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"Cross-Talk" Between the Dark Fringe Beam and the Secondary Beam

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1 Introduction

Two monochromatic fields which impinges on the same surface at different angles, make interference. This phenomena is due essentially by the coherent diffusion of the optical surfaces and it depends on the surface roughness and on the incidence angle of two fields wave vectors.

That is the case of the two beams coming from the VIRGO beam splitter, the so called dark-fringe beam and the other one, the "secondary beam" reflected by the anti-reflection coated beam splitter face.

The interference with the diffused secondary beam and the dark-fringe beam (beams "cross-talk"), and a noise on the position of the optical elements, can generate a noise on the dark-fringe demodulated signals.

For little angles of incidence, ($\sim 40\ mrad$) these behavior could be not negligible and it has to be investigated.

2 Beams "Cross-Talk"

The field expression at the interferometer output when the dark-fringe condition is imposed (see fig. 1) is

$$A_f = rA_1 - tA_2, \qquad A_1, A_2 \in R, \tag{1}$$

where r^2 and t^2 are the reflectivity and transmittivity of the beam splitter.

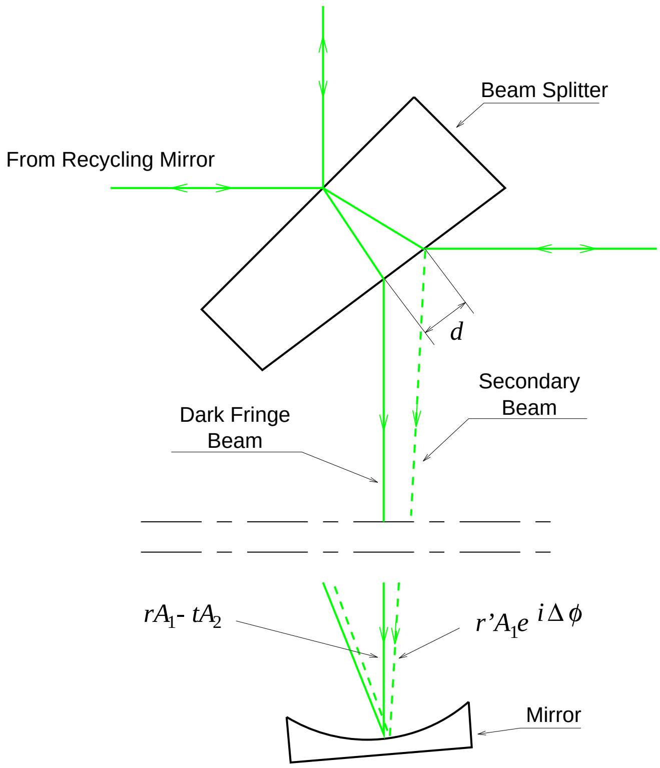

Figure 1: Beam Splitter and First Mirror of the VIRGO-1998 Detection Bench Telescope.

The diffusion of the secondary beam due to the reflection of the first mirror of the detection bench, produces a field on the same direction of the dark-fringe beam.

Neglecting a common phase, the dark-fringe beam reflected by the mirror will be

$$A_f = rA_1 - tA_2 + r'\beta e^{i\Delta\phi} A_1, \tag{2}$$

where r'^2 is the reflectivity of the beam splitter second face, β is the "cross-talk" real coefficient that depends on the angle between the two incident beams and on the roughness of the surface, and $\Delta\phi$ is the path difference between the two beams.

Considering a noise $\delta\phi \ll 1$ in the phase shift between the two beams, we have

$$\Delta\phi = \Delta\phi_0 + \delta\phi, \tag{3}$$

and the approximated expression of the (2) becomes

$$\begin{aligned} A_f &= rA_1 - tA_2 + r'\beta e^{i\Delta\phi_0} A_1 (1 + i\delta\phi) \\ &= a + r'\beta (\cos\Delta\phi_0 - \delta\phi \sin\Delta\phi_0) A_1 + ir'\beta (\sin\Delta\phi_0 + \delta\phi\cos\Delta\phi_0) A_1, \end{aligned} \tag{4}$$

where

$$a = rA_1 - tA_2 \qquad a \in R. \tag{5}$$

For a gravitational wave case and neglecting the "cross-talk", the reflected beam from the first detection bench mirror is

$$A_f = rA_1 e^{i\delta\phi_{GW}} - tA_2 e^{-i\delta\phi_{GW}}, \qquad A_1, A_2 \in R, \tag{6}$$

where $\delta\phi_{GW}$ is the phase shift produced by the gravitational wave.

