VIRGO NTS 95-40

Doppler effect and Periodic signal Search

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1 Introduction

The electromagnetic waves emitted by a stand alone pulsar have a very well defined frequency and therefore we can use very long integration time. But the earth motion introduce a Doppler effect changing over the time. To avoid losses of signal we have to correct this effect. That means that we have to known the earth position as well as the pulsar position in the sky. In this note we will compute the needed accuracy on these positions to keep the losses at a negligible level. In the case of the pulsar direction, or any kind of periodic sources of gravitational waves, this precision define a small area in the sky. This give us the number of search directions for a blind search.

2 Needed precision on the earth position

To discribe the effect of the earth motion around the sun on a periodic signal of frequency ν, and amplitude h_0 we should take into account the wave propagation from the sun to the earth. Therefore the phase observed on earth is $\phi = 2\pi\nu t + \vec{k}.\vec{r}(t)$ where $\vec{k}$ is the wave propagation vector and $\vec{r}(t)$ is the earth position with respect to the sun. If we know the earth and the pulsar position we can cancel the $\vec{k}.\vec{r}(t)$ term and recover the periodic signal.

Let assume that we have an error $\delta\vec{r}(t)$ on the earth position. Then the Fourier Transform $\hat{f}_e(\nu)$ of the corrected signal becomes:

$$
\begin{aligned}
\hat{f}_e(\nu) &= 2\frac{h_0}{T}\int_0^T e^{i2\pi\nu t}\sin(2\pi\nu t + \vec{k}.\delta\vec{r}(t))dt \\
&= \sum_{i=0}^{n-1} 2\frac{h_0}{T}\int_{\frac{i}{\nu}}^{\frac{i+1}{\nu}} e^{i2\pi\nu t}\left(\sin(2\pi\nu t)\cos(\vec{k}.\delta\vec{r}(t)) + \cos(2\pi\nu t)\sin(\vec{k}.\delta\vec{r}(t))\right) dt
\end{aligned}
$$

The function which describe the earth motion is continuous. We can assume that the error $\delta\vec{r}(t)$ will have a slow trend compared to the pulsar frequency and could be taken as constant over one pulsar period. Therefore with $t_i = \frac{i}{\nu}$ we have :

$$\begin{aligned}
\hat{f}_e(\nu) &= \sum_{i=0}^{n-1} \frac{h_0}{T\nu} \left(\cos(\vec{k}.\delta\vec{r}(t_i)) + i\sin(\vec{k}.\delta\vec{r}(t_i))\right) \\
&= \sum_{i=0}^{n-1} \frac{h_0}{n} \left(\cos(\vec{k}.\delta\vec{r}(t_i)) + i\sin(\vec{k}.\delta\vec{r}(t_i))\right)
\end{aligned}$$

So

$$|\hat{f}_e(\nu)| = \frac{h_0}{n}\sqrt{\left[\sum_{i=0}^{n-1}\cos(\vec{k}.\delta\vec{r}(t_i))\right]^2 + \left[\sum_{i=0}^{n-1}\sin(\vec{k}.\delta\vec{r}(t_i))\right]^2}$$

If we assume that $\vec{k}.\delta\vec{r}(t)$ is small then

$$\begin{aligned}
|\hat{f}_e(\nu)| &= h_0\sqrt{\left[\frac{1}{n}\sum_{i=0}^{n-1}\left(1-\frac{(\vec{k}.\delta\vec{r}(t_i))^2}{2}\right)\right]^2 + \left[\frac{1}{n}\sum_{i=0}^{n-1}\vec{k}.\delta\vec{r}(t_i)\right]^2} \\
&= h_0\sqrt{\left[1-\frac{1}{2} < (\vec{k}.\delta\vec{r})^2 >\right]^2 + < \vec{k}.\delta\vec{r} >^2} \\
&= h_0\sqrt{1-\left(< (\vec{k}.\delta\vec{r})^2 > - < \vec{k}.\delta\vec{r} >^2\right)}
\end{aligned}$$

