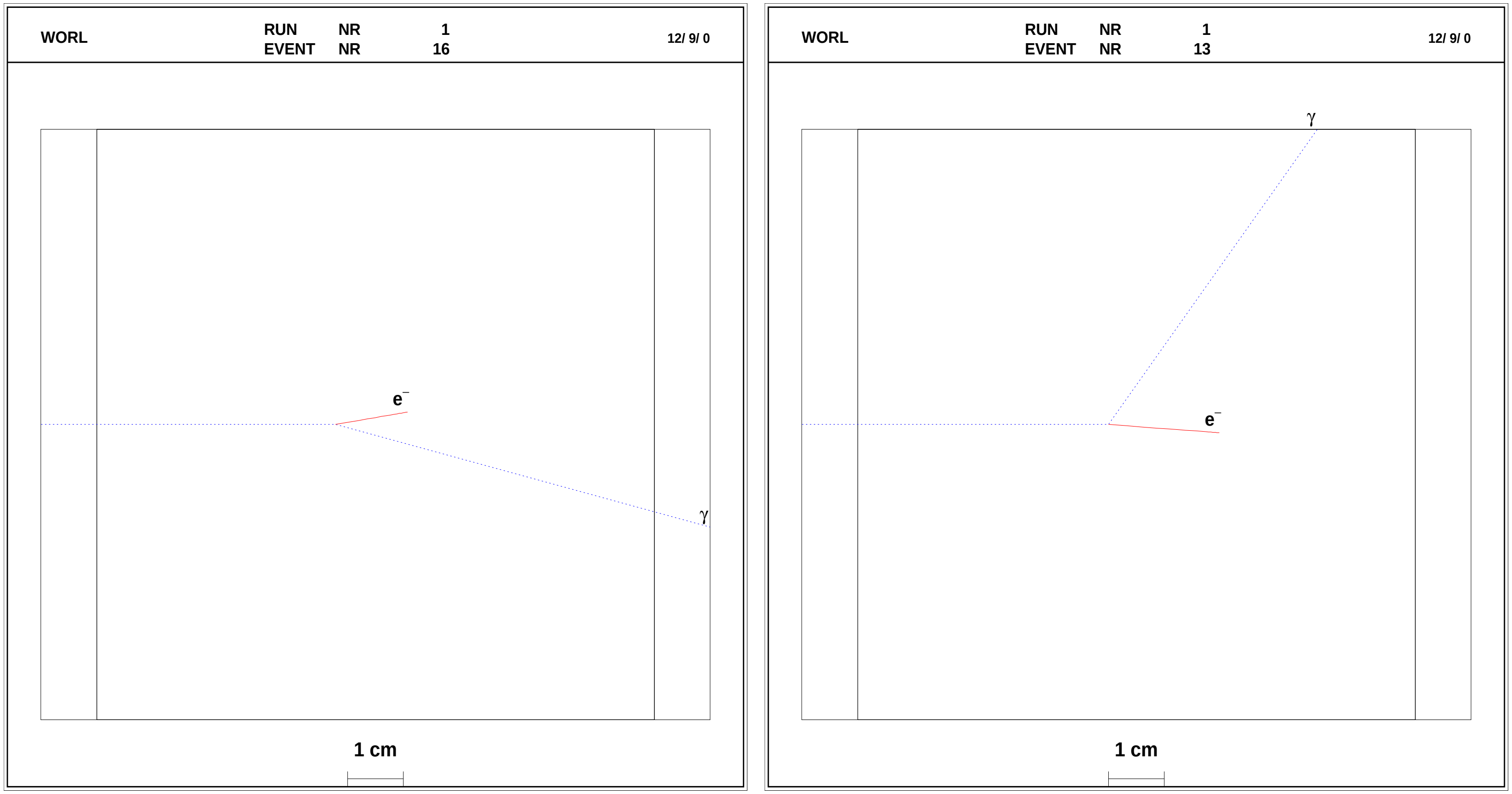
After the successful sampling of ϵ, the polar angles of the scattered photon with respect to the direction of the parent photon are generated :
θ is computed from ϵ, ϕ is generated isotropically.

The momentum vector of the scattered photon, $\overrightarrow{k'}$, is transformed into the *global* coordinate system.

