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FRICTION FORCES IN GENERAL RELATIVITY

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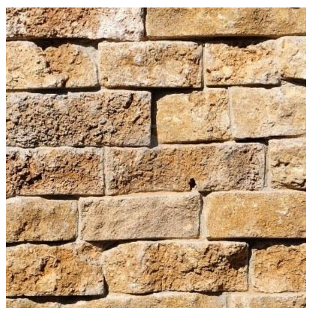
FRICTION EFFECTS IN EVERY DAYS LIFE

A big number of common actions are influenced by the presence or the absence of frictional effects:

Credit: petpassion.tv

•To walk on a frozen lake in Sweden during the winter season

• To build a wall



Credit: it.123rf.com

•Motion inside air

Credit: clipartof.com

Quite always our computations are only approximation

MODELING OF FRICTIONAL FORCES IN CLASSICAL MECHANICS

In classical mechanics usually friction forces are described by a linear relation with respect to the velocity of the point-like particle moving inside the medium where we put all the microscopic details of the scattering collisions with the fluid inside the coefficient of proportionality:

$$F_s = -\gamma v$$

For example a particle of mass *m* which starts at rest its motion inside a viscous fluid in the gravitational field will undergo the equation of motion

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

with solution

$$v = \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma t}{m}}\right)$$

The preceding parametrization can be changed to different ones involving higher power of the velocity, for example

$$F_s = -\gamma v^2$$

which gives different solution of the equations of motion.

FRICTION EFFECTS IN ASTROPHYSICS

•The Klein-Gordon equation for a scalar field (let us say inflaton) ϕ with potential *V* in an expanding background (FLRW metric) with Hubble function *H* can be written as:

$$\ddot{\phi} + V'(\phi) + 3H\dot{\phi} = 0$$

where the last term can be interpreted as the dissipative one proportional to the velocity

•Motion inside a photon gas considered as a test field superposed to a Schwarzschild black hole: when we consider the weak field limit we can study the importance of this phenomenon in the formation of the accretion disk of a star

•Motion inside the Tolman metric solution of the Einstein equations where the source is a photon gas: we can understand how important the curvature of the space is respect to the scattering processes between test particle and photons

•Motion inside a gas in equilibrium described by the Pant and Sah metric: also in this case we can compare the geodesic motion with the scattered one with applications to gravitational lensing

MODELING OF FRICTION FORCES IN GENERAL RELATIVITY

How do we extend the formulation of a friction force in a completely arbitrary space-time?

We follow the Poynting-Robertson modelization:

$$f_{(\mathrm{fric})}(U)^\alpha = -\sigma P(U)^\alpha{}_\beta T^{\beta\mu} U_\mu$$

Where:

U Four velocity of the particle

σ Cross section of the process (considered as a Thomson scattering)

$P(U) = g + U \otimes U$ Projects ortogonally to the velocity (*g* now is the metric)

And we put the information on the fluid inside its stress-energy tensor $T^{\beta\mu}$

We will now move to explicit examples

FIRST EXAMPLE: PHOTON GAS NEAR A SCHWARZSCHILD BLACK HOLE

Schwarzschild metric: $$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Mass of the black hole

Two possible modelizations for the test photon field:

$$T^{\beta\mu} = \Phi k^\beta k^\mu$$

The photons have a favourite direction of motion (done in litterature)

To be determined to have a divergence-free tensor

Four vector parallel to a null geodesic

$$T^{\beta\mu} = (p + \rho) u^\beta u^\mu + p g^{\beta\mu}$$

The photons do not have a favourite direction of motion (new analysis)

Energy density

Four velocity

pressure

Christoffel symbols

Equation of motion $$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = f^\alpha_{(fric)}$$

KINETIC THEORY IN A CURVED SPACETIME: AN OVERVIEW

We can derive the perfect fluid stress-energy tensor of the photons from the Boltzmann equation:

Functional generator

$$T^{\mu\nu} = \frac{1}{\beta^2}\frac{\partial^2 I}{\partial\xi_\mu\partial\xi_\nu}$$

$$\xi_\mu = -\delta^0_\mu$$

Boltzmann factor

Time-like Killing vector of the background metric

$$I = \int f(\xi_\mu P^\mu)2\delta^+(P^2+m^2)\sqrt{-g}d^4P$$

Solution of the Boltzmann equation

Mass shell condition

$$T^{\beta\mu} = (p+\rho)u^\beta u^\mu + pg^{\beta\mu} \qquad p = \frac{1}{3}\rho$$

Describes photons moving without a favourite direction

FINAL EQUATIONS OF MOTION IN THE RADIATION GAS CASE:

Coupling constant between particle and field which depends on the mass of the test particle, on the temperature of the photons and on the cross section of the interaction

Lorentz factor

$$\frac{d\nu^{\hat{r}}}{d\tau} = -\frac{A\nu^{\hat{r}}}{N^4(r)} - N(r)\frac{\gamma}{r}\left[\nu_K^2(1-(\nu^{\hat{r}})^2) - (\nu^{\hat{\phi}})^2\right]$$

Metric factor $\sqrt{1-\frac{2M}{r}}$

Keplerian velocity

$$\frac{dr}{d\tau} = \gamma N(r)\nu^{\hat{r}}$$

$$\frac{d\nu^{\hat{\phi}}}{d\tau} = -\frac{A\nu^{\hat{\phi}}}{N^4(r)} + \frac{\gamma N(r)}{r\gamma_K^2}\nu^{\hat{\phi}}\nu^{\hat{r}}$$

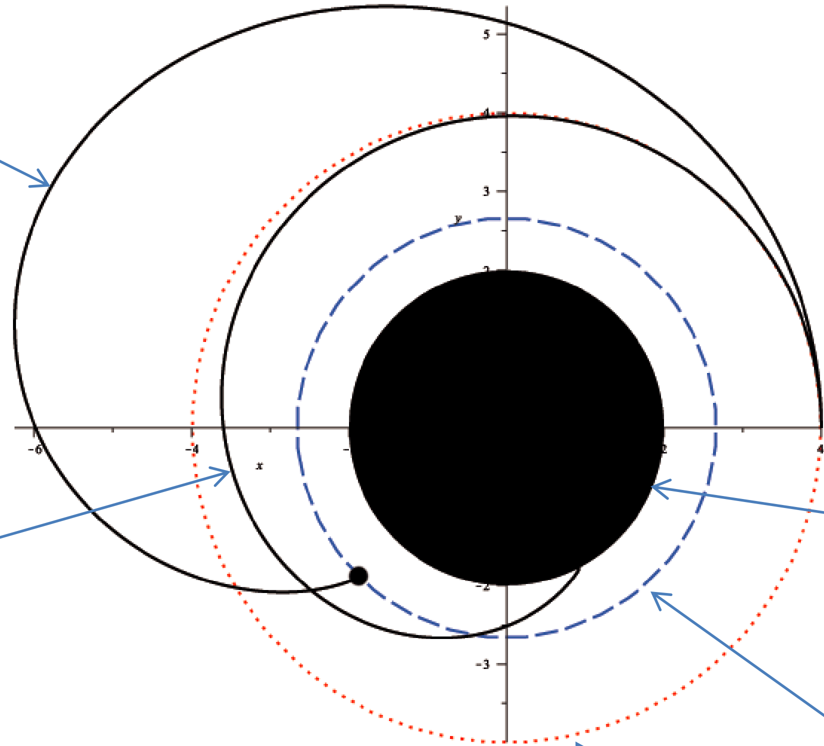
$$\frac{d\phi}{d\tau} = \frac{\gamma}{r}\nu^{\hat{\phi}}$$

Lorentz factor for the keplerian velocity

Bini, Gregoris "Kinetic theory in curved space-times: applications to black holes", proceeding for the 12th Italian-Korean Symposium on relativistic astrophysics

NUMERICAL INTEGRATION AND TYPE OF ORBITS

Particle motion for the previous description of the Poynting-Robertson effect



"New" particle orbit

Schwarzschild horizon

Equilibrium points for the previous description of the Poynting-Robertson effect

Orbit for a geodesic path

D. Bini, D. Gregoris, and S. Succi, EPL (Europhysics Letters) 97, 40007 (2012)

…AND IF THE PHOTONS ARE NOT A TEST FIELD?

