(e^+, e^-) pair creation by photon

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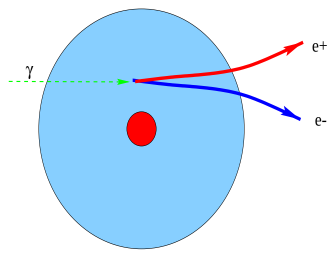
This is the transformation of a photon into an (e^+, e^-) pair in the Coulomb field of atoms (for momentum conservation).

To create the pair, the photon must have at least an energy of $2mc^2(1 + m/M_{rec})$.

Theoretically, (e^+, e^-) pair production is related to bremsstrahlung by crossing symmetry:

- incoming $e^- \leftrightarrow$ outgoing e^+
- outgoing $\gamma \leftrightarrow$ incoming γ

For $E_\gamma \geq$ few tens MeV, (e^+, e^-) pair creation is the dominant process for the photon, in all materials.



differential cross section

The differential cross section is given by the Bethe-Heitler formula [Heitl57], corrected and extended for various effects:

- the screening of the field of the nucleus
- the pair creation in the field of atomic electrons
- the correction to the Born approximation
- the LPM suppression mechanism
- ...

See Seltzer and Berger for a synthesis of the theories [Sel85].

corrected Bethe-Heitler cross section

Let E_γ the energy of the photon, E the total energy carried by one particle of the pair (e^+, e^-), and $\epsilon = E/E_\gamma$.

The kinematical limits of ϵ are :

$$\frac{m_e c^2}{E_\gamma} = \epsilon_0 \le \epsilon \le 1 - \epsilon_0$$

The corrected Bethe-Heitler formula is written as in [Egs4] :

$$\begin{aligned} \frac{d\sigma(Z,\epsilon)}{d\epsilon} &= \alpha r_e^2 Z[Z+\xi(Z)]\left\{[\epsilon^2+(1-\epsilon)^2]\left[\Phi_1(\delta(\epsilon)) - \frac{F(Z)}{2}\right]\right. \\ &\quad \left. + \frac{2}{3}\epsilon(1-\epsilon)\left[\Phi_2(\delta(\epsilon)) - \frac{F(Z)}{2}\right]\right\} \end{aligned} \tag{1}$$

where α is the fine-structure constant and r_e the classical electron radius.

screening effect

Depending of the energy of the projectile, the Coulomb field of the nucleus can be more on less screened by the electron cloud.

A screening parameter measures the ratio of an 'impact parameter' of the projectile to the radius of an atom, for instance given by a Thomas-Fermi approximation or a Hartree-Fock calculation.

Then, screening functions are introduced in the Bethe-Heitler formula.

Qualitatively:

- at low energy → no screening effect
- at ultra relativistic electron energy → full screening effect

screening effect. The *screening variable*, which is in a function of ϵ, measure the 'impact parameter' of the projectile :

$$\delta(\epsilon) = \frac{136}{Z^{1/3}} \frac{\epsilon_0}{\epsilon(1-\epsilon)}$$

Two screening functions are introduced in the Bethe-Heitler formula :

$$\begin{aligned}
\text{for } \delta \leq 1 \quad \Phi_1(\delta) &= \quad 20.867 - 3.242\delta + 0.625\delta^2 \\
\Phi_2(\delta) &= \quad 20.209 - 1.930\delta - 0.086\delta^2 \\
\text{for } \delta > 1 \quad \Phi_1(\delta) &= \quad \Phi_2(\delta) = 21.12 - 4.184\ln(\delta + 0.952)
\end{aligned}$$

The formula 1 is symetric to the exchange $\epsilon \leftrightarrow (1-\epsilon)$ we can restrict the range of ϵ :

$$\epsilon \in [\epsilon_0, 1/2]$$

Born approximation

The Bethe-Heitler formula is calculated with plane waves, while it would be with Coulomb waves, for $\beta \leq \alpha Z$.