With $\delta\phi_{GW} \ll 1$ the previous expression becomes

$$A_f = a + i(rA_1 + tA_2)\,\delta\phi_{GW}. \tag{7}$$

The imaginary part carries on the gravitational signal.

Comparing the (4) and the (7) we deduce that effects of the "cross-talk" are the following:

- a noise on the gravitational wave signal proportional to $r'\beta$,
- a worst dark-fringe.

Supposing that

$$A_1 \simeq A_2, \qquad t \simeq r \qquad \text{(perfect contrast)}, \tag{8}$$

we obtain the following condition to neglect the noise induced by the "cross-talk"

$$\beta < \frac{\sqrt{2}}{2}\frac{1}{r'}\frac{\delta\phi_{GW}}{\delta\phi}. \tag{9}$$

The β parameter and the noise on the phase shift $\delta\phi$ have to be estimated.

3 Beams “Cross-Talk” versus Phase Modulation

In this section, we want to calculate the effect of the “cross-talk” in the phase-modulated detection technique case.

For little modulation index, the general expression of a phase-modulated field is the following

$$A = \left\{ J_0 A^{(0)} + J_1 \left[A^{(+)} e^{i\Omega t} - A^{(-)} e^{-i\Omega t} \right] \right\} e^{i\omega t}, \tag{10}$$

where$A^{(0)}, A^{(+)}, A^{(-)}$ are respectively the field expressions of the carrier, of the right and left side-bands, $\omega/2\pi$ the carrier frequency, $\Omega/2\pi$ the modulation frequency and J_0, J_1 the Bessel functions.

The in-phase and in-quadrature expressions at the modulation frequency are respectively

$$\left[|A|^2 \right]_{\sin \Omega t} = -2 J_0 J_1 \, Im \left\{ A^{(0)^*} A^{(+)} - A^{(0)} A^{(-)^*} \right\}, \tag{11}$$

$$\left[|A|^2 \right]_{\cos \Omega t} = +2 J_0 J_1 \, Re \left\{ A^{(0)^*} A^{(+)} - A^{(0)} A^{(-)^*} \right\}. \tag{12}$$

In our case, the approximated field expressions of the reflected beam by the first detection bench mirror are the following

$$A^{(k)} \simeq a^{(k)} + r'\beta \left(1 + i\delta\phi\right) A_1^{(k)} e^{i\Delta\phi_0}, \qquad k = -, 0, +, \tag{13}$$

where k is the frequnecy index and

$$a^{(0)} = A_1^{(0)} r - A_2^{(0)} t \qquad \in R, \tag{14}$$

$$a^{(\pm)} = A_1^{(\pm)} r + A_2^{(\pm)} t \qquad \in C. \tag{15}$$

The phase shift due to the frequency differences are neglected $(\Delta\phi(\omega \pm \Omega) \simeq \Delta\phi(\omega))$.

After some algebra and neglecting terms of the second order in $r'\beta$, the argument of the real and the imaginary part of the (11) and of the (12), becomes

$$\begin{aligned} A^{(0)^*} A^{(+)} - A^{(0)} A^{(-)^*} \simeq \; & a^{(0)} \left[(1 + i\delta\phi) A_1^{(+)} e^{i\Delta\phi_0} - (1 - i\delta\phi) A_1^{(-)^*} e^{-i\Delta\phi_0} \right] r'\beta \\ & \left[(1 - i\delta\phi) a^{(+)} A_1^{(0)} e^{i\Delta\phi_0} - (1 + i\delta\phi) a^{(-)^*} e^{-i\Delta\phi_0} \right] r'\beta \\ & a^{(0)} \left(a^{(+)} - a^{(-)^*} \right). \end{aligned} \tag{16}$$