POYNTING-ROBERTSON EFFECT IN THE TOLMAN METRIC

Since the Tolman metric is a solution of the Einstein equation for a photon gas as a source we can study the interaction between massive test particle and photon gas in three different ways: as a geodesic motion in the Tolman metric which takes into account the curvature of the space because of the presence of photons (0th order approximation), as a scattered motion modeled as a Poynting-Robertson effect in the same metric (1st order approximation) and as a scattered motion inside a photon test field in the minkowskian space. This analysis let us understand the relative importance of curvature of the space and friction .

We obtain that in the presence of collisions the massive test particle always stops its motion because it dissipates all its initial energy, while when we consider the geodesic motion the velocity exhibits an oscillating behavior.

Thus the approximation of a test field for the photons gives the same asymptotic result of the "exact" computation.

Bini, Gregoris, Rosquist, Succi, "Particle motion in a photon gas: friction matters", General Relativity and Gravitation: Volume 44, Issue 10 (2012), Page 2669-2680

BASIC EQUATIONS: THE TOLMAN METRIC

Stress-energy tensor for the photons:
(conserved and trace-free)

$$T^{\mu\nu} = \frac{\rho}{3}\left[g^{\mu\nu} + 4u^{\mu}u^{\nu}\right], \qquad u^{\mu} = \frac{1}{\sqrt{-g_{tt}}}\partial_t$$

Einstein equations:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = kT_{\alpha\beta}$$

Tolman metric:

$$ds^2 = -ar\,dt^2 + \frac{1}{a}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad a = \frac{4}{7}$$

EQUATIONS OF MOTION (SCATTERED CASE)

1-st order approximation:

Coupling constant between test particle and photons

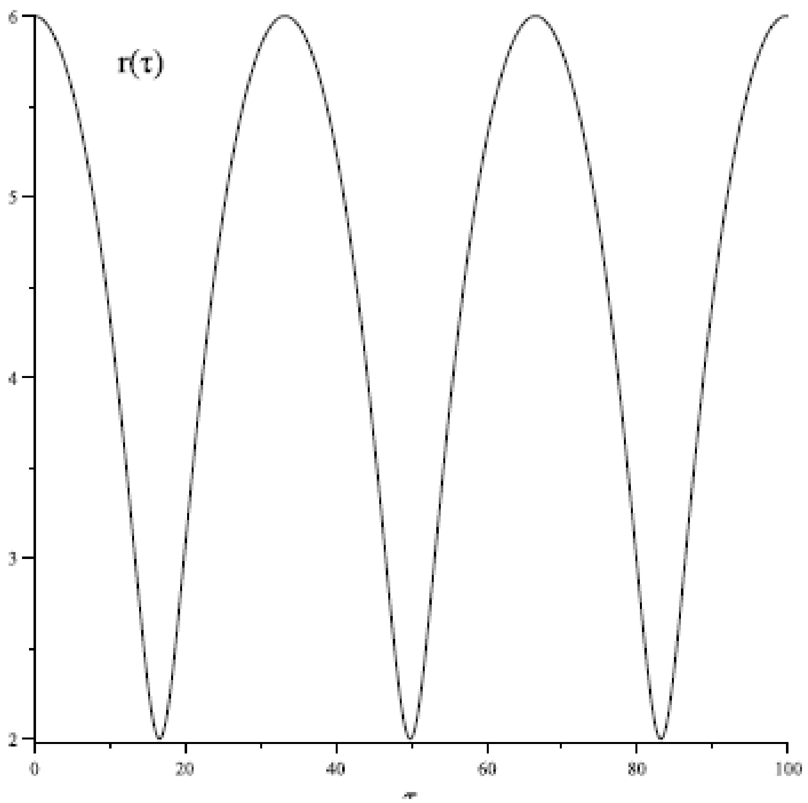
$$\frac{d\nu}{d\tau} = -\frac{\sqrt{a}}{2}\frac{\sin\alpha}{\gamma r} - \frac{\tilde{A}\nu}{r^2}, \quad \frac{dr}{d\tau} = \sqrt{a}\gamma\nu\sin\alpha\,,$$