To correct for this, a Coulomb correction function is introduced in the Bethe-Heitler formula :

$$\text{for } E_\gamma < 50 \text{ MeV} : \quad F(z) = \quad 8/3 \ln Z$$
$$\text{for } E_\gamma \geq 50 \text{ MeV} : \quad F(z) = \quad 8/3 \ln Z + 8 f_c(Z)$$

with :

$$f_c(Z) = (\alpha Z)^2 \left[\frac{1}{1 + (\alpha Z)^2} + 0.20206 - 0.0369(\alpha Z)^2 + 0.0083(\alpha Z)^4 - 0.0020(\alpha Z)^6 + \cdots \right]$$

triplet creation in the electron field

The projectile feels not only the Coulomb field of the nucleus (charge Ze), but also the fields of the atomic electrons (Z electrons of charge e).

As for bremsstrahlung, the amplitude is roughly the same in both cases, except the charge.

Thus the electron cloud gives an additional contribution to the pair creation, proportional to Z (instead of Z^2).

This is taken into account through the expression :

$$\xi(Z) = \frac{\ln(1440/Z^{2/3})}{\ln(183/Z^{1/3}) - f_c(Z)}$$

The recoil electron may be ejected from the atom, thus the final state can be a triplet (e^+, e^-, e^-). The kinetic energy of this e^- is small. It is not generated in Geant4.

ϵ range

After these additions the cross section becomes negative if :

$$\delta > \delta_{max}(\epsilon_1) = \exp\left[\frac{42.24 - F(Z)}{8.368}\right] - 0.952$$

This gives an additional constraint on ϵ :

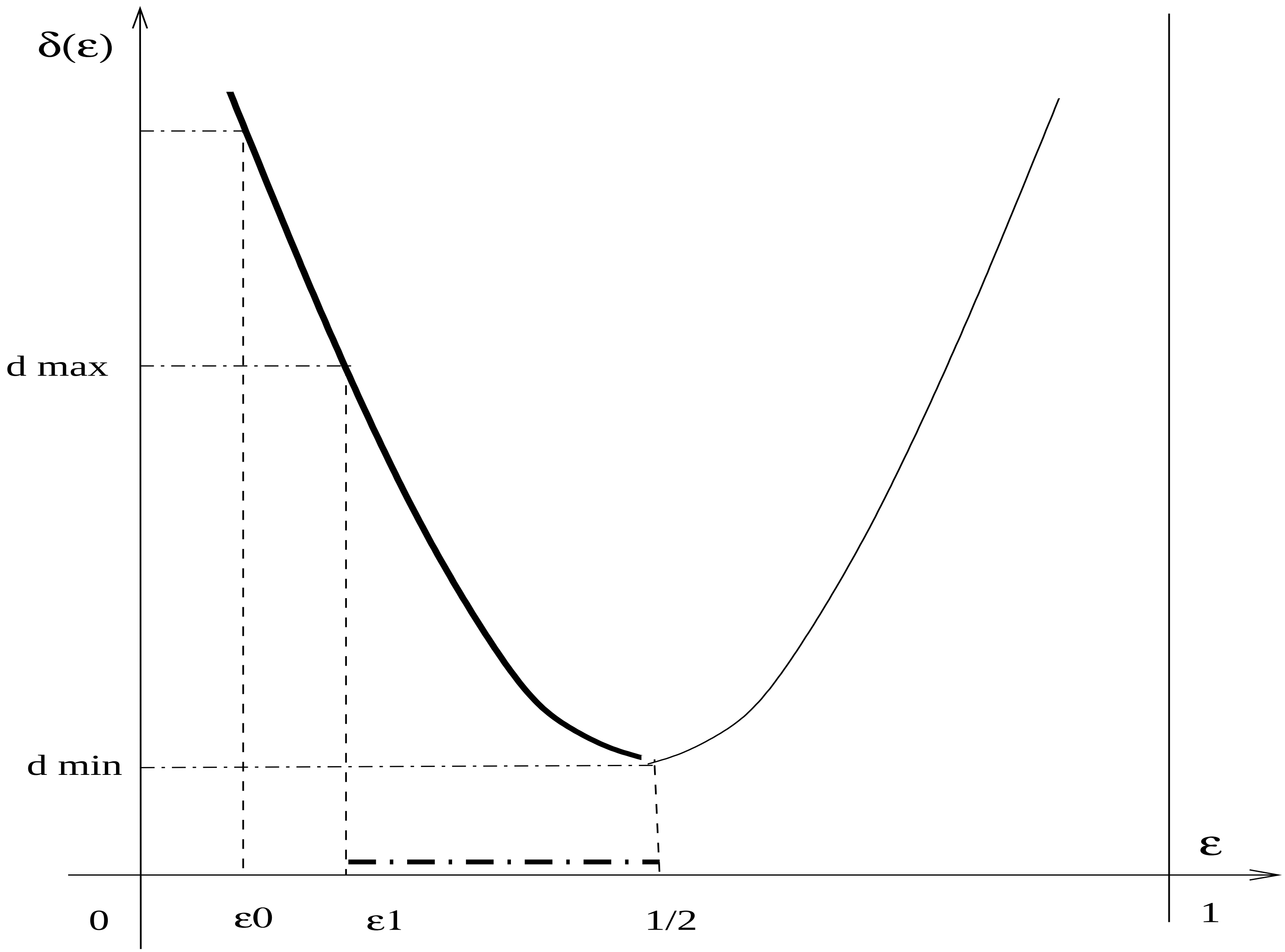
$$\delta \leq \delta_{max} \Longrightarrow \epsilon \geq \epsilon_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{\delta_{min}}{\delta_{max}}}$$