For the dark-fringe condition (which impose a null demodulated signal when the “cross-talk” is not considered) we have

$$a^{(+)} = a^{(-)^*}. \tag{17}$$

Supposing that

$$\left| A_k^{(+)} \right| = \left| A_k^{(-)} \right|, \qquad k = 1, 2, \tag{18}$$

and considering that the phase of the side bands reflected from the Fabry-Perot cavities fulfill the following condition

$$\varphi_k^{(+)} = \varphi_k^{(-)}, \qquad k = 1, 2, \tag{19}$$

we obtain

$$\begin{aligned} A^{(0)*}A^{(+)} - A^{(0)}A^{(-)*} &= 2a^{(0)}r'\beta \left|A_1^{(\pm)}\right| \left[\cos\left(\Delta\phi_0 \ + \varphi_1^{(\pm)}\right) - \delta\phi \sin\left(\Delta\phi_0 \ + \varphi_1^{(\pm)}\right)\right] \\ & -2a^{(+)}r'\beta A_1^{(0)} \left(\sin\Delta\phi_0 + \delta\phi\cos\Delta\phi_0\right). \end{aligned} \tag{20}$$

Finally, the in-phase and in-quadrature expressions result to be

$$\begin{aligned} |A_f|_I^2 = & -4J_0J_1A_1^{(0)} Im\left\{a^{(+)}\right\} r'\beta \left(\sin\Delta\phi_0 + \delta\phi\cos\Delta\phi_0\right), \\ |A_f|_Q^2 = & +4J_0J_1A_1^{(0)} Re\left\{a^{(+)}\right\} r'\beta \left(\sin\Delta\phi_0 + \delta\phi\cos\Delta\phi_0\right) \\ & +4J_0J_1a^{(0)}r'\beta \left|A_1^{(\pm)}\right| \left[\cos\left(\Delta\phi_0 \ + \varphi_1^{(\pm)}\right) - \delta\phi\sin\left(\Delta\phi_0 \ + \varphi_1^{(\pm)}\right)\right]. \end{aligned} \tag{21}$$

For a gravitational wave case, we obtain the following expressions

$$\begin{aligned} |A_{GW}|_I^2 = & \ 4J_0J_1\left(A_1^{(0)}r + tA_2^{(0)}\right) Re\left\{a^{(+)}\right\} \delta\phi_{GW}, \\ |A_{GW}|_Q^2 = & \ 4J_0J_1\left(A_1^{(0)}r + tA_2^{(0)}\right) Im\left\{a^{(+)}\right\} \delta\phi_{GW}. \end{aligned} \tag{22}$$

Comparing the (21) with the and (22), we deduce the following points :

- a DC offset in the demodulated signal (due to the worst dark fringe),
- a noise on the demodulated signal (due to the phase noise between the two beams).

As we expected and considering $a^{(0)} \simeq 0$, with the phase modulation technique of signal extraction, we have the same results obtained with those given in the previous section (eq. (9)).

4 Noise on the Path Difference

The phase noise $\delta\phi$ on the path difference $\Delta\phi$ between the to beams, is due essentially to the angular noise on the VIRGO beam splitter.

If d is the distance between the two beams measured on the beam splitter second face plane (see fig. 1), and $\tilde{\alpha}(\omega)$ is spectral density of the angular noise, we have

$$\delta\phi = \frac{2\pi}{\lambda}\tilde{\alpha}(\omega)d. \tag{23}$$

For the low frequency range (dominated by the seismic noise), the amplitude of $\tilde{\alpha}(\omega)$ can be estimated using the maximum tilt value allowed for the detection bench [1] and at "higher" frequencies [2] the thermal angular noise :

$$\begin{aligned} \delta\alpha_{rms} &\simeq 1.0\cdot 10^{-6}\ rad, & \nu < 1\ Hz, \\ \delta\tilde{\alpha} &\simeq 2\cdot 10^{-17}\ rad/\sqrt{Hz}, & \nu \simeq 10\ Hz. \end{aligned} \tag{24}$$