Let's define δr as the root mean square of $\delta\vec{r}(t)$ projected along $\vec{k}$, then

$$\begin{aligned}
|\hat{f}_e(\nu)| &= h_0\sqrt{1-(k\delta r)^2} \\
&= h_0\left(1-\frac{(k\delta r)^2}{2}\right)
\end{aligned}$$

Let assume we want less than 5 % of losses. Then we want

$$\begin{aligned}
\frac{(k\delta r)^2}{2} &\leq 0.05 \\
\delta r &\leq \frac{c\sqrt{0.1}}{2\pi\nu} \\
\delta r &\leq 15km\left(\frac{1kHz}{\nu}\right)
\end{aligned}$$

This is a tight constraint on the earth position. As an example the figure 1 represents the earth position out of the ecliptic plane over ten years. We have an effect several order of magnitude larger than our constraint. This effect is coming from the influence of the moon and the other planets. To correct this effect we use a function from the 'Bureau des longitudes' in Paris which gives the earth position with an error of about $2km$. This function is now included in Siesta (Function MEarth in MEtricL.c). Therefore the losses due to the earth position uncertainty will be less than 5% (about 0.3% for $1kHz$) once we know the pulsar position.

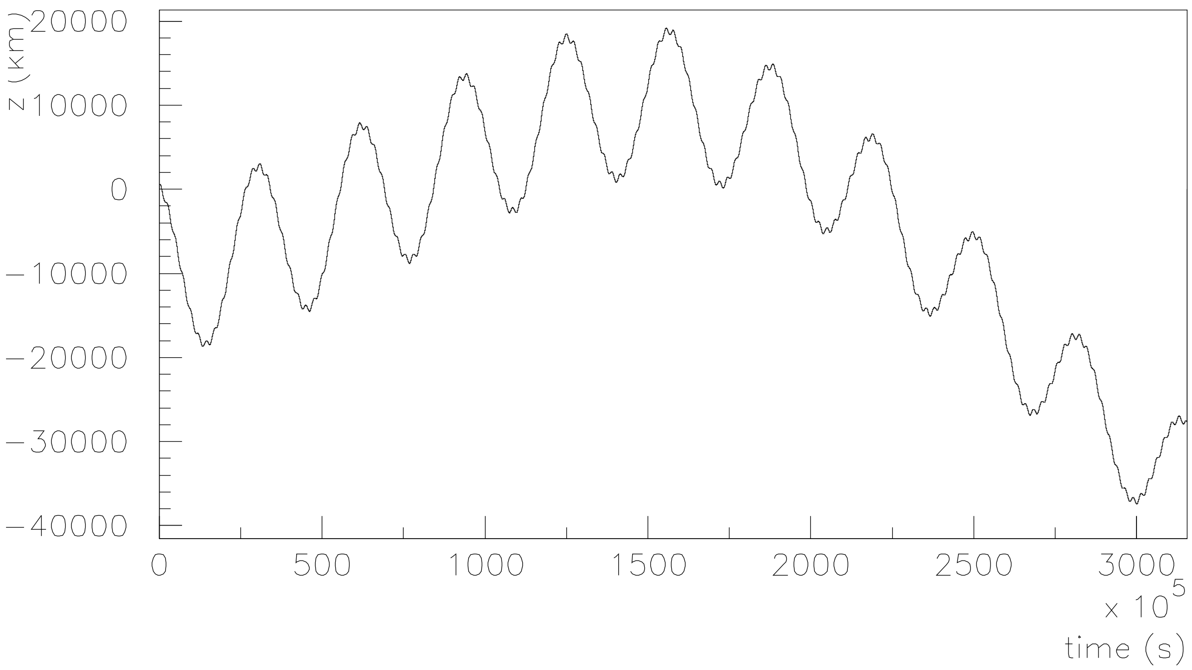

Figure 1: Earth position out of the ecliptic plane

3 Evaluation of the number of direction for a blind search

Let us now compute the error $\delta\vec{k}$ for a given $\vec{k}$ and a given level of losses. This $\delta\vec{k}$ define a cell in the sky where the signal search will give the same result (within the losses). This value will be converted to the number of cells in the sky for a blind search.