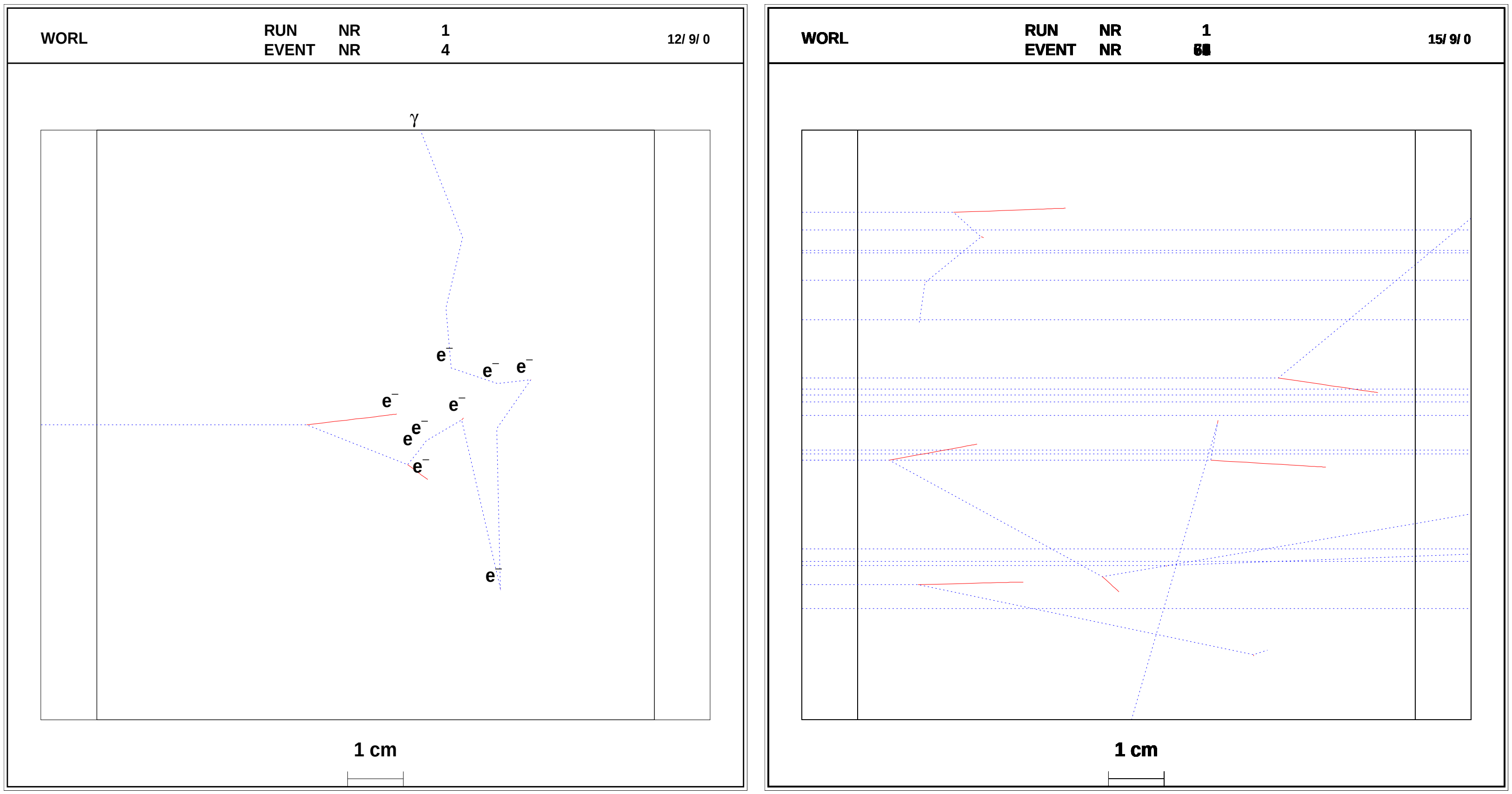
The kinetic energy and momentum of the recoil electron are :

$$\begin{aligned} T_{el} &= k - k' \\ \overrightarrow{P_{el}} &= \overrightarrow{k} - \overrightarrow{k'} \end{aligned}$$