Considering that $d \simeq .06\ m$, we obtain

$$\begin{array}{lcll} \delta\phi_{rms} & \simeq & 3.6 \cdot 10^{-1}\ rad, & \nu < 1\ Hz, \\ \delta\tilde{\phi} & \simeq & 7.4 \cdot 10^{-12}\ rad/\sqrt{Hz}, & \nu \simeq 10\ Hz. \end{array} \tag{25}$$

Using the VIRGO-2000 phase sensitivities values

$$\begin{array}{lclll} \langle \delta\phi_{GW} \rangle_{rms} & \simeq & 1.3 \cdot 10^{-8}, & \nu < 1\ Hz, & \text{(overestimated)} \\ \delta\tilde{\phi}_{GW} & \simeq & 1.1 \cdot 10^{-9}/\sqrt{Hz}, & \nu \simeq 10\ Hz, & \end{array} \tag{26}$$

and considering $r' = 10^{-2}$, from the (9), we obtain

$$\begin{array}{ll} \beta \ll 2.5 \cdot 10^{-6}, & \nu < 1\ Hz, \\ \beta \ll 1.1 \cdot 10^{+4}, & \nu \simeq 10\ Hz. \end{array} \tag{27}$$

For VIRGO-1998, considering that the phase sensitivity has to be almost equals to VIRGO-2000, we obtain roughly, the same values for β.

5 "Cross-Talk" Coefficient β

The idea for the β estimation, is to compute the projection of the diffused beam on the TEM_{00} dark-fringe beam; the square modulus of the projection is proportional to β.

The method for computing the diffused beam, is based on the statistic model of the surface roughness [3]. In this case, the expression of a TEM_{00} beam $|\psi_{00}\rangle$ reflected by a rough surface, with an incidence angle θ in the mirror reference frame (see. fig. 2) is

$$|\Psi(\underline{x})\rangle = \sqrt{P_0} e^{2ik\,[f(\underline{x}) + x\theta]}\, |\psi_{00}(\underline{x})\rangle\,, \tag{28}$$

where $\underline{x} = (x, y)$ is a bi-dimensional random variable and $f(\underline{x})$ is a stochastic, stationary function; P_0 is the secondary-beam power. The expectation value and the variance and the autocorrelation function of f, are respectively $\langle f(\underline{x}) \rangle = 0$, $\langle f^2(\underline{x}) \rangle = \sigma^2$, $\langle f^2(\underline{x}) f^2(\underline{x}') \rangle / \sigma^2 = C(|\underline{x} - \underline{x}'|)$.

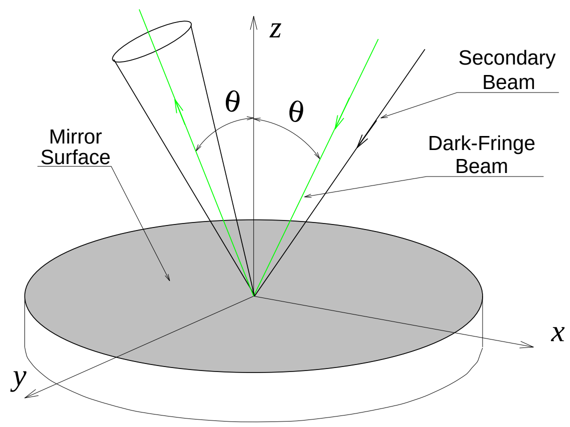

Figure 2: Mirror reference frame.

Supposing $\sigma << \lambda$, the expression of the projection on the TEM_{00} $\sqrt{P_{d.f.}}\,|\psi_{00}(\underline{x})\rangle$ dark-fringe beam becomes

$$\gamma = \sqrt{P_0 P_{d.f.}}\,\langle\psi_{00}\,|\Psi\rangle = 2ik\sqrt{P_0 P_{d.f.}}\int_{R^2} d\underline{x}\;\psi_{00}^*(\underline{x})\psi_{00}(\underline{x})f(\underline{x})e^{2ikx\theta}, \tag{29}$$

and calculating the expectation value for the square modulus of γ, we obtain

$$\langle\gamma^*\gamma\rangle = \epsilon P_0 P_{d.f.}\int d\underline{x}\int d\underline{x}'\;I(\underline{x})I(\underline{x}')e^{2ik(x-x')\theta}C\left(|\underline{x}-\underline{x}'|\right), \tag{30}$$

where $I(\underline{x}) = |\psi_{00}(\underline{x})|^2$ is the TEM_{00} intensity and $\epsilon = 4k^2\sigma^2$ is the scattering loss rate.