3.1 Principle of calculation

Let's $\hat{f}_e(\nu)$ be the value of the Fourier Transform at the frequency ν with the error.

$$\hat{f}_e(\nu) = \frac{2h_0}{T}\int_0^T e^{i2\pi\nu t}\sin(2\pi\nu t + \delta\vec{k}.\vec{r}(t))dt$$

T is bigger than $\frac{1}{\nu}$ so we can use the same approximation as before. Therefore, we can assume that the earth position $\vec{r}(t)$ has slow variation compared to frequency ν of a pulsar (a few Hz to one kHz). During period, $\delta\vec{k}.\vec{r}(t)$ will be assumed to be constant.

$$\begin{aligned}
\hat{f}_e(\nu) &= \sum_{i=0}^{n-1}\frac{2h_0}{T}\int_{\frac{i}{\nu}}^{\frac{i+1}{\nu}} e^{i2\pi\nu t}\sin(2\pi\nu t + \delta\vec{k}.\vec{r}(t))dt \\
&= \sum_{i=0}^{n-1}\frac{2h_0}{T}\int_{\frac{i}{\nu}}^{\frac{i+1}{\nu}} e^{i2\pi\nu t}\left[\sin(2\pi\nu t)\cos(\delta\vec{k}.\vec{r}(t)) + \cos(2\pi\nu t)\sin(\delta\vec{k}.\vec{r}(t))\right]dt
\end{aligned}$$

Using the slow variation approximation we can take out of the integral the terms $\cos(\delta\vec{k}.\vec{r}(t))$ and $\sin(\delta\vec{k}.\vec{r}(t))$. Where $t_i = \frac{i}{\nu}$

$$\hat{f}_e(\nu) = \sum_{i=0}^{n-1} \frac{h_0}{n} \left[\sin(\delta\vec{k}.\vec{r}(t_i)) + i\cos(\delta\vec{k}.\vec{r}(t))\right]$$

So, the signal losses L is :

$$L = 1 - \frac{1}{n}\sqrt{\left[\sum_{i=0}^{n-1}\cos(\delta\vec{k}.\vec{r}(t_i))\right]^2 + \left[\sum_{i=0}^{n-1}\sin(\delta\vec{k}.\vec{r}(t_i))\right]^2} \tag{1}$$

$$L = 1 - \frac{1}{n}\sqrt{\sum_{i,j=0}^{n-1}\cos(\delta\vec{k}.(\vec{r}(t_i) - \vec{r}(t_j))} \tag{2}$$

(2) shows that the calculations are not dependent of the coordinates origin. And (1) is used for the numerical calculations.

3.2 Numerical calculation

We can assume that $\vec{r}(t) \simeq \vec{r_0} + \vec{V}t + \frac{\vec{a}}{2}t^2 + o(t^2)$. We should notice that the value of $\vec{V}$ is a function of the time origin used for the development of $\vec{r}$. We define $\vec{u} = \frac{\vec{k}}{2\pi\nu}$. With such an approximation the signal $\sin(2\pi\nu t + \delta\vec{k}.\vec{r}(t))$ become $\sin(2\pi\nu(1 + \delta\vec{u}.\vec{V})t + \frac{\vec{a}}{2}t^2 + \delta\vec{k}.\vec{r_0} + o(t^2))$. So the Doppler effect shift the signal at another frequency : $\nu(1 + \delta\vec{u}.\vec{V})$ and the variation of the Doppler effect (t^2 term) is responsible of the signal losses. To compute the best frequency or the best speed $\vec{V}$ we will minimize the signal losses introduced by the t^2 term. If we look for the signal at the shift frequency the earth position become : $\frac{\vec{a}}{2}t^2 + o(t^2)$. So with $f(t) = \delta\vec{k}.\vec{r} = \delta\vec{k}.\frac{\vec{a}}{2}t^2 + o(t^2)$ and the low losses approximation, using the results of section 2, the signal losses L is given by : $L = \frac{<(\delta\vec{k}.\vec{r})^2> - <\delta\vec{k}.\vec{r}>^2}{2} = \frac{<f^2> - <f>^2}{2}$. To simplify the calculation we will note : $f(t) = bt^2$. In this case we choose as time of origin the time where the motion is tangent to the speed we look for. T is the total integration time and α is the time before the origin.