where we have introduced :

$$\delta_{min} = \delta\left(\epsilon = \frac{1}{2}\right) = \frac{136}{Z^{1/3}}\ 4\epsilon_0$$

Finally the range of ϵ is :

$$\epsilon \in [\epsilon_{min} = \max(\epsilon_0, \epsilon_1),\ 1/2]$$


δ(ε)
d max
d min
ε
0
ε0
ε1
1/2
1

factorization of the differential cross section

Let introduce two auxiliary screening functions :

$$
\begin{aligned}
F_1(\delta) &= 3\Phi_1(\delta) - \Phi_2(\delta) - F(Z) \\
F_2(\delta) &= \frac{3}{2}\Phi_1(\delta) - \frac{1}{2}\Phi_2(\delta) - F(Z)
\end{aligned}
$$

$F_1(\delta)$ and $F_2(\delta)$ are decreasing functions of δ, $\forall \delta \in [\delta_{min}, \delta_{max}]$.
They reach their maximum for $\delta_{min} = \delta(\epsilon = 1/2)$:

$$
F_{10} = \max F_1(\delta) = F_1(\delta_{min}) \qquad F_{20} = \max F_2(\delta) = F_2(\delta_{min})
$$

After some algebraic manipulations the formula 1 can be written :

$$
\begin{aligned}
\frac{d\sigma(Z,\epsilon)}{d\epsilon} &= \alpha r_e^2 Z[Z + \xi(Z)] \frac{2}{9} \left[\frac{1}{2} - \epsilon_{min}\right] \\
&\quad \times [N_1 \; f_1(\epsilon) \; g_1(\epsilon) + N_2 \; f_2(\epsilon) \; g_2(\epsilon)]
\end{aligned} \tag{2}
$$

where :

$$N_1 = \left[\frac{1}{2} - \epsilon_{min}\right]^2 F_{10} \qquad f_1(\epsilon) = \frac{3}{\left[\frac{1}{2} - \epsilon_{min}\right]^3} \left[\frac{1}{2} - \epsilon\right]^2 \qquad g_1(\epsilon) = \frac{F_1(\epsilon)}{F_{10}}$$

$$N_2 = \frac{3}{2} F_{20} \qquad f_2(\epsilon) = \text{const} = \frac{1}{\left[\frac{1}{2} - \epsilon_{min}\right]} \qquad g_2(\epsilon) = \frac{F_2(\epsilon)}{F_{20}}$$

$f_1(\epsilon)$ and $f_2(\epsilon)$ are density probability functions on $\epsilon \in [\epsilon_{min}, 1/2]$:

$$\int_{\epsilon_{min}}^{1/2} f_i(\epsilon)\, d\epsilon = 1$$

$g_1(\epsilon)$ and $g_2(\epsilon)$ are valid rejection functions : $0 < g_i(\epsilon) \leq 1$

choose an Element

The differential cross section depends of Z_i. In a compound material one choose randomly an Element according :

$$Prob(Z_i, E_\gamma) = \frac{n_{ati}\sigma(Z_i, E_\gamma)}{\sum_i [n_{ati} \cdot \sigma_i(E_\gamma)]}$$

n_{ati} : nb of atoms per volume of the i^{th} element in the material.