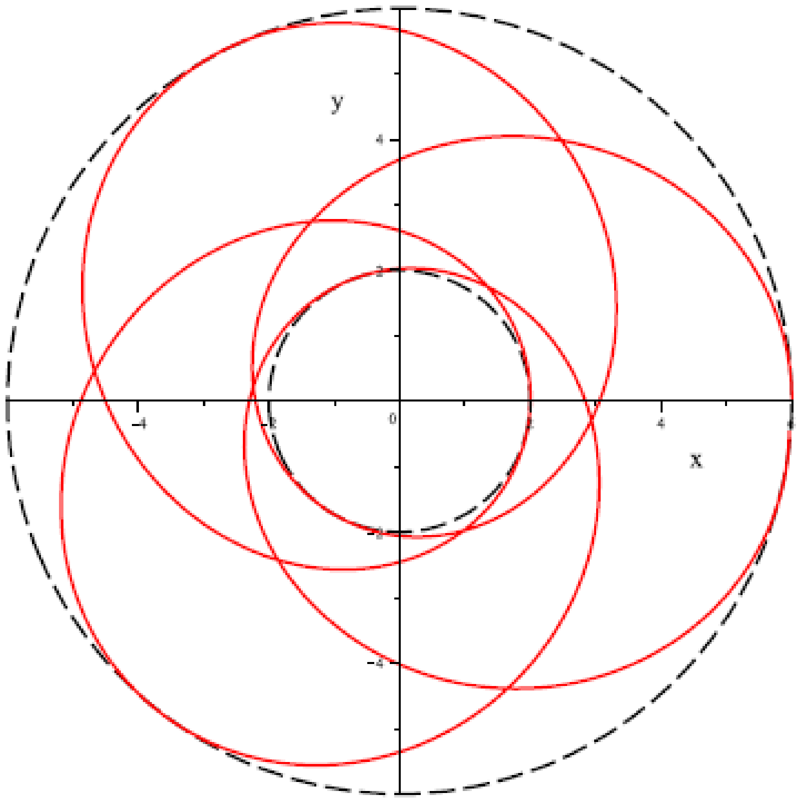
$$\frac{d\phi}{d\tau} = \frac{\gamma\nu\cos\alpha}{r}, \quad \frac{d\alpha}{d\tau} = \frac{\sqrt{a}}{2}\frac{\gamma(2\nu^2-1)\cos\alpha}{\nu r}$$

Where we used a polar representation of the velocity:

$$\nu^{\hat{r}} = \nu(\tau)\sin\alpha(\tau), \qquad \nu^{\hat{\phi}} = \nu(\tau)\cos\alpha(\tau)$$