$$\begin{aligned}
<f> &= \frac{1}{T}\int_{-\alpha}^{T-\alpha} bt^2 dt = \frac{b}{3T}\left[t^3\right]_{-\alpha}^{T-\alpha} = \frac{b}{3}\left(T^2 - 3\alpha T + 3\alpha^2\right) \\
<f>^2 &= \frac{b^2}{9}\left(T^4 - 6\alpha T^3 + 15\alpha^2 T^2 - 18\alpha^3 T + 9\alpha^4\right) \\
<f^2> &= \frac{1}{T}\int_{-\alpha}^{T-\alpha} b^2 t^4 dt = \frac{b^2}{5T}\left[t^5\right]_{-\alpha}^{T-\alpha} = \frac{b^2}{5}\left(T^4 - 5\alpha T^3 + 10\alpha^2 T^2 - 10\alpha^3 T + 5\alpha^4\right) \\
L &= \frac{1}{2}\left(\frac{b^2}{5}\left(T^4 - 5\alpha T^3 + 10\alpha^2 T^2 - 10\alpha^3 T + 5\alpha^4\right) - \frac{b^2}{9}\left(T^4 - 6\alpha T^3 + 15\alpha^2 T^2 - 18\alpha^3 T + 9\alpha^4\right)\right) \\
&= \frac{b^2}{2}\left(\frac{4T^4}{45} - \frac{\alpha T^3}{3} + \frac{\alpha^2 T^2}{3}\right)
\end{aligned}$$

To minimize the losses we need :

$$\frac{\partial L}{\partial \alpha} = 0 \Rightarrow$$
$$\frac{T^3}{3} - \frac{2T^2\alpha}{3} = 0$$
$$\alpha = \frac{T}{2}$$

The best speed is the speed at half the integration time which is also the average speed.

Using formula (1) and the right time origin we can numerically compute the losses as function of $\delta\vec{k}$. As we can see from figure 2, a given level of losses define an elliptic cell in the sky.

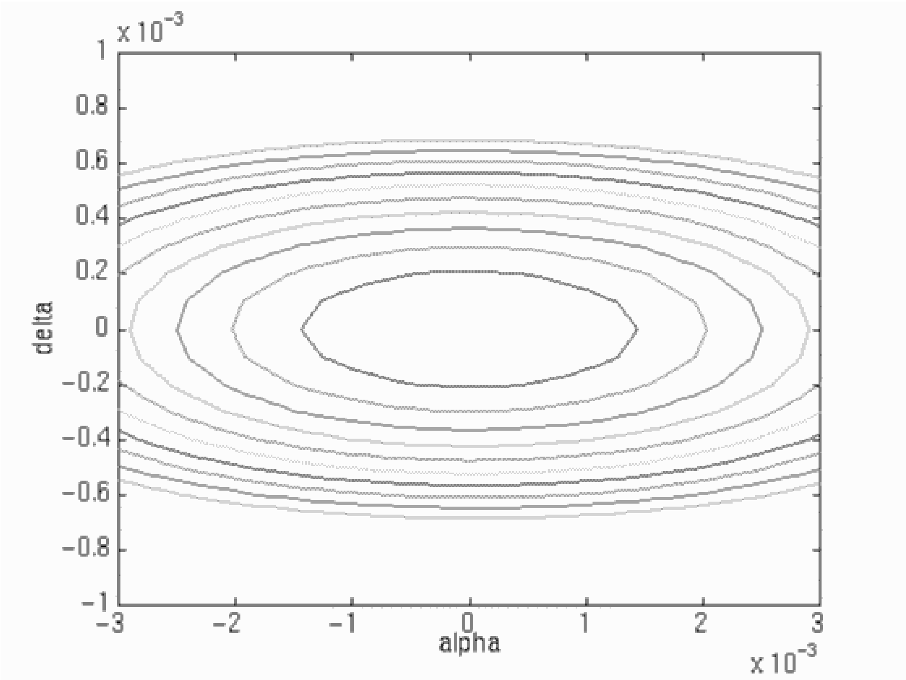

Figure 2: Contour plot of the losses computed for $\nu = 50Hz$, $\alpha = \frac{\pi}{4}, \delta = \frac{\pi}{4}$, integration time: $10^6 seconds$. Each contour line correspond to 2% of losses changes.