sample the energy of e^{+-}

Given a triplet of uniform random numbers (r_a, r_b, r_c) :

1. choose the decomposition term in 2 with r_a :

$$\text{if } r_a < N_1/(N_1 + N_2) \rightarrow f_1(\epsilon)\ g_1(\epsilon) \text{ else } f_2(\epsilon)\ g_2(\epsilon)$$

2. sample ϵ from $f_1(\epsilon)$ or $f_2(\epsilon)$ with r_b :

$$\epsilon = \frac{1}{2} - \left(\frac{1}{2} - \epsilon_{min}\right) {r_b}^{1/3} \quad \text{or} \quad \epsilon = \epsilon_{min} + \left(\frac{1}{2} - \epsilon_{min}\right) r_b$$

3. reject ϵ if $g_1(\epsilon)$ or $g_2(\epsilon) < r_c$

NOTE : below $E_{gamma} = 2$ MeV it is enough to sample ϵ uniformly on $[\epsilon_0,\ 1/2]$, without rejection.

charge : the charge of each particle of the pair is fixed randomly.

final state

The polar angle θ of e^{+-} is defined w. r. to the direction of the parent photon. The energy-angle is sampled from a density function suggested by Urban [geant3], as an approximation of the Tsai distribution [Tsai74] :

$$\forall u \in [0,\ \infty[: \quad f(u) = \frac{9a^2}{9+d}\left[ue^{-au} + d\ ue^{-3au}\right]$$

with : $\theta_\pm = (mc^2/E_\pm)\ u$

The azimuthal angle ϕ is generated isotropically.

The e^+ and e^- momenta are assumed to be coplanar with the parent photon.
This information, together with energy conservation, is used to calculate the momentum vectors of (e^+, e^-) and to rotate them to the global reference system.

high energies regime : $E_\gamma \gg m_e c^2/(\alpha Z^{1/3})$

Above few GeV the energy spectrum formula becomes simple :

$$\left.\frac{d\sigma}{d\epsilon}\right]_{Tsai} \approx 4\alpha\, r_e^2 \times \left\{ \left[1 - \frac{4}{3}\epsilon(1-\epsilon)\right] \left(Z^2\left[L_{rad} - f(Z)\right] + Z L'_{rad}\right) \right\} \tag{3}$$