With the aid of the Fourier transform, of the Wiener-Khintchine theorem and of some algebra, the previous expression becomes

$$\langle\gamma^*\gamma\rangle = \frac{\epsilon}{4\pi^2}P_0 P_{d.f.}\int d\underline{p}\;\tilde{C}(p-2k\theta)\tilde{I}^2(p). \tag{31}$$

For a TEM_{00}, $\tilde{I}^2(p)$ is a sharply picked function compared with $\tilde{C}(p-2k\theta)$ and the (31) can be approximated with the following one

$$\langle\gamma^*\gamma\rangle \simeq \frac{\epsilon}{4\pi^2}P_0 P_{d.f.}\tilde{C}(2k\theta)\int d\underline{p}\;\tilde{I}^2(p). \tag{32}$$

Solving the integral and considering that [1]

$$\tilde{C}(k\theta) = \frac{\lambda^2}{2\pi}p(\theta), \tag{33}$$

we obtain the almost final result

$$\beta = \frac{\sqrt{\langle\gamma^*\gamma\rangle}}{P_{d.f.}} \simeq \sqrt{\frac{1}{2}\epsilon\frac{P_0}{P_{d.f.}}p(2\theta)}\;\theta_\infty. \tag{34}$$

For angles $\theta > 10^{-3}\ rad$ we have $p(\theta) \simeq a/\theta^2$ and the previous expression becomes

$$\beta \simeq \frac{1}{2}\sqrt{\frac{1}{2}a\epsilon\frac{P_0}{P_{d.f.}}}\;\frac{\theta_\infty}{\theta}. \tag{35}$$

The ratio $P_0/P_{d.f.}$ can be estimated as follows

$$\frac{P_0}{P_{d.f.}} = \frac{r'^2}{1-\mathcal{C}} \tag{36}$$

where $\mathcal{C}$ is the contrast and r'^2 is the reflectivity of the beam splitter secondary face.

Using the following values for VIRGO-2000

$$\begin{array}{ll} \theta = 5^\circ\ , & \\ a\epsilon = 10^{-5}\ , & \theta_\infty = 1.7\cdot 10^{-5}\ rad \\ 1-\mathcal{C} = 10^{-3}\ , & r' = 10^{-2} \end{array}$$

[1] the function $p(\theta)$ has not to be confused with the vector $\underline{p}$ or with his modulus p

we obtain

$$\beta \simeq 6.9 \cdot 10^{-8}, \qquad \text{(VIRGO-2000)}. \tag{37}$$

For VIRGO-1998 instead, we have

$$\begin{array}{ll} \theta = 2.5^\circ \ , & \\ a\epsilon = 10^{-5} \ , & \theta_\infty = 9.68 \cdot 10^{-5} \ rad \\ 1 - \mathcal{C} = 10^{-3} \ , & r' = 10^{-2} \end{array}$$

and we obtain

$$\beta \simeq 7.8 \cdot 10^{-7}, \qquad \text{(VIRGO-1998)}. \tag{38}$$

6 Conclusions

At frequencies $\nu \geq 10\ Hz$, we can largely neglect the "Cross-Talk" effect on the mirrors for VIRGO-1998 and VIRGO-2000 telescopes.

Considering the fact that the virgo phase sensitivity it has been overestimated at $\nu \sim 1\ Hz$, (were the seismic noise dominates) the cross talk effect, continues to be negligible.

References

[1] Virgo Note on the detection bench position sensor, to be published.

[2] **Alignment of the Virgo Interferometer**, *Virgo F.D.* item 4400 version 0. June 1995.

[3] J.-Y. Vinet, V. Brisson, S. Braccini. **Scattered Light Noise in Gravitational Wave Interferometric Detectors: Coherent Effects**, *Virgo Note* NTS96-005 (1996) and *Physical Review D* preprint 54,2 (1966)