For each direction in the sky, we compute the size of the cell. To calculate the surface of the ellipse we parametrise $\delta\vec{k}$ like a function of an angle, θ, and a norm, $\delta(\theta)$.

$$\delta\vec{k}(\theta) = \frac{2\pi\nu}{c}.\delta(\theta).(\cos\theta\vec{i} + \sin\theta\vec{j})$$

The losses is a function of two parameters : ν and θ. For a given calculation, ν doesn't change, we choose the losses and we calculate δ for a few θ_i and we fit an ellipse on this value. The number of cells in this direction is proportional to the inverse of the ellipse's surface.

The figure 3 shows the cells density in an Galactic sky for ten days integration time starting the first of January for a pulsar with a frequency below $50Hz$. As we can see the density in the galactic center is not so high, that mean the best region for pulsars will not be the hardest to search. The figure 4 presents the distribution of the cells density which is raiser flat. The total number of cells is in this case 8.4×10^4.

The low density area in figure 3 (as in figure 5 & figure 6) correspond to the ecliptic plane which contain the earth acceleration and therefore which minimize the $\delta\vec{k}.\vec{a}$ term.

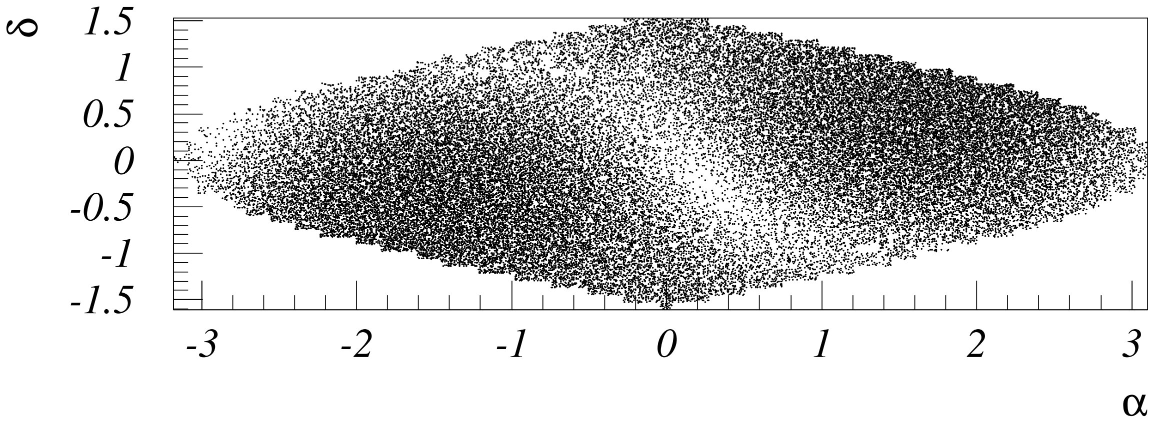

Figure 3: Cells density in galactic coordinates for ten days of integration

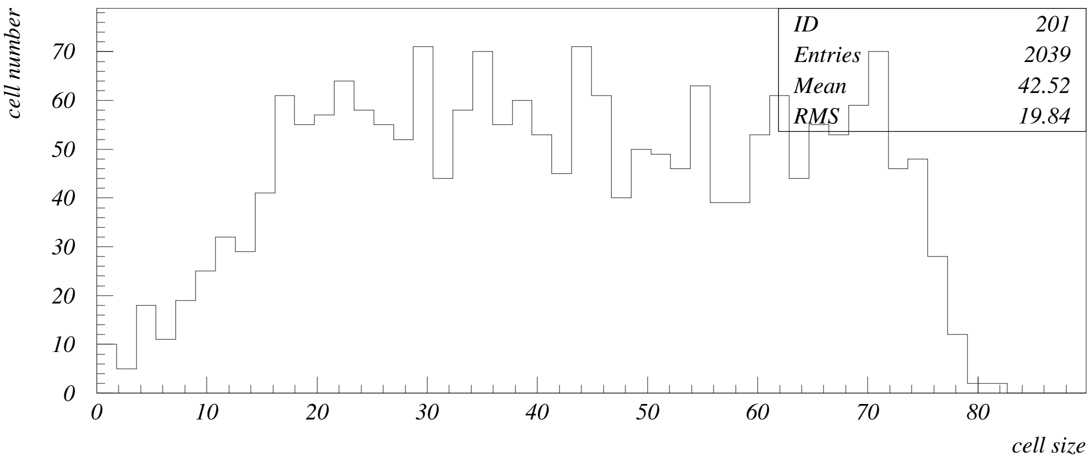

Figure 4: Distribution of the number of cells

The integration time for figure 5 is starting three months after the one from figure 3, it gives approximately the same result, except for the very low density regions which are not the same. The figure 6 (three months of integration) shows that in mean the low density areas are in the ecliptic plane.

Figure 7 presents the total number of cells for two pulsar frequencies versus the integration time. This number increase dramatically with the integration time underling the technical challenge of a blind search. Two regimes could be observed in figure 7. The first part, up to few days, is dominated by the daily rotation and the second one by the motion of the earth around the sun. For even longer integration time we would have to take into account the solaris system motion within the galaxy.

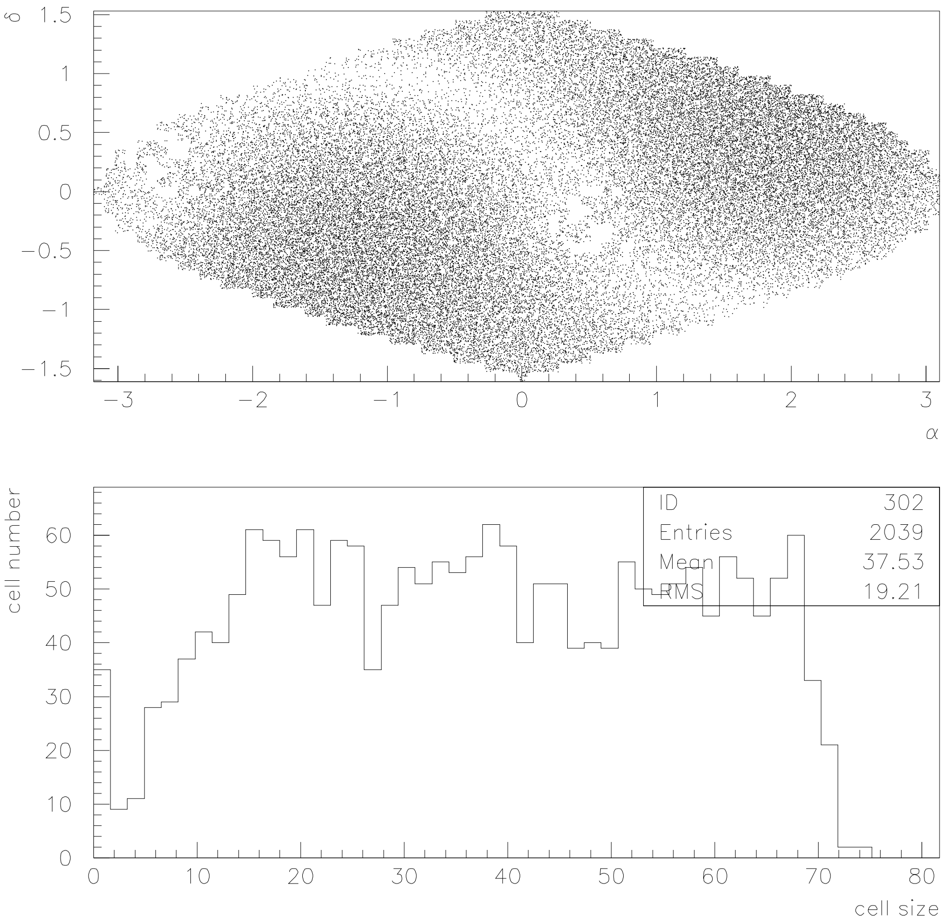

Figure 5: Number of cells for each direction in galactic coordinates for 10 days integration three months later than figure 3