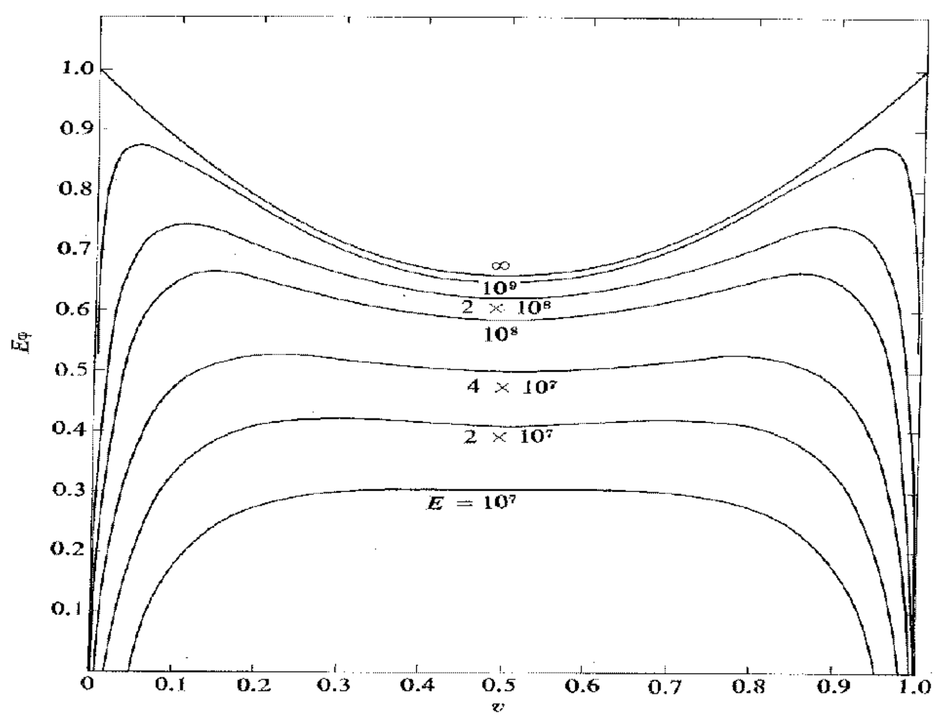
where

E_γ	energy of the incident photon
E	total energy of the created e^+ (or e^-) ; $\epsilon = E/E_\gamma$
$L_{rad}(Z)$	$\ln(184.15/Z^{1/3})$ (for $Z \geq 5$)
$L'_{rad}(Z)$	$\ln(1194/Z^{2/3})$ (for $Z \geq 5$)
$f(Z)$	Coulomb correction function

energy spectrum

limits: $E_{min} = mc^2$: no infrared divergence. $E_{max} = E_\gamma - mc^2$.

The partition of the photon energy between e^+ and e^- is flat at low energy ($E_\gamma \leq 50\ MeV$) and increasingly asymmetric with energy. For $E_\gamma > TeV$ the LPM effect reinforces the asymmetry.



In the high energies regime, one can use the complete screened expression 3 of $d\sigma/d\epsilon$ to compute the total cross section.

$$\sigma(E_\gamma) = \int_{\epsilon_0 \approx 0}^{\epsilon_{max} \approx 1} \frac{d\sigma}{d\epsilon}\, d\epsilon$$

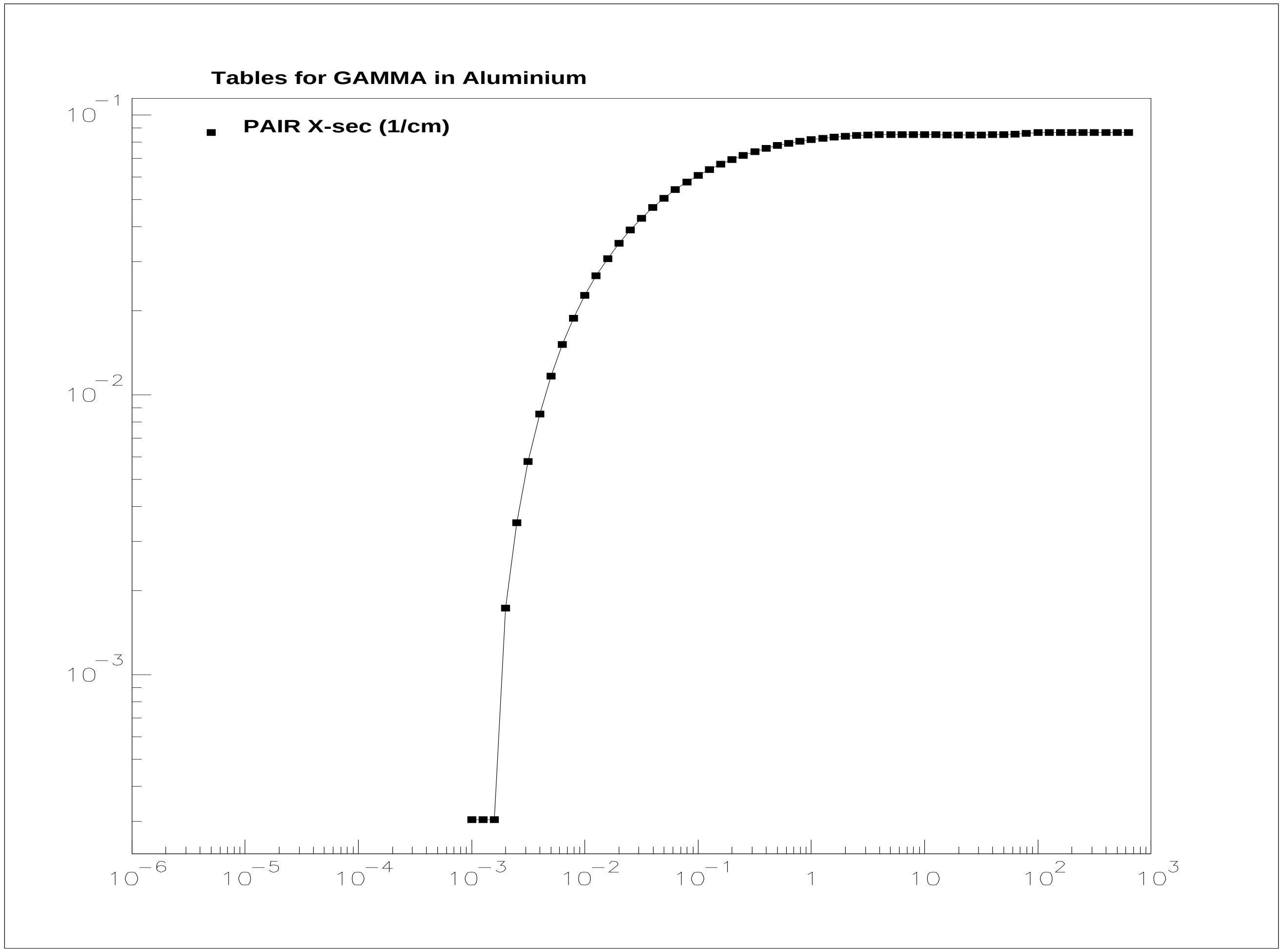
which gives:

$$\sigma_{pair}(E_\gamma) \approx \frac{7}{9}\, \frac{1}{n_{at}\; X_0}$$

n_{at} is the number of atoms per volume.

The total cross section is approximately constant above few GeV, for at least 4 decades (then, LPM effect).

number of interactions per cm in Aluminium



photon energy (GeV)

total cross section per atom in GEANT4

E_γ = incident gamma energy, and $X = \ln(E_\gamma / m_e c^2)$

The total cross-section has been parameterised as :

$$\sigma(Z, E_\gamma) = Z(Z+1) \left[F_1(X) + F_2(X)\, Z + \frac{F_3(X)}{Z} \right]$$

with :

$$\begin{aligned} F_1(X) &= a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 X^4 + a_5 X^5 \\ F_2(X) &= b_0 + b_1 X + b_2 X^2 + b_3 X^3 + b_4 X^4 + b_5 X^5 \\ F_3(X) &= c_0 + c_1 X + c_2 X^2 + c_3 X^3 + c_4 X^4 + c_5 X^5 \end{aligned}$$

The parameters a_i, b_i, c_i were fitted to the data [hubb80].

This parameterisation describes the data in the range :

$$\left. \begin{array}{c} 1 \leq Z \leq 100 \\ E_\gamma \in [1.5 \text{ MeV}, 100 \text{ GeV}] \end{array} \right\} \frac{\Delta \sigma}{\sigma} \leq 5\% \quad \text{with a mean value of} \approx 2.2\%$$