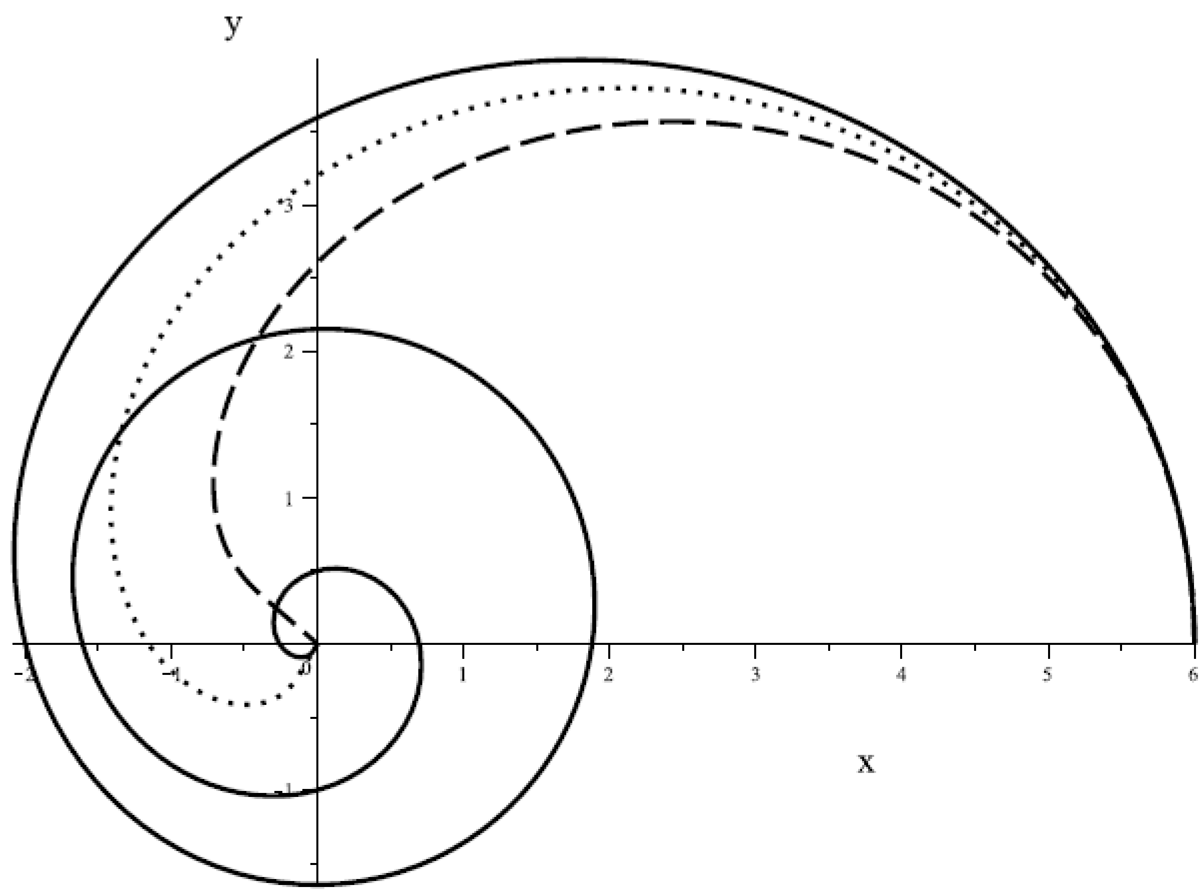
GEODESIC MOTION IN THE TOLMAN METRIC VS...





The plots show respectively the behavior of the radial coordinate respect to the proper time and the equatorial geodesic for the Tolman metric

… SCATTERED MOTION IN THE TOLMAN METRIC



Polar plot of the equatorial orbits of the Tolman metric undergoing the Poynting-Robertson-like effect corresponding to different coupling constants between particle and field.

WHAT ABOUT WHEN THE TEST PARTICLE IS MOVING INSIDE A MASSIVE GAS?

•The Pant-Sah and Buchdahl metrics are solutions of the Einstein equations when a massive gas undergoing a polytropic equation of state with index n=5 is considered

•Also in this case we can compare and contrast the geodesic motion with the scattered one where the stress-energy tensor appearing in the friction force is the same appearing in the Einstein equations

•In the presence of friction the test particle dissipates all its initial energy and stops its motion at the center of the configuration with null velocity

•This analysis let us to study the behavior of a gas cloud as a gravitational lensing.

CONCLUSIONS

•Friction plays an important role in a wide class of phenomena
•We have seen how the friction has been modeled in classical mechanics…
•… and how to extend it to a general relativistic context (Poynting-Robertson effect)
•We have derived the equation of motion inside a photon gas near a Schwarzschild black hole…
•… and we have integrated them to obtain the type of orbits
•We have studied how the curvature of the space affects the results of our analysis
•We have studied a possible extension of the Poynting-Robertson formalism when we have a massive fluid…
•… to study the scattered motion inside the Pant -Sah metric which is a solution of the Einstein equations when a massive gas in equilibrium is considered as a sorce

AND OUTLOOK

•We would like to include the noise effects in our equations to satisfy the fluctuation-dissipation theorem.