The one hundred factor between $50Hz$ and $500Hz$ is explain by formula (1). In fact if you change the frequency of the pulsar by a factor ten, the number of cells increase in a factor hundred. As shown before with the parametrisation of $\delta\vec{k}$, if we change ν in $\nu * 10$ and δ in $\frac{\delta}{10}$, $\delta\vec{k}$ doesn't change. So the losses are the same. A factor 0.1 on δ give a factor 0.01 on the ellipse's surface. So a factor 10 on the pulsar frequency put a factor 100 on the number of cells.

4 Conclusion

The number of cells for a blind search of periodic source strongly depend of the integration time and of the maximum frequency. The goal of a search algorithm will be to minimize this number of cells.

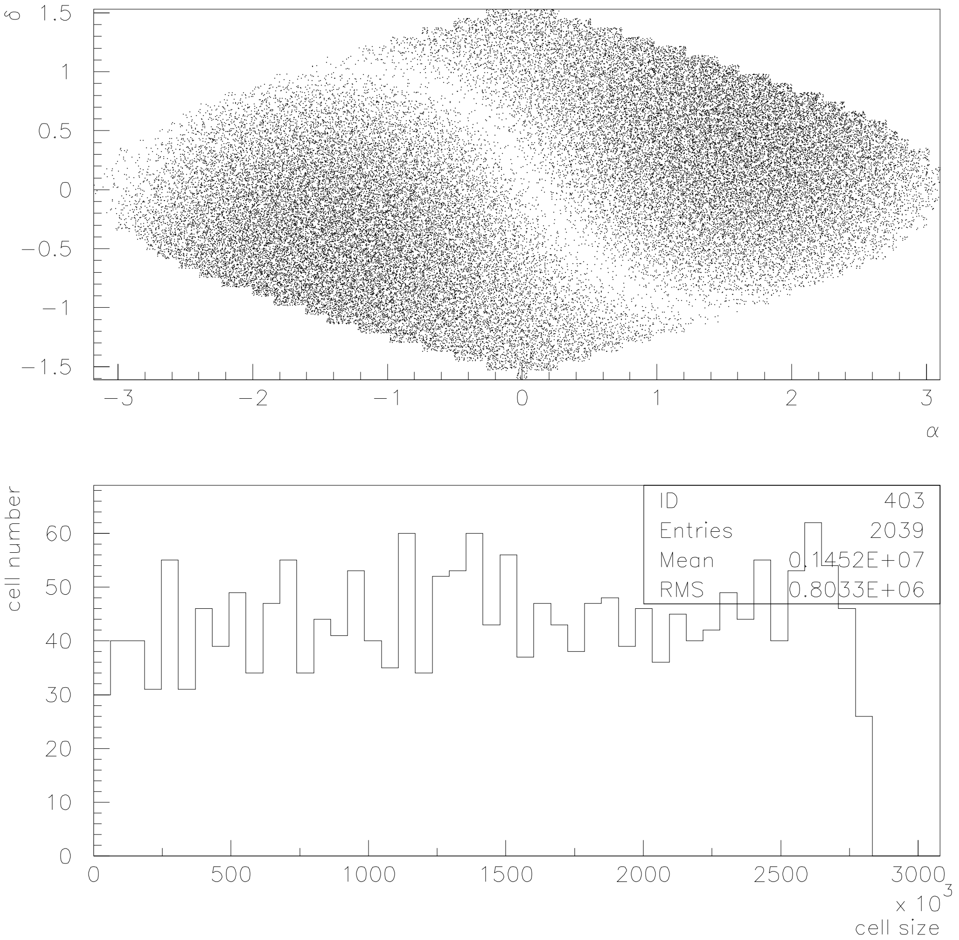

Figure 6: Number of cells for each direction in galactic coordinates for three months of integration