Mean free path

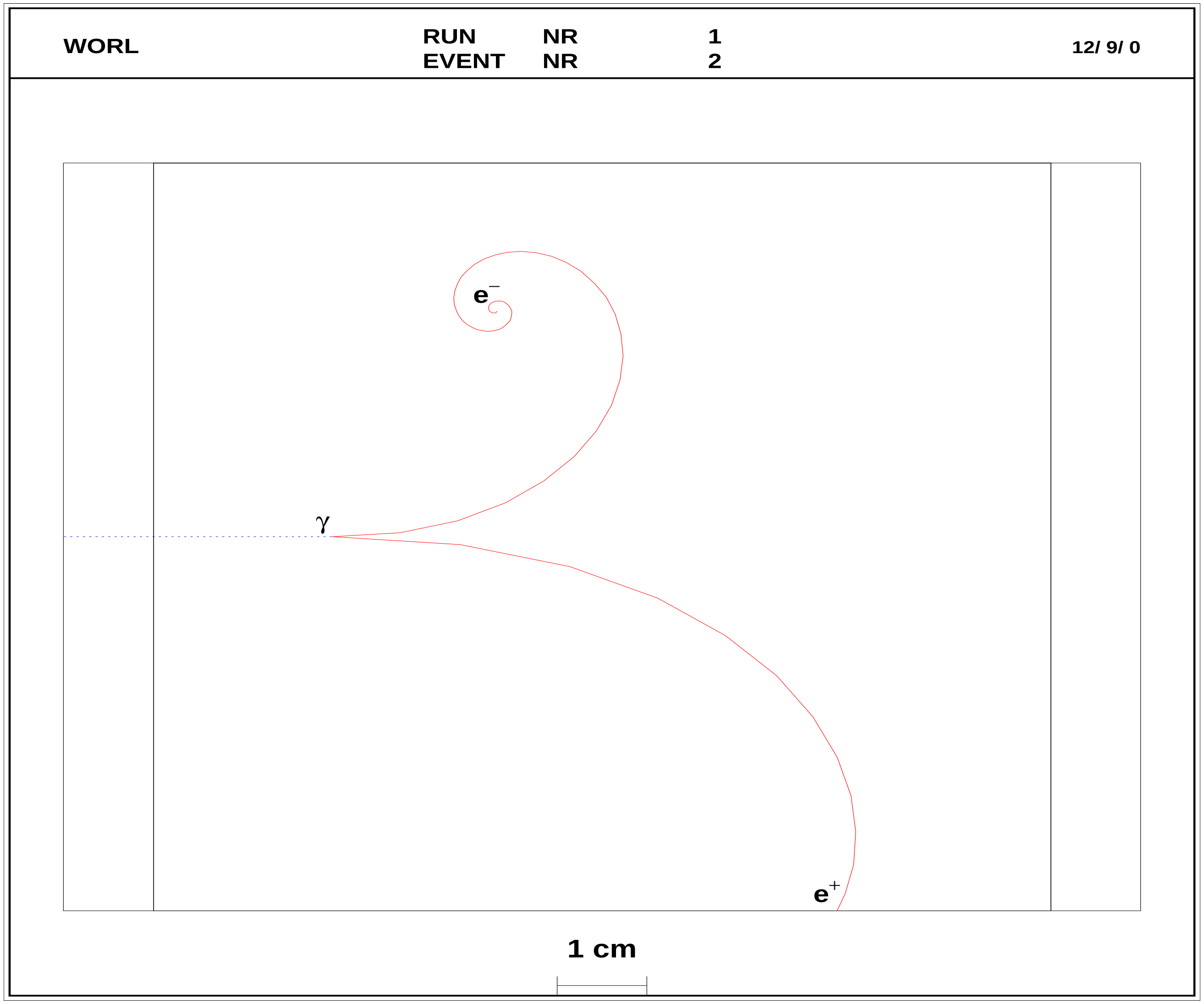
$$\lambda(E_\gamma) = \left(\sum_i n_{ati} \cdot \sigma(Z_i, E_\gamma) \right)^{-1}$$

n_{ati} : nb of atoms per volume of the i^{th} element in the material.

At initialization stage, the function `BuildPhysicsTables()` computes and tabulates :

- crossSectionPerAtom for all elements
- meanFreePath for all materials

γ 200 MeV in 10 cm Aluminium. Field 5 tesla



Landau-Pomeranchuk-Migdal suppression mechanism

Due to the LPM mechanism, the (e^+, e^-) pair creation is reduced for [PDG00] :

$$E(E_\gamma - E) > E_\gamma \; E_{lpm} \qquad \Longleftrightarrow \; \epsilon(1-\epsilon) > \frac{E_{lpm}}{E_\gamma}$$