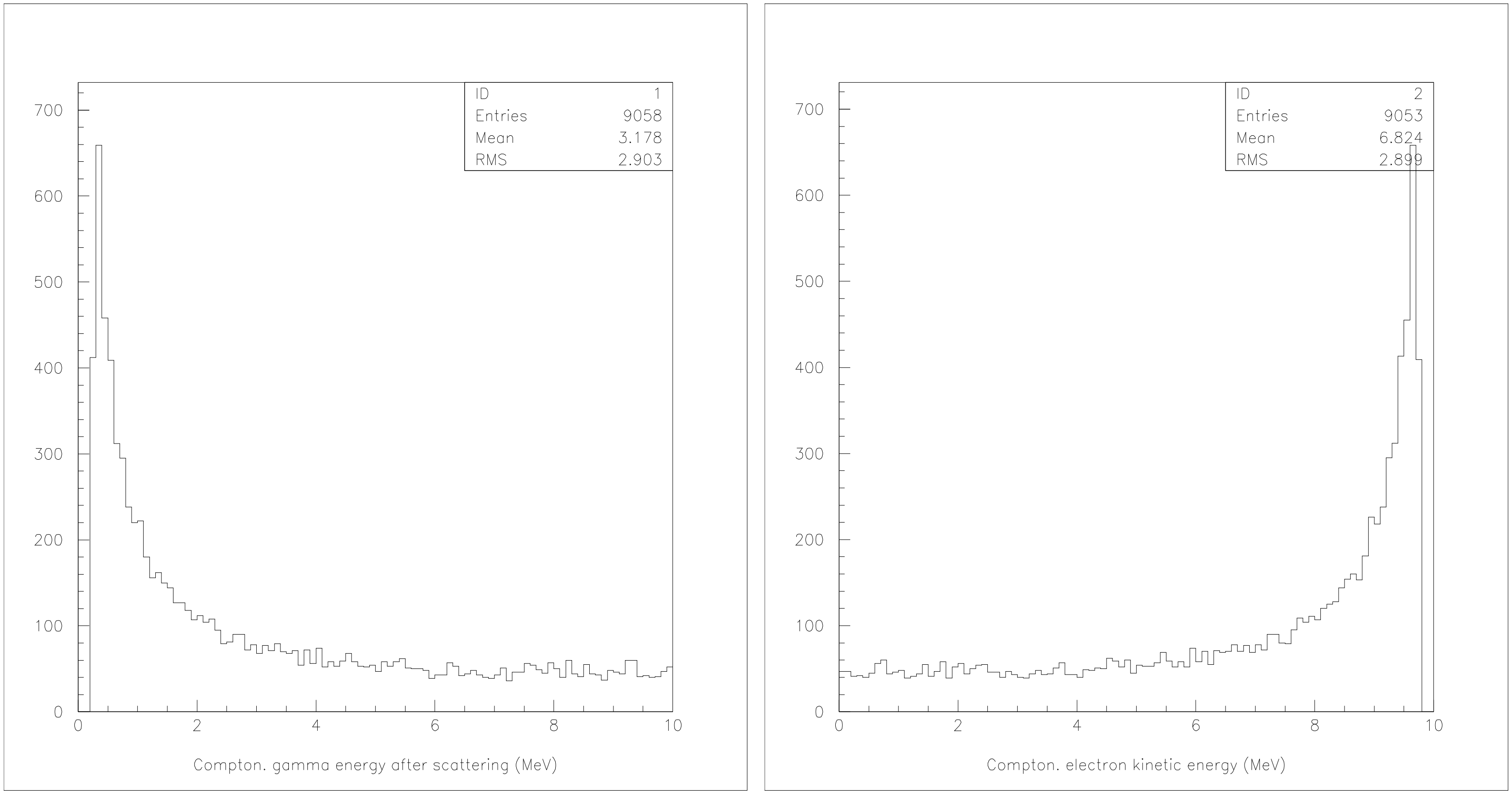
γ 10 MeV in 10 cm Aluminium: Compton scattering



γ 10 MeV in 10 cm Aluminium: Compton scattering



Compton scattering : γ 10 MeV in Aluminium. *Compton edge*:
energy spectrum of scattered photon (left) and emitted e^- (right)



transfer, scattering, attenuation

Only a fraction of the energy of the incident photon is transferred to the recoil electron, which is generally stopped into the material. The mean kinetic energy of the electron is :

$$\langle T(k) \rangle = \frac{1}{\sigma(k)} \int_{k'_{min}}^{k} T \, \frac{d\sigma}{dk'} \, dk'$$

Then the energy transfer coefficient of Compton scattering is defined as

$$\mu_{tr} = n_{at} \, \sigma(k) \, \frac{\langle T(k) \rangle}{k} \qquad (n_{at} \text{ is the nb of atoms per volume})$$

$\frac{\mu_{tr}}{\rho}$ is the mass energy transfer coefficient.

Similar, from the mean energy of the scattered photon one defines the energy scattered coefficient of Compton scattering

$$\langle k' \rangle = \frac{1}{\sigma(k)} \int_{k'_{min}}^{k} k' \, \frac{d\sigma}{dk'} \, dk' \qquad \longrightarrow \mu_{sca} = n_{at} \, \sigma(k) \, \frac{\langle k' \rangle}{k}$$

The attenuation coefficient of Compton scattering is

$$\mu_{att} \stackrel{def}{=} n_{at}\sigma(k) = \mu_{tr} + \mu_{sca}$$

and similar relations for the mass coefficients.

Rayleigh scattering

Rayleigh scattering is the scattering of photons by an atom as a whole : all the electrons of the atom participate in a coherent manner.

It is an elastic collision: no energy transfer from photon to atom (no ionisation nor excitation).

At x-rays and γ-rays energy region, Rayleigh scattering is small compared to the photo electric effect, and can be generally neglected.

References

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J.H.Hubbell et al. Rad. Phys. Chem. vol50, 1 (1997)

[Salvat96] F.Salvat et al. Penelope, Informes Técnicos Ciemat 799, Madrid (1996)

[Geant3] Geant3 writeup, *Cern Program Library* (1993)