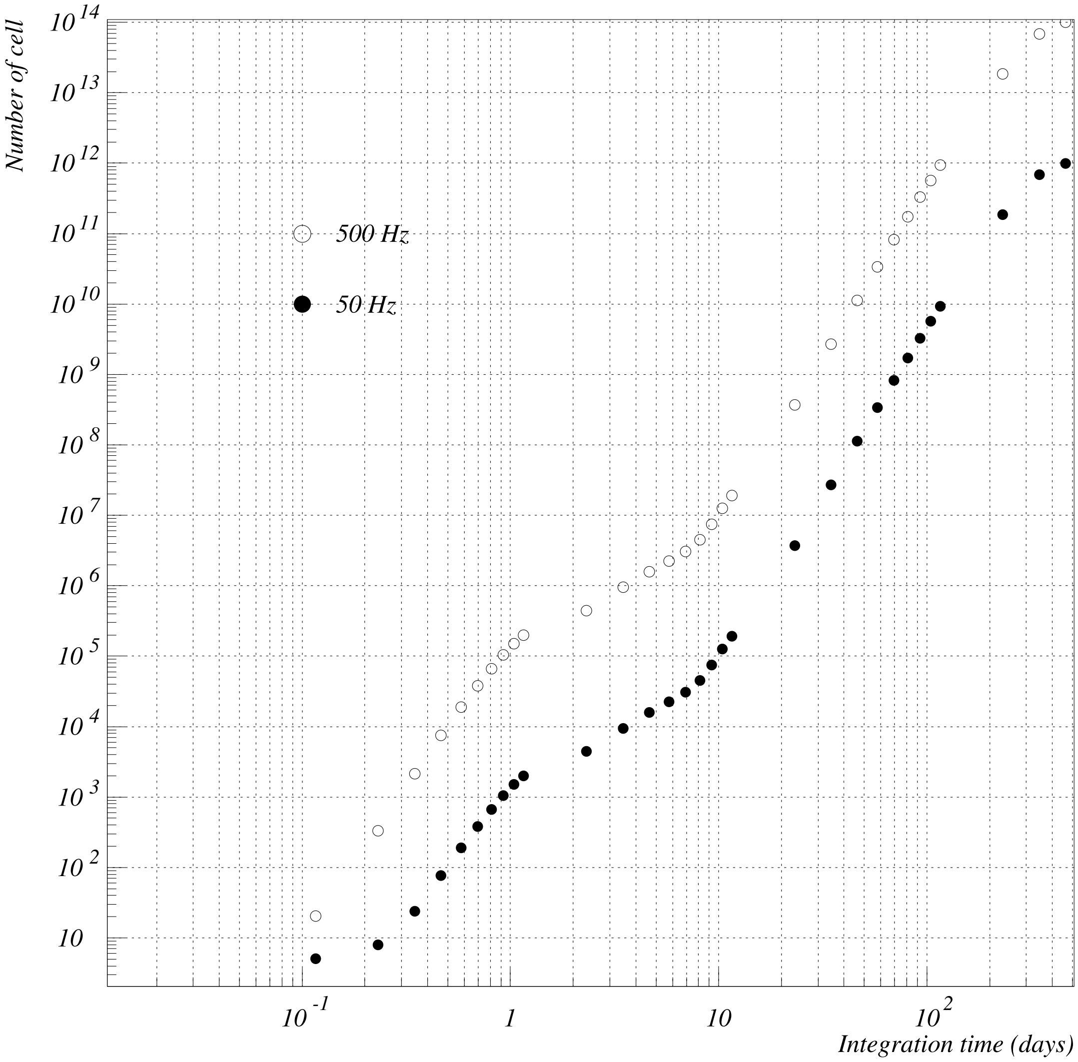

Figure 7: Losses for different Pulsar's frequency