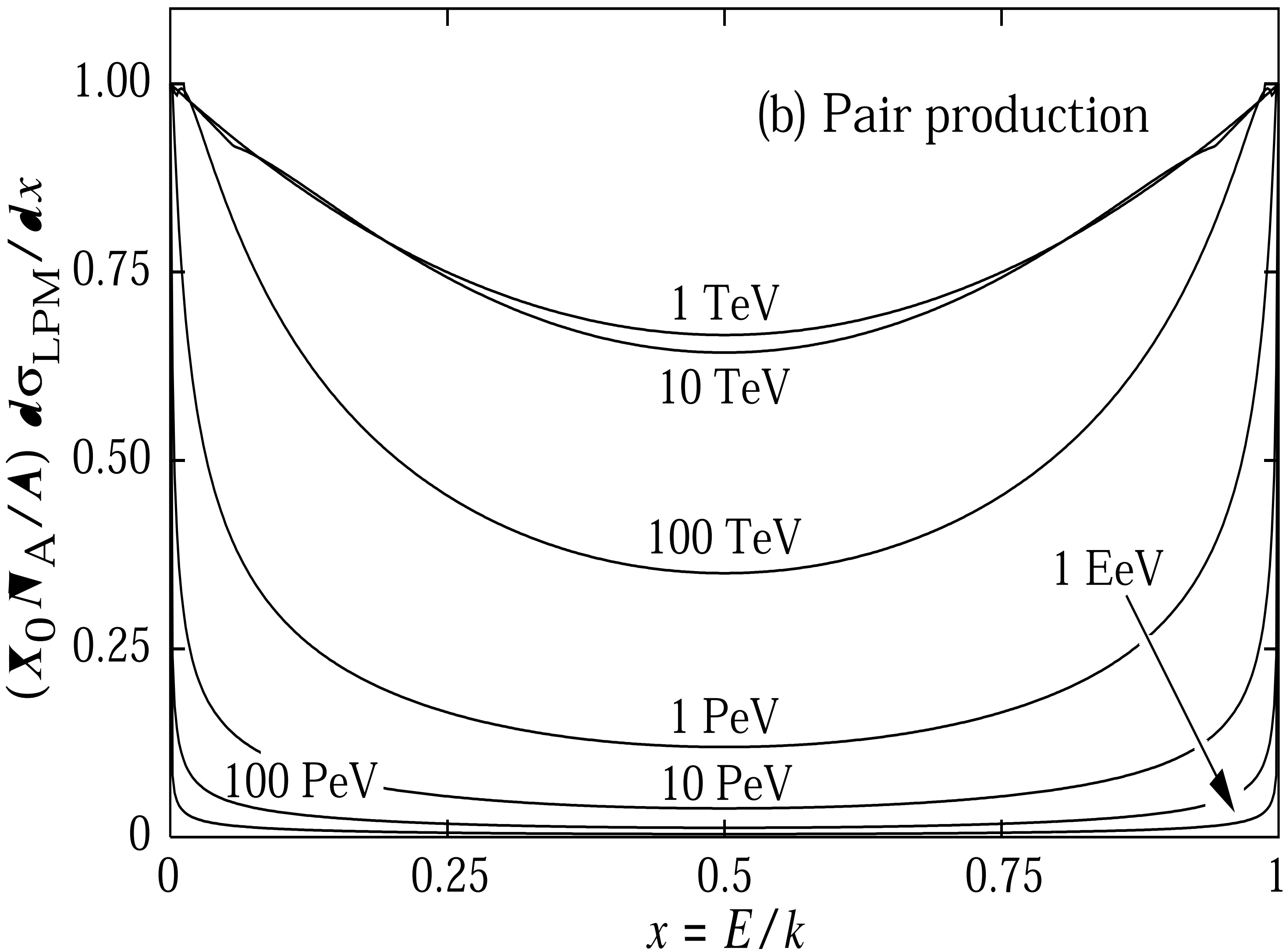
where: $\epsilon = E/E_\gamma \quad \Longrightarrow E_\gamma > 4 \; E_{lpm}$

E_{lpm} is a characteristic energy of the effect :

$$E_{lpm} = \frac{\alpha^2}{4\pi} \frac{mc^2}{r_e} \; X_0 \qquad\qquad \sim (7.7 \; TeV/cm) \times X_0 \; (cm)$$

The suppression function S_{lpm} is:

$$S_{lpm}(\epsilon) = \sqrt{\frac{E_\gamma \; E_{lpm}}{E(E_\gamma - E)}} \equiv \sqrt{\frac{E_{lpm}/E_\gamma}{\epsilon(1-\epsilon)}}$$


(b) Pair production
1.00
0.75
0.50
0.25
0
0
0.25
0.5
0.75
1
$x = E/k$
$(X_0 N_A/A)\, d\sigma_{LPM}/dx$
1 TeV
10 TeV
100 TeV
1 PeV
10 PeV
100 PeV
1 EeV

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